Machine Learning

4771

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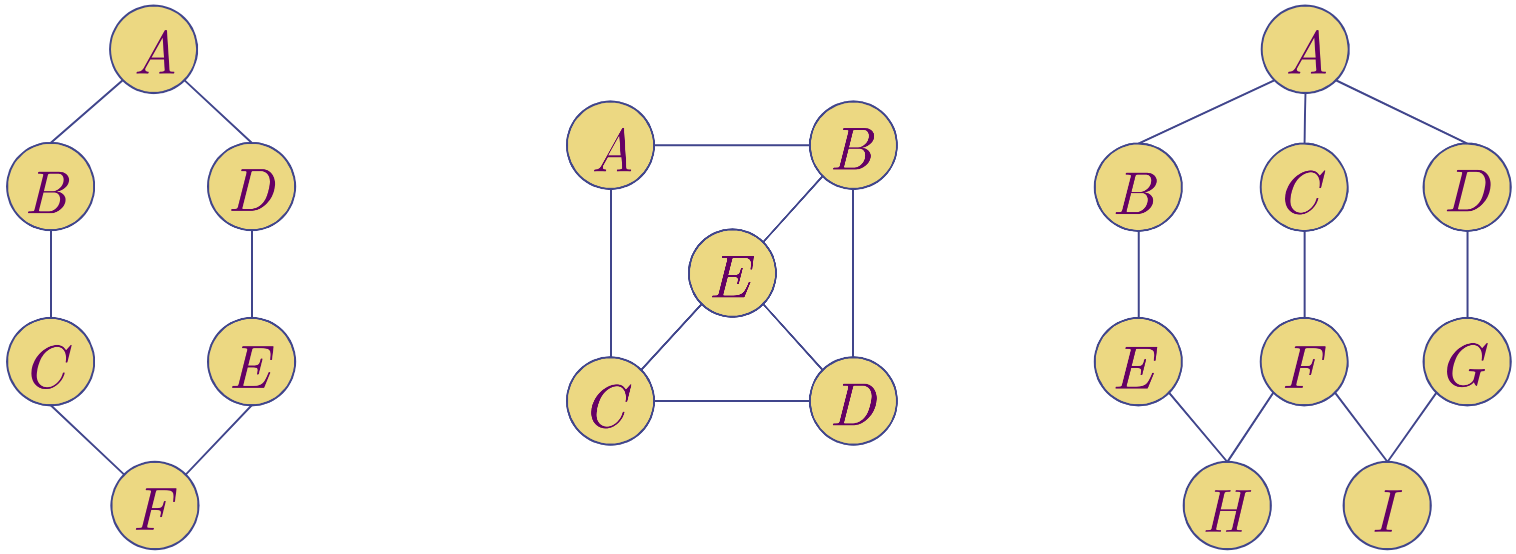
Lecture 22

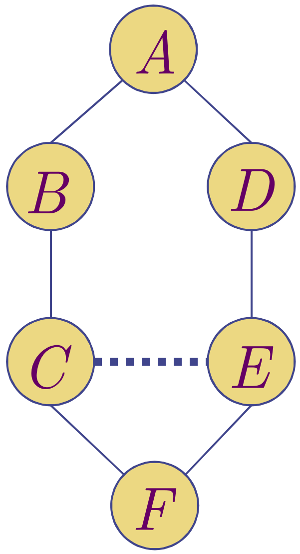
- Triangulation Examples
- Running Intersection Property
- Building a Junction Tree
- The Junction Tree Algorithm

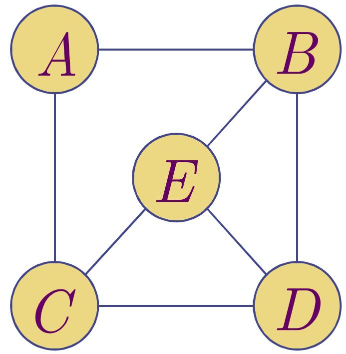
Triangulation Examples

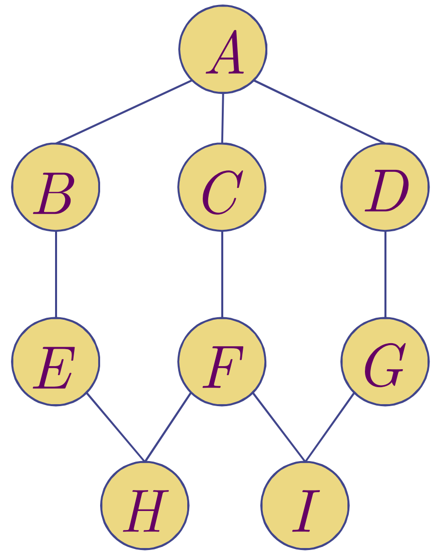


- Cycle: A closed (simple) path, with no repeated vertices other than the starting and ending vertices
- Chordless Cycle: a cycle where no two non-adjacent vertices on the cycle are joined by an edge.
- Triangulated Graph: a graph that contains no chordless cycle of four or more vertices (aka a Chordal Graph).

Triangulation Examples







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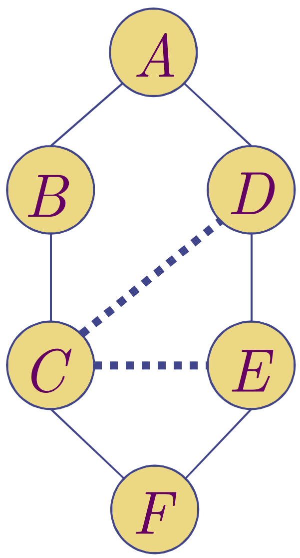

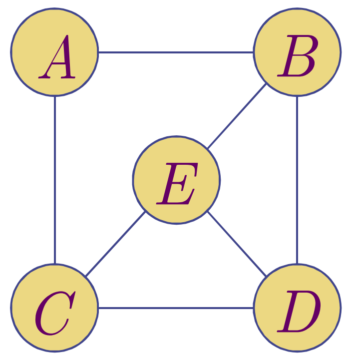

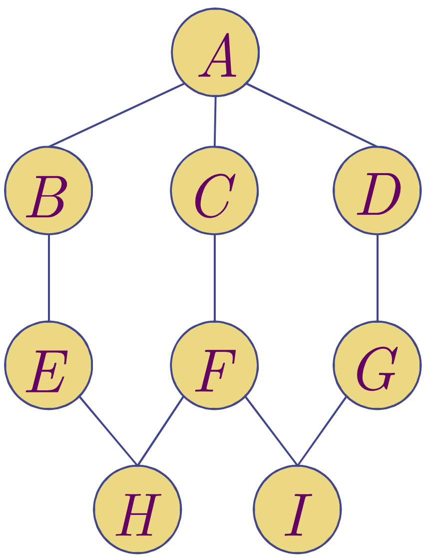

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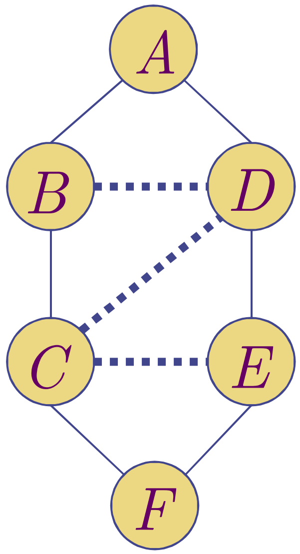

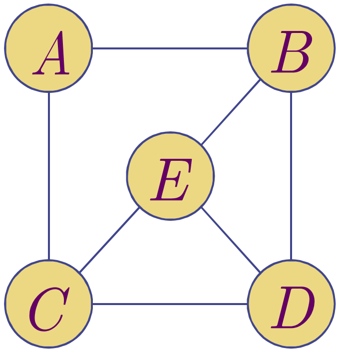

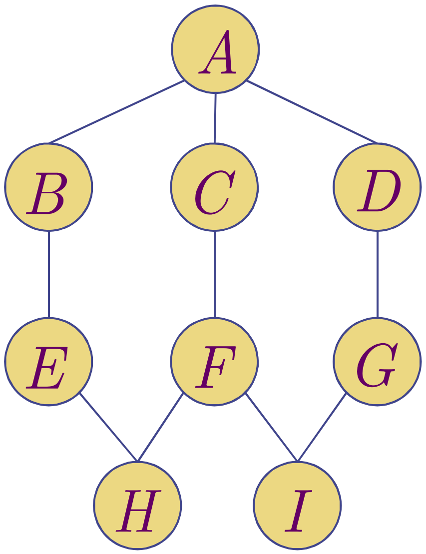

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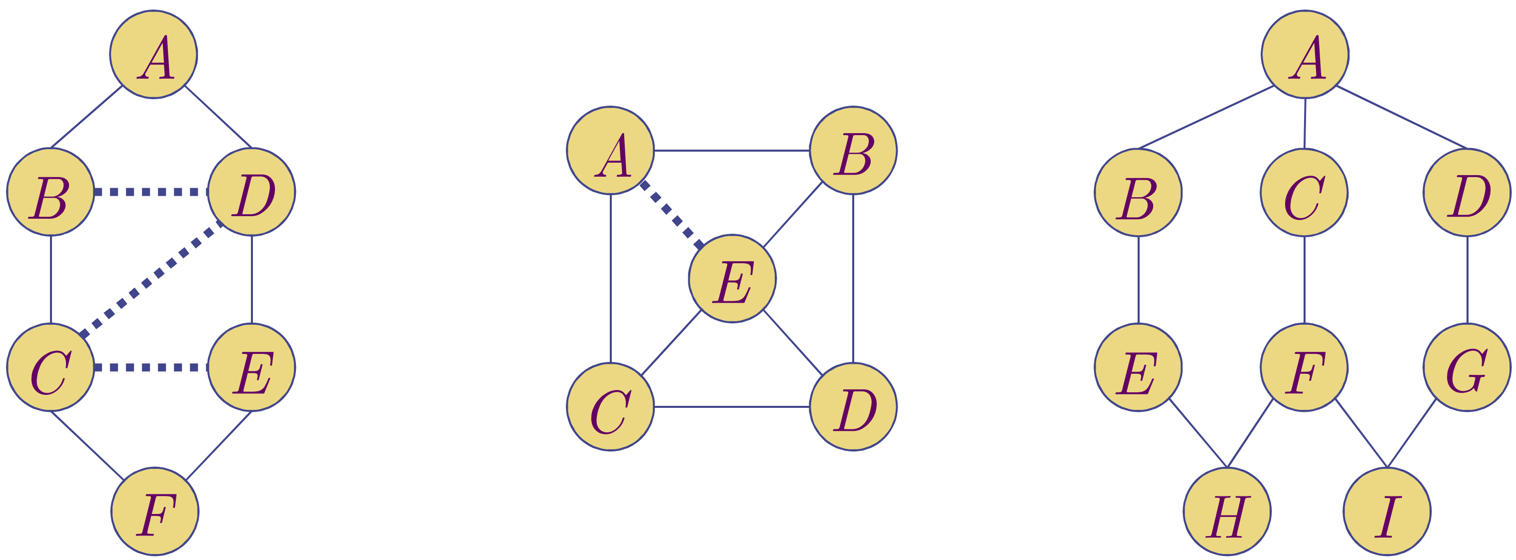

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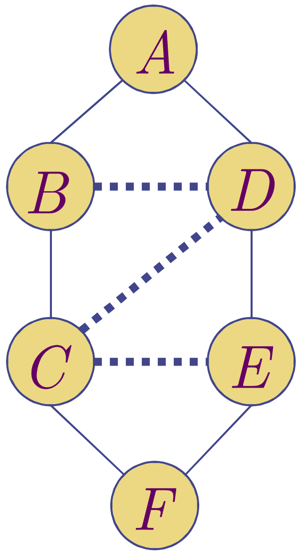

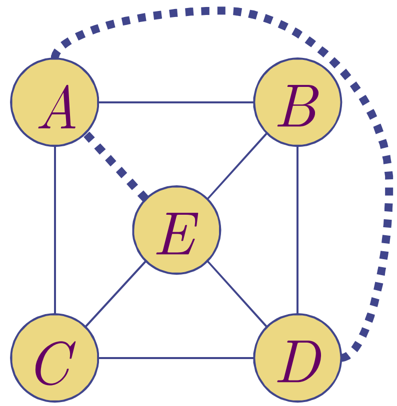

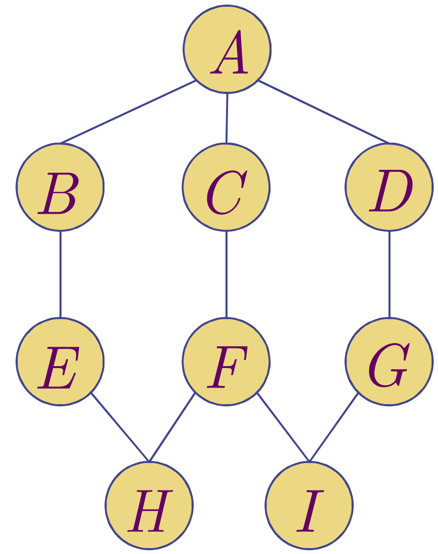

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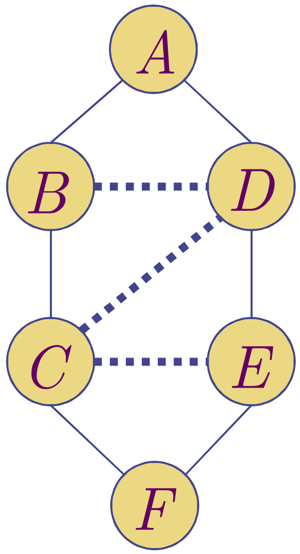

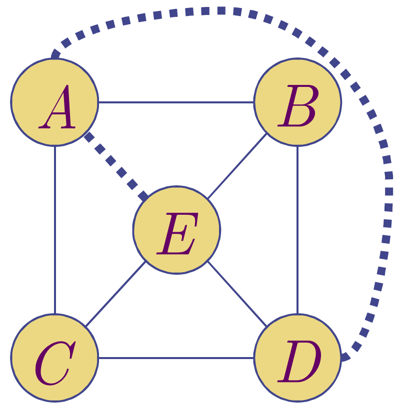

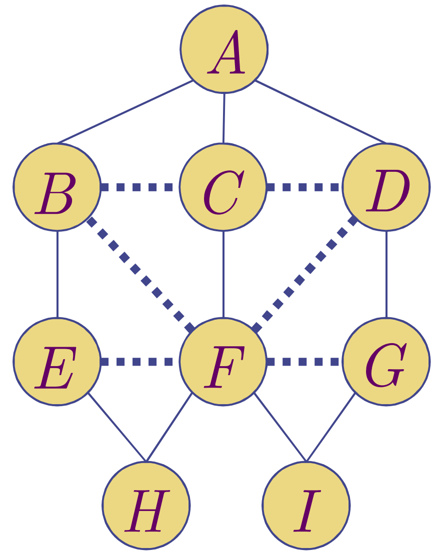

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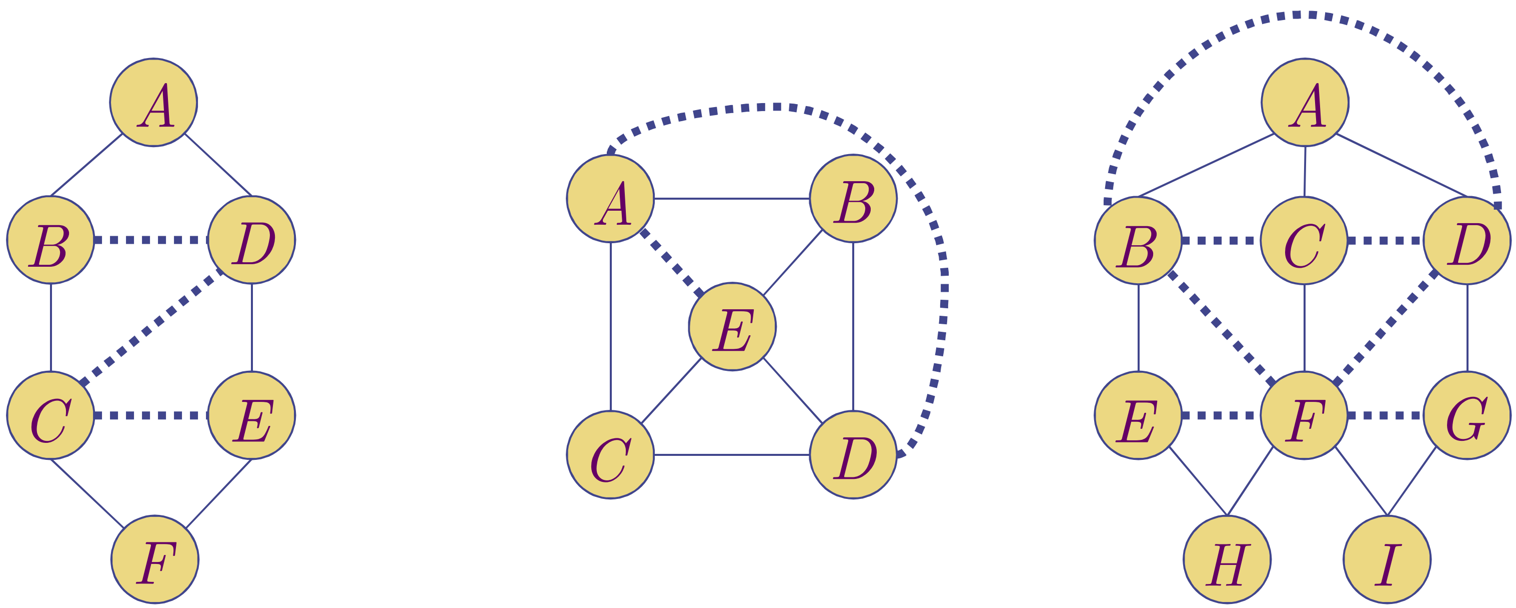
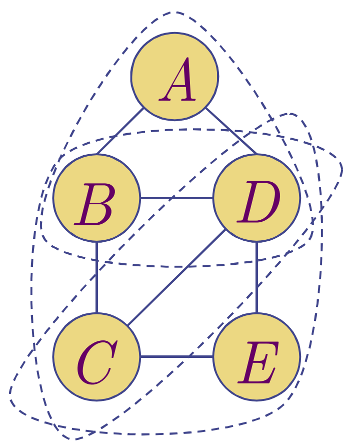
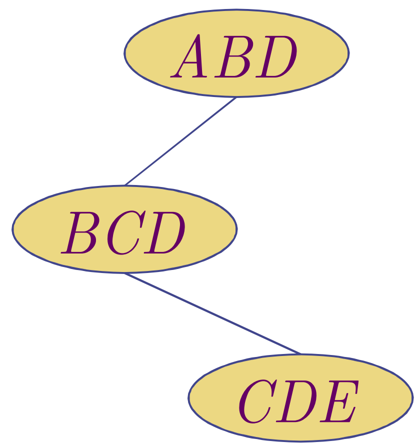
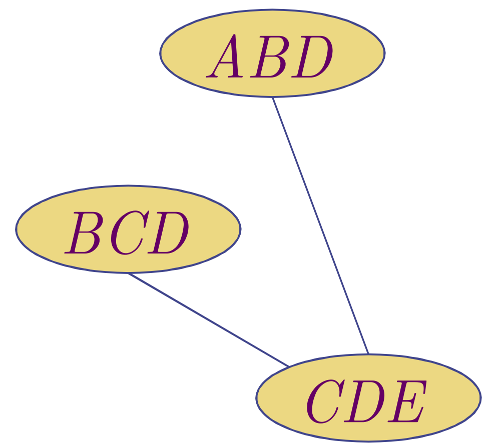


Running Intersection Property

- Junction Tree must satisfy Running Intersection Property
- RIP: On unique path connecting clique V to clique W, all other cliques share nodes in $V \cap W$







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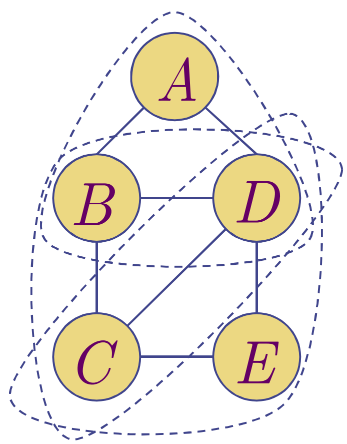
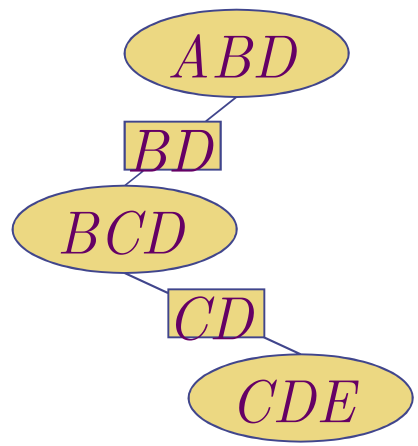
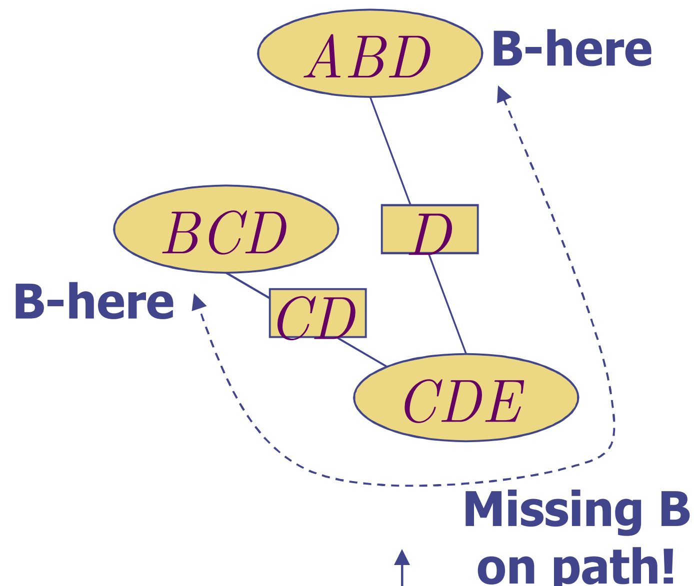






HINT: Junction Tree has largest total separator cardinality

$$|\Phi| = |\phi(B,D)| + |\phi(C,D)| = 2 + 2$$

$$|\Phi| = |\phi(C,D)| + |\phi(D)| = 2 + 1$$

Forming the Junction Tree

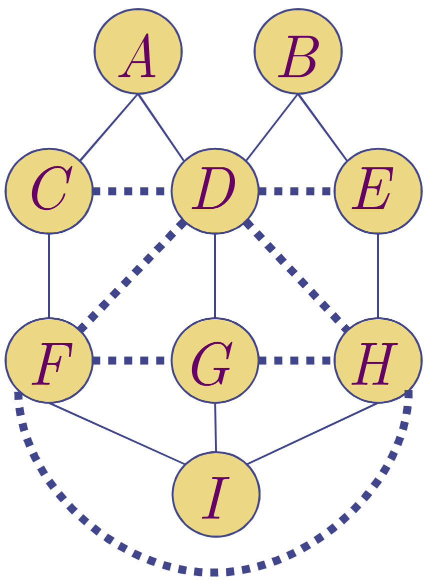
- Goal: connect k cliques into a tree… k^{k-2} possibilities!
- For each, check Running Intersection Property, too slow…
- Theorem: a valid (RIP) Junction Tree connection is one that maximizes the cardinality of the separators

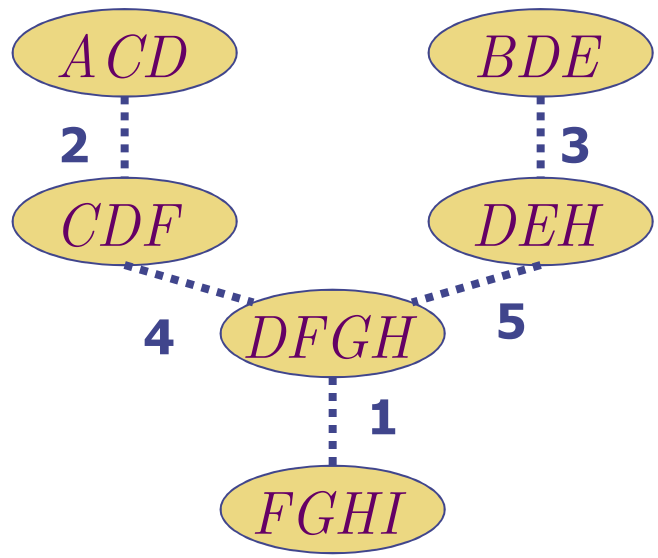
$$
\begin{aligned}
JT^* &= \arg\max_{TREE\,STRUCTURES} \left|\Phi\right| \\
&= \arg\max_{TREE\,STRUCTURES} \sum_S \left|\phi\left(X_S\right)\right|
\end{aligned}
$$

- Use very fast Kruskal algorithm:
  1) Init Tree with all cliques unconnected (no edges)
  2) Compute size of separators between all pairs
  3) Connect the two cliques with the biggest separator cardinality which doesn’t create a loop in current Tree (maintains Tree structure)
  4) Stop when all nodes are connected, else goto 3

Kruskal Example

- Start with unconnected cliques (after triangulation)





Connection with triangulation?

	ACD	BDE	CDF	DEH	DFGH	FGHI
ACD	-	1	2	1	1	0
BDE		-	1	2	1	0
CDF			-	1	2	1
DEH				-	2	1
DFGH					-	3
FGHI						-

Junction Tree Probabilities

- We now have a valid Junction Tree!
- What does that mean?
- Recall probability for undirected graphs:

$$p(X) = p(x_1, \ldots, x_M) = \tfrac{1}{Z} \prod_C \psi(X_C)$$

- Can write junction tree as potentials of its cliques:

$$p(X) = \tfrac{1}{Z} \prod_C \tilde{\psi}(X_C)$$

- Alternatively: clique potentials over separator potentials:

$$p(X) = \frac{1}{Z} \frac{\prod_C \psi(X_C)}{\prod_S \phi(X_S)}$$

- This doesn't change/do anything! Just less compact…
- Like *de-absorbing* smaller cliques from maximal cliques:

$$\tilde{\psi}(A,B,D) = \frac{\psi(A,B,D)}{\phi(B,D)} \longleftarrow$$

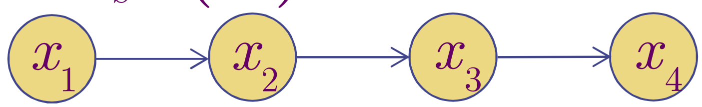
…gives back original formula if $\phi(B,D) \triangleq 1$

Junction Tree Probabilities

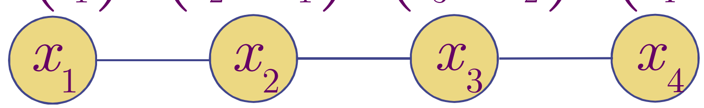
- Can quickly converted directed graph into this form:

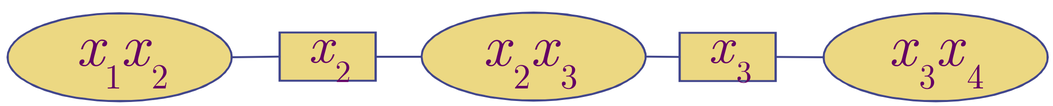
$$p(X) = \frac{1}{Z} \frac{\prod_C \psi(X_C)}{\prod_S \phi(X_S)}$$

- Example:



$$p(X) = p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_2)p(x_4 \mid x_3)$$





$$p(X) = \frac{1}{1} \frac{p(x_1, x_2)p(x_3 \mid x_2)p(x_4 \mid x_3)}{1 \times 1}$$

$$= \frac{1}{Z} \frac{\psi(x_1, x_2)\psi(x_2, x_3)\psi(x_3, x_4)}{\phi(x_2)\phi(x_3)}$$

By inspection, can just cut & paste CPTs as cliques and add separator potential functions

Junction Tree Algorithm

- Running the JTA converts clique potentials & separator potentials into marginals over their variables … and does not change p(X)

$$\psi(A,B,D) \to p(A,B,D)$$
$$\phi(B,D) \to p(B,D)$$
$$\psi(B,C,D) \to p(B,C,D)$$

- Don't want just normalization!

$$\frac{\psi(A,B,D)}{\sum_{A,B,D}\psi(A,B,D)} \neq p(A,B,D)$$

- These marginals should all agree & be consistent

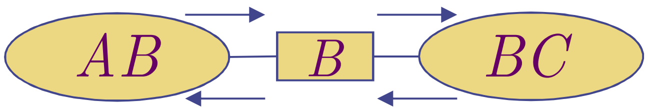
$$\psi(A,B,D) \to p(A,B,D) \quad \to \sum_A p(A,B,D) = \tilde{p}(B,D)$$
$$\phi(B,D) \to p(B,D) \quad \to p(B,D)$$
$$\psi(B,C,D) \to p(B,C,D) \quad \to \sum_C p(B,C,D) = \tilde{\tilde{p}}(B,D)$$

ALL EQUAL

- Consistency: all distributions agree on submarginals
- JTA sends messages between cliques & separators dividing each by the others marginals until consistency…

Junction Tree Algorithm

- Send message from each clique *to* its separators of what it thinks the submarginal on the separator is.
- Normalize each clique by incoming message *from* its separators so it agrees with them



$V = \{A, B\} \quad S = \{B\} \quad W = \{B, C\}$

If agree: $\sum_{V \setminus S} \psi_V = \phi_S = p(S) = \phi_S = \sum_{W \setminus S} \psi_W$ **...Done!**

Else:

Send message From V to W...

$$\phi_S^* = \sum_{V \setminus S} \psi_V$$
$$\psi_W^* = \frac{\phi_S^*}{\phi_S} \psi_W$$
$$\psi_V^* = \psi_V$$

Send message From W to V...

$$\phi_S^{**} = \sum_{W \setminus S} \psi_W^*$$
$$\psi_V^{**} = \frac{\phi_S^{**}}{\phi_S^*} \psi_V^*$$
$$\psi_W^{**} = \psi_W^*$$

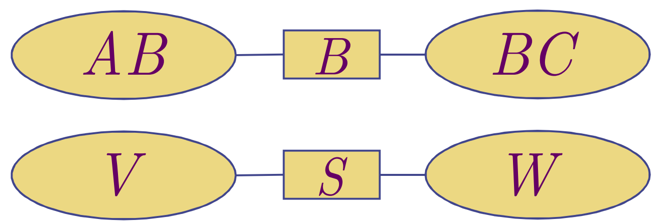
Now they Agree...Done!

$$\sum_{V \setminus S} \psi_V^{**} = \sum_{V \setminus S} \frac{\phi_S^{**}}{\phi_S^*} \psi_V^* = \frac{\phi_S^{**}}{\phi_S^*} \sum_{V \setminus S} \psi_V^* = \phi_S^{**} = \sum_{W \setminus S} \psi_W^{**}$$

Junction Tree Algorithm

- When "Done", all clique potentials are marginals and all separator potentials are submarginals!
- Note that p(X) is unchanged by message passing step:

$$\boxed{\begin{aligned} \phi_S^* &= \sum\nolimits_{V \backslash S} \psi_V \\ \psi_W^* &= \frac{\phi_S^*}{\phi_S} \psi_W \\ \psi_V^* &= \psi_V \end{aligned}}$$



$$p\left(X\right) = \frac{1}{Z} \frac{\psi_V^* \psi_W^*}{\phi_S^*} = \frac{1}{Z} \frac{\psi_V \frac{\phi_S^*}{\phi_S} \psi_W}{\phi_S^*} = \frac{1}{Z} \frac{\psi_V \psi_W}{\phi_S}$$

- Potentials set to conditionals (or slices) become marginals!

$$\begin{aligned} \psi_{AB} &= p\left(B \mid A\right) p\left(A\right) \\ &= p\left(A, B\right) \end{aligned} \quad \longrightarrow \quad \phi_B^* = \sum\nolimits_A \psi_{AB} = \sum\nolimits_A p\left(A, B\right) = p\left(B\right)$$

$$\psi_{BC} = p\left(C \mid B\right) \quad \longrightarrow \quad \psi_{BC}^* = \frac{\phi_S^*}{\phi_S} \psi_{BC} = \frac{p\left(B\right)}{1} p\left(C \mid B\right) = p\left(B, C\right)$$

$$\phi_B = 1$$