# Machine Learning

## 4771

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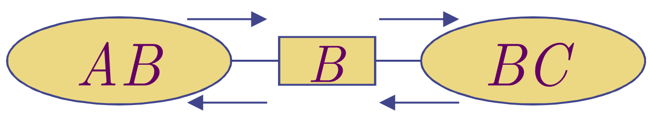
# Lecture 23

- The Junction Tree Algorithm
- Collect & Distribute
- Algorithmic Complexity
- ArgMax Junction Tree Algorithm

# Review: Junction Tree Algorithm

- Send message from each clique *to* its separators of what it thinks the submarginal on the separator is.
- Normalize each clique by incoming message *from* its separators so it agrees with them




$$V = \{A, B\} \quad S = \{B\} \quad W = \{B, C\}$$

**If agree:** $\sum_{V \setminus S} \psi_V = \phi_S = p(S) = \phi_S = \sum_{W \setminus S} \psi_W$ **...Done!**

**Else:**

**Send message From V to W...**

$$\phi_S^* = \sum_{V \setminus S} \psi_V$$
$$\psi_W^* = \frac{\phi_S^*}{\phi_S} \psi_W$$
$$\psi_V^* = \psi_V$$

**Send message From W to V...**

$$\phi_S^{**} = \sum_{W \setminus S} \psi_W^*$$
$$\psi_V^{**} = \frac{\phi_S^{**}}{\phi_S^*} \psi_V^*$$
$$\psi_W^{**} = \psi_W^*$$

**Now they Agree...Done!**

$$\sum_{V \setminus S} \psi_V^{**} = \sum_{V \setminus S} \frac{\phi_S^{**}}{\phi_S^*} \psi_V^* = \frac{\phi_S^{**}}{\phi_S^*} \sum_{V \setminus S} \psi_V^* = \phi_S^{**} = \sum_{W \setminus S} \psi_W^{**}$$

# JTA with Evidence

- Example: if *evidence* is observed, say variable A=1

Initialize as before…

$$\psi_{AB} = p\left(A,B\right) \qquad \psi_{BC} = p\left(C \mid B\right) \qquad \phi_B = 1$$

Update with slice…

$$\phi_B^* = \sum_A \psi_{AB}\delta\left(A=1\right) = \sum_A p\left(A,B\right)\delta\left(A=1\right) = p\left(A=1,B\right)$$

$$\psi_{BC}^* = \frac{\phi_S^*}{\phi_S}\psi_{BC} = \frac{p\left(A=1,B\right)}{1}p\left(C \mid B\right) = p\left(A=1,B,C\right)$$

$$\psi_{AB}^* = \psi_{AB} = p\left(A=1,B\right)$$

If normalized, all $\psi, \phi$ become marginals *conditioned* on evidence

$$p\left(B,C \mid A=1\right) = \frac{\psi_{BC}^*}{\sum_{B,C}\psi_{BC}^*}$$

# JTA with many cliques

- Problem: what if we have more than two cliques?

1) Update AB & BC


2) Update BC & CD


- Problem: AB has not heard about CD!
  After BC updates, it will be inconsistent for AB

- Need to iterate the pairwise updates many times
- This will eventually converge to consistent marginals
- But, inefficient… can we do better?

# JTA: Collect & Distribute

- Trees: recursive, no need to reiterate messages mindlessly!
- Send your message only after hearing from all neighbors…

**initialize(DAG){ Pick root**
**Set all variables as:** $\forall i,\ \text{assign each } p\left(x_i \mid \pi_i\right) \text{ to } 1\ \psi_{C_i}$ **}**
$\forall S, \phi_S = 1$

always exists at least 1, why? can be more than 1?

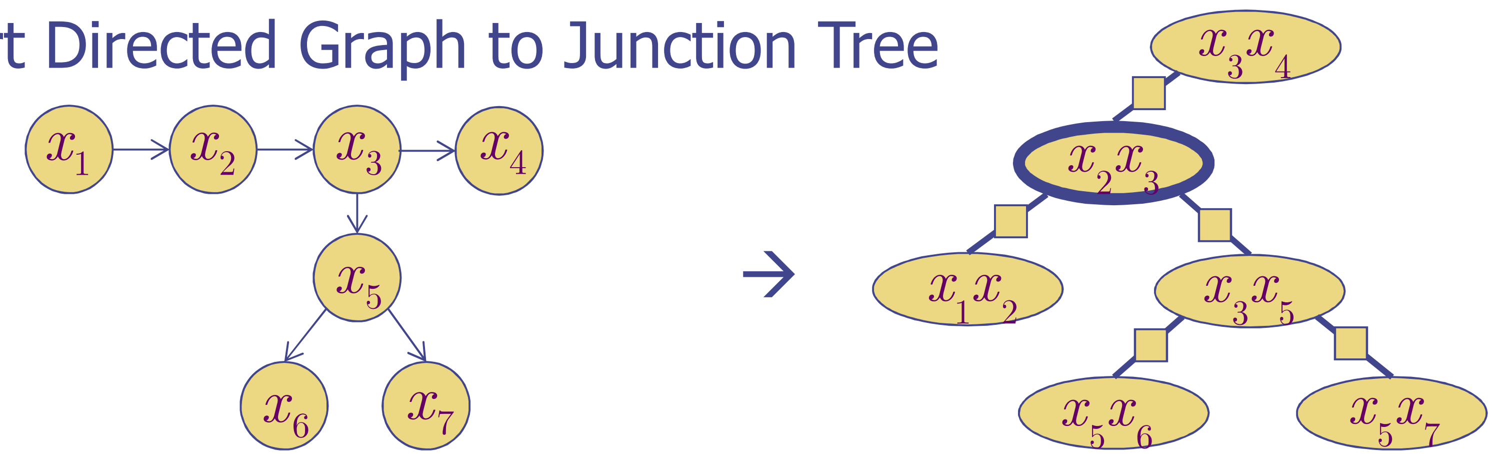
```
collectEvidence(node) {
   for each child of node  {
      update(node,collectEvidence(child)); }
   return(node); }

distributeEvidence(node) {
   for each child of node  {
      update(child,node);
      distributeEvidence(child); } }
```

**normalize(DAG) {** $p\left(X_C\right) = \frac{\psi_C}{\sum_{X_C} \psi_C},\ p\left(X_S\right) = \frac{\phi_S}{\sum_{X_S} \phi_S}$ **}** (optional, depends on application)

**update(node $\psi$,evidence $\phi$) {** $\psi_C^* = \frac{\phi_S^*}{\sum_{C \setminus S} \psi_C} \psi_C$ **}**

# Junction Tree Algorithm

- Convert Directed Graph to Junction Tree



- *Initialize* separators to 1 (and Z=1) and set clique tables to appropriate CPTs in the Directed Graph

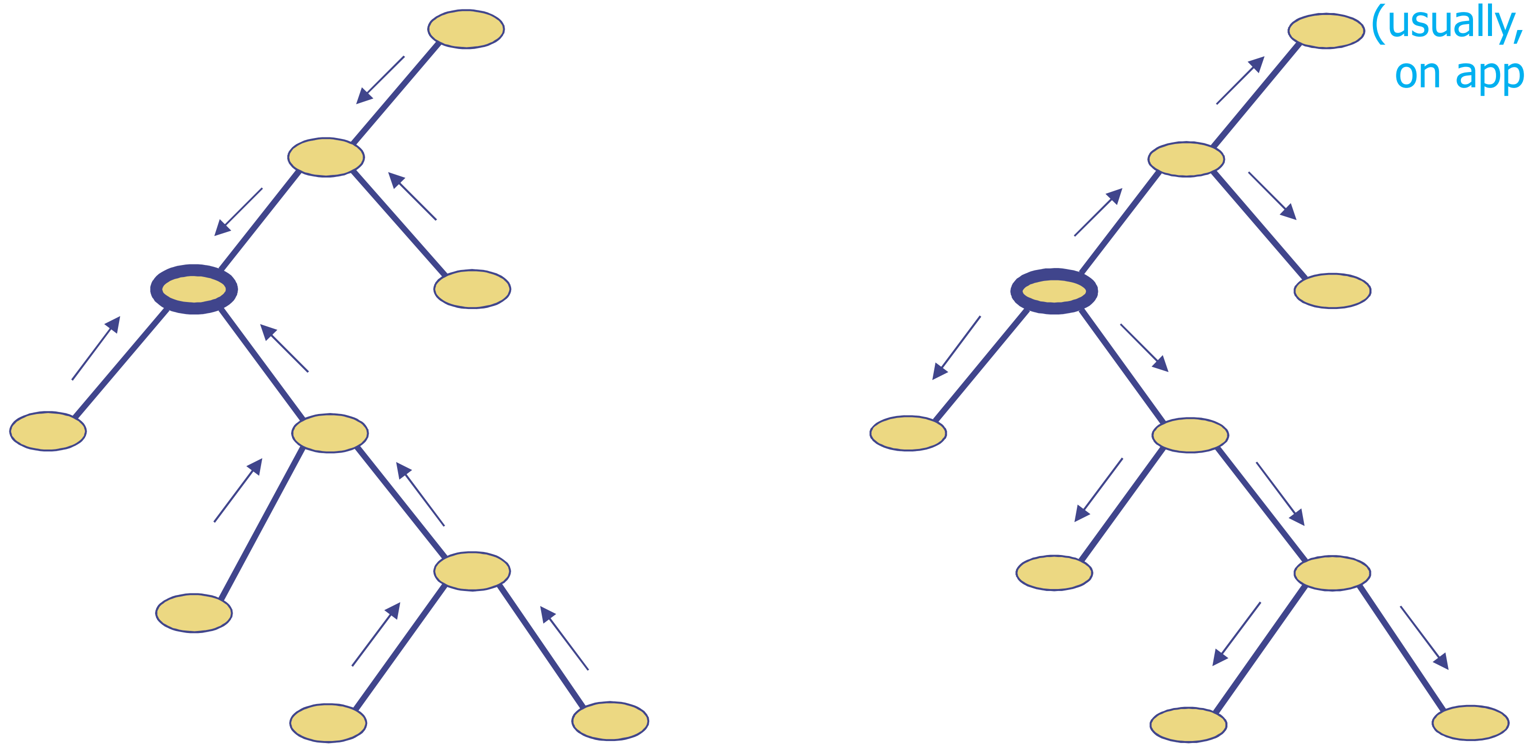
$$p(X) = p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_2)p(x_4 \mid x_3)p(x_5 \mid x_3)p(x_6 \mid x_5)p(x_7 \mid x_5)$$

$$p(X) = \frac{1}{Z}\frac{\prod_C \psi(X_C)}{\prod_S \phi(X_S)}$$

$$= \frac{1}{1}\frac{p(x_1, x_2)p(x_3 \mid x_2)p(x_4 \mid x_3)p(x_5 \mid x_3)p(x_6 \mid x_5)p(x_7 \mid x_5)}{1 \times 1 \times 1 \times 1 \times 1}$$

# Junction Tree Algorithm

- JTA: 1)*Initialize* 2)*Collect* 3)*Distribute* 4)*Normalize* (usually, depends on application)



- Note: leaves do not change their $\psi$ during *collect*
- Note: first cliques to *collect* changes are parents of leaves
- Note: root does not change its $\psi$ during *distribute*

# Algorithmic Complexity

- The 5 steps of JTA are all efficient:

OFFLINE

1) Moralization

Why? Min richness to capture cond indep,
Gtee can assign CPTs to a psi function

Polynomial in # of nodes

2) Introduce Evidence (fixed or constant)

Or can do later

Polynomial in # of nodes (convert pdf to slices)

3) Triangulate (Tarjan & Yannakakis 1984)

Why?
Cycles/RIP

Suboptimal=Polynomial, Optimal=NP

4) Construct Junction Tree (Kruskal)

Polynomial in # of cliques

ONLINE (for each query, new evidence, etc.)

5) Propagate Probabilities (Junction Tree Algorithm)

Polynomial (linear) in # of cliques, *Exponential* in Clique Cardinality

# ArgMax Junction Tree Algorithm

- We can also use JTA for finding the max not the sum over the joint to get argmax of marginals & conditionals
- Say have some evidence: $p\left(X_F, \bar{X}_E\right) = p\left(x_1, \ldots, x_n, \bar{x}_{n+1}, \ldots, \bar{x}_N\right)$
- Most likely (highest p) $X_F$? $X_F^* = \arg\max_{X_F} p\left(X_F, \bar{X}_E\right)$
- What is most likely state of patient with flu & headache?

$$p_F^* = \max_{x_2,x_3,x_4,x_5} p\left(x_1 = 1, x_2, x_3, x_4, x_5, x_6 = 1\right)$$

$$= \max_{x_2} p\left(x_2 \mid x_1 = 1\right) p\left(x_1 = 1\right) \max_{x_3} p\left(x_3 \mid x_1 = 1\right)$$

$$\max_{x_4} p\left(x_4 \mid x_2\right) \max_{x_5} p\left(x_5 \mid x_3\right) p\left(x_6 = 1 \mid x_2, x_5\right)$$

See slide 17:23

max(ab,ac)=a max(b,c)
a,b,c≥0
Can move max like Σ
Others? min?

- Solution: update in JTA uses max instead of sum:

$$\phi_S^* = \max_{V \setminus S} \psi_V \qquad \psi_W^* = \frac{\phi_S^*}{\phi_S} \psi_W \qquad \psi_V^* = \psi_V$$

- Final potentials aren't marginals: $\psi\left(X_C\right) = \max_{U \setminus C} p\left(X\right)$
- Highest value in potential is most likely: $X_C^* = \arg\max_C \psi\left(X_C\right)$