Optimal Distribution Testing via Reductions

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Distribution Testing

Given samples from one or more unknown probability distributions, decide whether they satisfy a certain property.

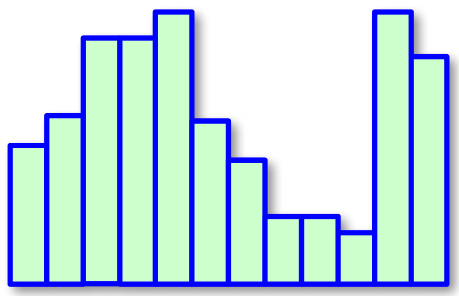
- Introduced by Karl Pearson (1899).
- Classical Problem in Statistics
  [Neyman-Pearson'33, Lehman-Romano'05]
- Last fifteen years (TCS): property testing
  [Goldreich-Ron'00, Batu *et al.* FOCS'00/JACM'13]

Notation

Basic object of study:

Probability distributions over finite domain.

$[n]$ or $[n]^d$

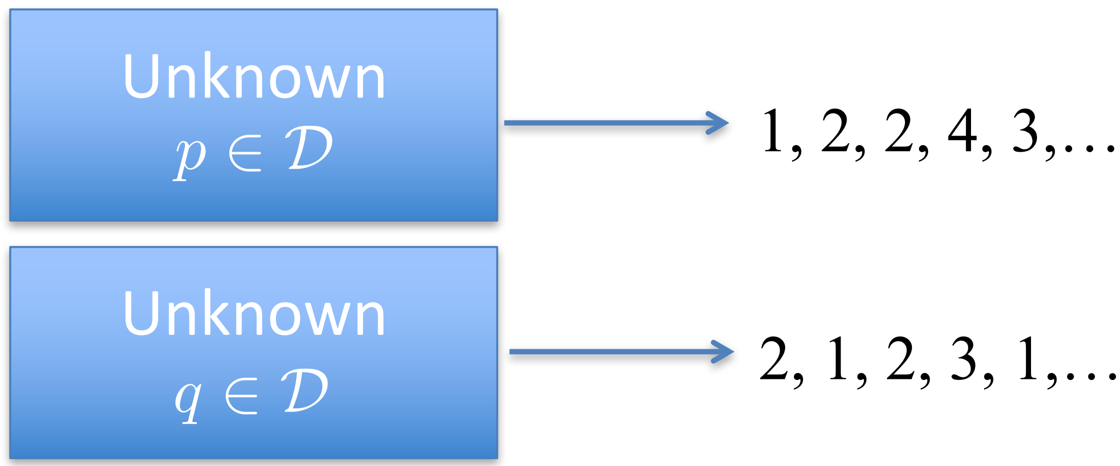


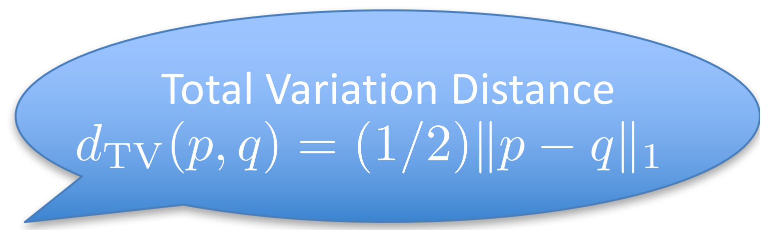
Notation:

p, q: probability mass function

Example: Testing Closeness

- Let $\mathcal{D}$ be a family of probability distributions





Example:

Testing Closeness Problem:

– Distinguish between the cases p=q and dist (p, q) > ε

– Minimize **sample size**, computation time

This Work

Simple Framework for Distribution Testing:
Leads to *sample-optimal and computationally efficient* estimators
for a variety of properties

Primarily based on:

A New Approach for Testing Properties of Discrete Distributions
(I. Diakonikolas and D. Kane, FOCS'16)

Outline

- Related and Prior Work
- Framework Overview and Statement of Results
- Case Study: Testing Identity, Closeness, and Independence
- Future Directions and Concluding Remarks

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Prior Work: Identity Testing

Focus has been on arbitrary distributions over support of size n .

Testing Identity to a *known* Distribution:

- [Goldreich-Ron'00]: $O(\sqrt{n}/\epsilon^4)$ upper bound for *uniformity testing* (collision statistics)
- [Batu *et al.,* FOCS'01]: $\widetilde{O}(\sqrt{n}) \cdot \mathrm{poly}(1/\epsilon)$ upper bound for testing identity to any *known* distribution.
- [Paninski '03]: upper bound of $O(\sqrt{n}/\epsilon^2)$ for uniformity testing, assuming $\epsilon = \Omega(n^{-1/4})$. Lower bound of $\Omega(\sqrt{n}/\epsilon^2)$.
- [Valiant-Valiant, FOCS'14, D-Kane-Nikishkin, SODA'15]: upper bound of $O(\sqrt{n}/\epsilon^2)$ for identity testing to any known distribution.
- [D-Gouleakis-Peebles-Price'16]: [GR'00] tester is optimal!

Prior Work: Closeness Testing

Focus has been on arbitrary distributions over support of size n .

Testing Closeness between two *unknown* distributions:

- [Batu *et al.,* FOCS'00]: $O(n^{2/3} \log n / \epsilon^{8/3})$ upper bound for testing closeness between two unknown discrete distributions.

- [P. Valiant, STOC'08]: lower bound of $\Omega(n^{2/3})$ for constant error.

- [Chan-D-Valiant-Valiant, SODA'14]: tight upper and lower bound of

$$O(\max\{n^{2/3}/\epsilon^{4/3}, n^{1/2}/\epsilon^2\})$$

- [Bhatacharya-Valiant, NIPS'15]: tight bounds for different sample sizes (assuming $\epsilon > n^{-1/12}$).

Prior Work: Testing Independence

Focus has been on arbitrary distributions over support of size n .

Testing Independence of a distribution on $[n] \times [m]$**.:**

- [Batu *et al.,* FOCS'01]: $\widetilde{O}(n^{2/3}m^{1/3} \cdot \mathrm{poly}(1/\epsilon))$ upper bound.
- [Levi-Ron-Rubinfeld, ICS'11]: lower bounds for constant error $\Omega(m^{1/2}n^{1/2}) \quad \text{and} \quad \Omega(n^{2/3}m^{1/3}), \text{ for } n = \Omega(m \log m)$
- [Acharya-Daskalakis-Kamath, NIPS'15]: upper bound of $O(n/\epsilon^2)$ for n=m.

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L2 Closeness Testing

Lemma 1: Let p, q be unknown distributions on a domain of size n . There is an algorithm that uses

$$O(\min\{\|p\|_2, \|q\|_2\}n/\epsilon^2)$$

samples from each of p, q, and with probability at least 2/3 distinguishes between the cases that $p = q$ and $\|p - q\|_1 \geq \epsilon$.

Basic Tester [Chan-D-Valiant-Valiant'14]**:**

- Calculate $Z = \sum_i \{(X_i - Y_i)^2 - X_i - Y_i\}$
- If $Z > \varepsilon^2 m^2$ then output "No" (different), otherwise, output "Yes" (same)

Collision-based estimator also works [D-Gouleakis-Peebles-Price'16]

Main New Idea

Solve *all* problems by reducing to this as a black-box.

Framework and Results

- **Approach**: Reduction of L1 Testing to L2 testing

1) Transform given distribution(s) to new distribution(s) (over potentially larger domain) with small L2 norm.

2) Use standard L2 tester as a black-box.

- Circumvents method of explicitly learning heavy elements [Batu et al., FOCS'00]

Algorithmic Applications

Sample Optimal Testers for:

- Identity to a Fixed Distribution
- Closeness between two Unknown Distributions
- (Nearly) Instance-optimal Identity Testing

Simpler Proofs of Known Results

- Closeness with unequal sample size
- *Adaptive* Closeness Testing
- Independence (in any dimension)
- Properties of Collections of Distributions (Sample & Query model)
- Testing Histograms
- Other Metrics (chi-squared, Hellinger)

New Results

All algorithms follow same pattern. Very simple analysis.

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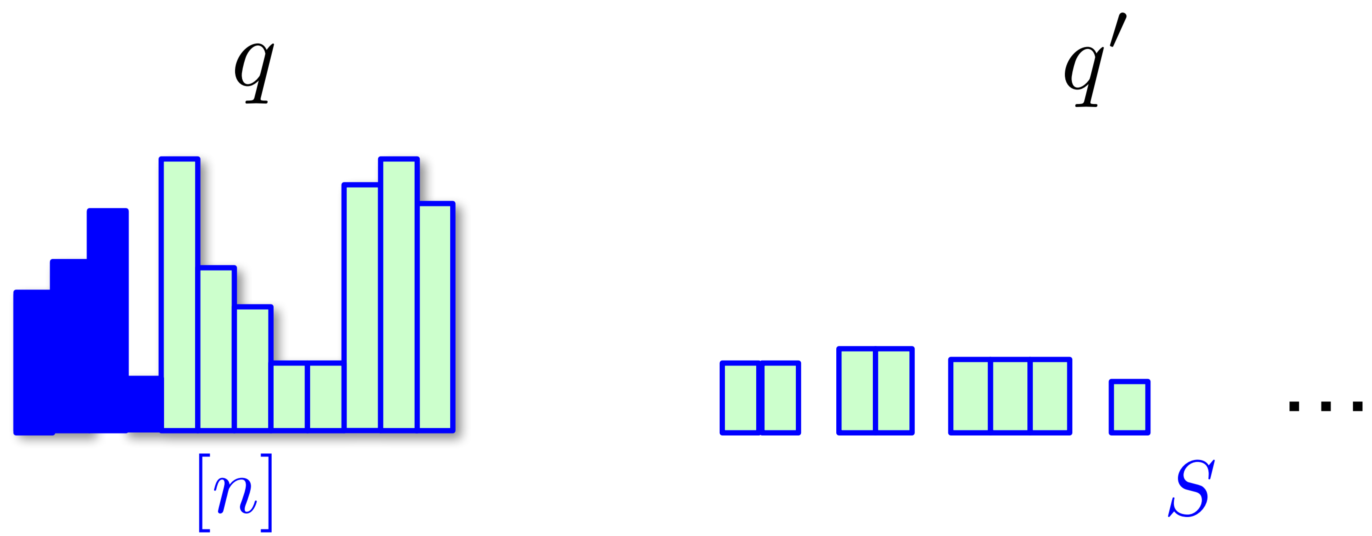
Warm-up: Testing Identity to Fixed Distribution (I)

Let p be unknown distribution and q known distribution on $[n]$.

Main Idea: "Stretch" the domain size to make L_2 norm of q small.

- For every bin $i \in [n]$ create set S_i of $\lceil nq_i \rceil$ new bins.
- Subdivide the probability mass of bin i equally within S_i.

Let S be the new domain and p', q' the resulting distributions over S.



Warm-up: Testing Identity to Fixed Distribution (II)

Let p be unknown distribution and q known distribution on $[n]$.

L1 Identity Tester

- Given q, construct new domain S.
- Use basic tester to distinguish between $p' = q'$ and $\|p' - q'\|_1 \geq \epsilon$.

We construct q' explicitly. Can sample from p' given sample from p.

Analysis:

Observation 1: $\|p' - q'\|_1 = \|p - q\|_1$

Observation 2: $|S| \leq 2n$ and $\|q'\|_2 = O(1/\sqrt{n})$

By Lemma 1, we can test identity between p' and q' with sample size

$$O(\|q'\|_2 |S|/\epsilon^2) = O(\sqrt{n}/\epsilon^2)$$

Identity Reduces to Uniformity

- **Summary of Previous Slides:**

Identity reduces to its special case when the explicit distribution has max probability $O(1/n)$.

- **Recent Improvement:**

[Oded Goldreich'16]:

Identity Reduces to Uniformity.

Testing Closeness (I)

Let p, q be unknown distributions on $[n]$.

Main Idea: Use samples from q to "stretch" the domain size.

- Draw a set S of $\mathrm{Poi}(k)$ samples from q.
- Let a_i be the number of times we see $i \in [n]$ in S.
- Subdivide the mass of bin i equally within $a_i + 1$ new bins.

Let S' be the new domain and p', q' the resulting distributions over S'.

We can sample from p', q'.

Observation: $\|p' - q'\|_1 = \|p - q\|_1$

Testing Closeness (II)

Let p, q be unknown distributions on $[n]$.

L1 Closeness Tester

- Draw a set S of $\mathrm{Poi}(k)$ samples from q, construct new domain S'.
- Use basic tester to distinguish between $p' = q'$ and $\|p' - q'\|_1 \geq \epsilon$.

Claim: Whp $|S'| \leq n + O(k)$ and $\|q'\|_2 = O(1/\sqrt{k})$.

Proof :

$$\|p'\|_2^2 = \sum_{i=1}^n p_i^2/(1+a_i), \quad \mathbb{E}[1/(1+a_i)] \leq 1/(kp_i). \quad \Box$$

By Lemma 1, we can test identity between p' and q' with sample size

$$O(\|q'\|_2 |S'|/\epsilon^2) = O(k^{-1/2} \cdot (n+k)/\epsilon^2).$$

Total sample size

$$O(k + k^{-1/2} \cdot (n+k)/\epsilon^2).$$

Set $\quad k := \min\{n, n^{2/3}\epsilon^{-4/3}\}$.

Closeness with Unequal Samples

Let p, q be unknown distributions on $[n]$.
Have $m_1 + m_2$ samples from q and m_2 samples from p.

L1 Closeness Tester Unequal

- Set $k := \min\{n, m_1\}$.
- Draw $\mathrm{Poi}(k)$ samples from q, construct new domain S'.
- Use basic tester to distinguish between $p' = q'$ and $\|p' - q'\|_1 \geq \epsilon$.

Claim: Whp $|S'| \leq n + O(k)$ and $\|q'\|_2 = O(1/\sqrt{k})$.

By Lemma 1, we can test identity between p' and q' with sample size

$$m_2 = O(\|q'\|_2 |S'|/\epsilon^2) = O(k^{-1/2} \cdot (n+k)/\epsilon^2).$$

By our choice of k, it follows

$$m_2 = O(\max\{n m_1^{-1/2} \epsilon^2, n^{1/2}/\epsilon^2\}).$$

Testing Independence in 2-d

Let p be unknown distribution on $[n] \times [m]$.
Let $q = p_1 \times p_2$.

L1 Independence Tester

- Set $k := \min\{n, n^{2/3}m^{1/3}\epsilon^{-4/3}\}$.
- Draw a set S_1 of $\mathrm{Poi}(k)$ samples from p_1, and S_2 of $\mathrm{Poi}(m)$ samples from p_2.
- Stretch domain in each dimension to obtain new support.
- Use basic tester to distinguish between $p' = q'$ and $\|p' - q'\|_1 \geq \epsilon$.

By Lemma 1, we can test identity between p' and q' with sample size

$$O(\|q'\|_2|S'|/\epsilon^2) = O(k^{-1/2}m^{-1/2} \cdot mn/\epsilon^2)$$

$$= O(\max\{n^{2/3}m^{1/3}\epsilon^{-4/3}, (mn)^{1/2}/\epsilon^2\})$$

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Future Directions (I)

This Talk: Unified Technique for Testing *Unstructured* Discrete Distributions.

Gives sample-optimal estimators for many properties in the literature.

Game Over?

- Recent line of work on Testing *Structured* Distributions [D-Kane-Nikishkin, SODA'15 / FOCS'15 / ICALP'16]

- Dependence on error probability? [D-Gouleakis-Peebles-Price'17]
  E.g., identity testing

$$O(\sqrt{n \log(1/\delta)}/\epsilon^2 + \log(1/\delta)/\epsilon^2)$$

- Optimal Constants? Practically relevant question; requires new insights. [Huang-Meyn IEEE ToIT'14]

Future Directions (II)

This Talk: Unified Technique for Testing *Unstructured* Discrete Distributions.

Future Directions:

- High-Dimensional *Structured* Distributions
  [Canonne-**D**-Kane-Stewart'16, Daskalakis-Pan'16, Daskalakis-Dikkala-Kamath'16, **D**-Kane-Stewart'17]

- Other criteria (privacy, communication, etc.)
  [Cai-Daskalakis-Kamath'17, Aliakbarpour-**D**-Rubinfeld'17, Acharya-Sun-Zhang'17, **D**-Grigorescu-Onak-Natarajan'16]

- Beyond Worst-Case Analysis

Thank you for your attention!