FIFTY SHADES OF ADAPTIVITY (IN PROPERTY TESTING)

An Adaptivity Hierarchy Theorem for Property Testing

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Joint work with **Tom Gur** (Weizmann Institute)

“PROPERTY TESTING?”

WHY?

Sublinear,

WHY?

Sublinear, approximate,

why?

Sublinear, approximate, randomized

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Sublinear, approximate, randomized decision algorithms that make queries

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Sublinear, approximate, randomized decision algorithms that make queries

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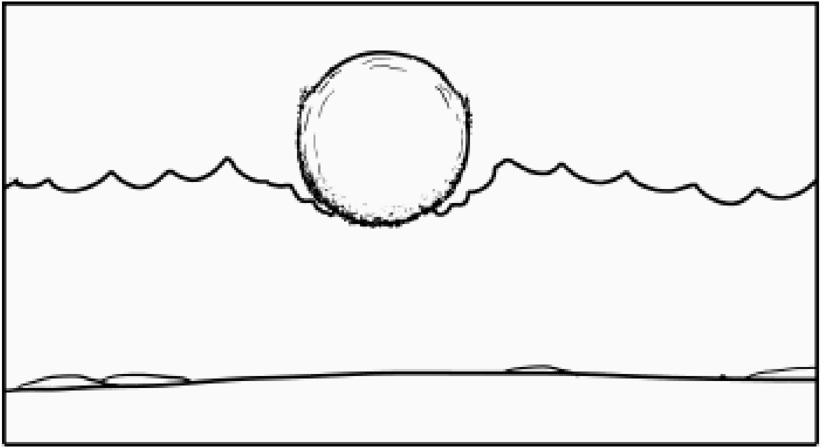
- Expensive access: pricey data
- “Model selection”: many options
- Good Enough: a priori knowledge

Need to infer information – one bit – from the data: quickly, or with very few lookups.



HOW?

Known space (say, $\{0,1\}^N$)

Property $\mathcal{P} \subseteq \{0,1\}^N$)

Query (oracle) access to unknown $x \in \{0,1\}^N$

Proximity parameter $\varepsilon \in (0,1]$

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Must decide:

$$x \in \mathcal{P}, \text{ or } d(x,\mathcal{P}) > \varepsilon?$$

(and be correct on any x with probability at least 2/3)

Property Testing:

HOW?

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Property Testing:

in an (egg)shell.

Many flavors...

... one-sided vs. two-sided,

IT'S COMPLICATED

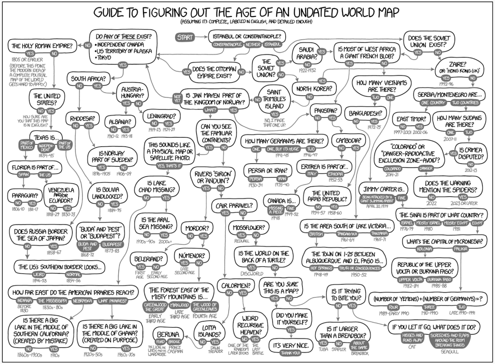
Many flavors...

... one-sided vs. two-sided, query-based vs. sample-based, Many flavors...

... one-sided vs. two-sided, query-based vs. sample-based, uniform vs. distribution-free,

Many flavors...

... one-sided vs. two-sided, query-based vs. sample-based, uniform vs. distribution-free, adaptive vs. non-adaptive



ADAPTIVITY

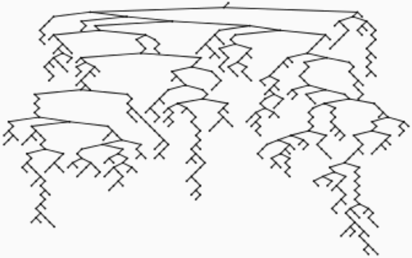
Non-adaptive algorithm

Makes all its queries upfront:

$$Q \subseteq [N] = Q(\varepsilon, r) = \{i_1, \ldots, i_q\}$$

Adaptive algorithm

Each query can depend arbitrarily on the previous answers:



Dense graph model

At most a quadratic gap between adaptive and non-adaptive algorithms: q vs. $2q^2$ [AFKS00, GT03],[GR11]

Boolean functions

At most an exponential gap between adaptive and non-adaptive algorithms: q vs. 2^q

Bounded-degree graph model

Everything is possible: $O(1)$ vs. $\Omega(\sqrt{n})$. [RS06]

Of course

Fewer queries is always better.

WHY SHOULD WE CARE?

Of course

Fewer queries is always better.

But

Many parallel queries can beat few sequential ones.

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Understanding the benefits and tradeoffs of adaptivity is crucial.

A closer look

Does the power of testing algorithms smoothly grow with the "amount of adaptivity?"

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Does the power of testing algorithms smoothly grow with the "amount of adaptivity?"

(and what does "amount of adaptivity" even mean?)

Definition (Round-Adaptive Testing Algorithms)

Let Ω be a domain of size n, and $k, q \leq n$. A randomized algorithm is said to be a (k, q)-round-adaptive tester for $\mathcal{P} \subseteq 2^{\Omega}$, if, on input $\varepsilon \in (0, 1]$ and granted query access to $f\colon \Omega \to \{0, 1\}$:

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(i) Query Generation: The algorithm proceeds in $k + 1$ rounds, such that at round $\ell \geq 0$, it produces a set of queries $Q_\ell := \{x^{(\ell),1}, \ldots, x^{(\ell),|Q_\ell|}\} \subseteq \Omega$, based on its own internal randomness and the answers to the previous sets of queries $Q_0, \ldots, Q_{\ell-1}$, and receives $f(Q_\ell) = \{f(x^{(\ell),1}), \ldots, f(x^{(\ell),|Q_\ell|})\}$;

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(ii) Completeness: If $f \in \mathcal{P}$, then it outputs **accept** with probability 2/3;

(iii) Soundness: If $\mathrm{dist}(f, \mathcal{P}) > \varepsilon$, then it outputs **reject** with probability 2/3.

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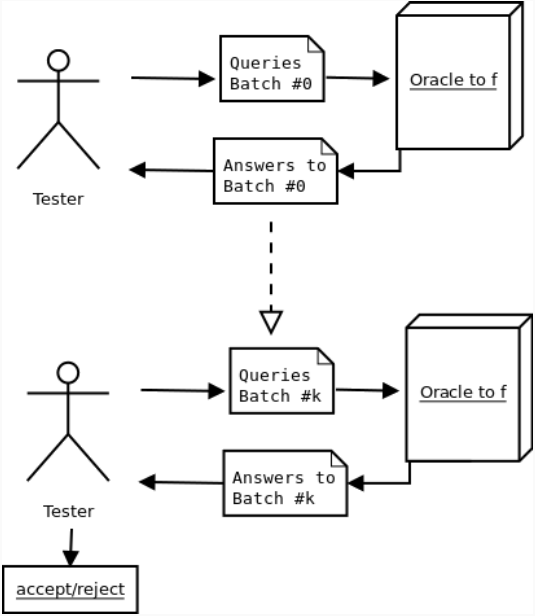
The query complexity q of the tester is the total number of queries made to f, i.e., $q = \sum_{\ell=0}^{k} |Q_\ell|$.

THAT WAS A MOUTHFUL, BUT... (I CAN'T DRAW)



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- Similar in spirit to...

... now, what do we do with it?

Does the power of testing algorithms smoothly grow with the ~~“amount of adaptivity”~~ number of rounds of adaptivity?

OUR RESULTS

... and we have an answer.

Yes, the power of testing algorithms smoothly grows with the number of rounds of adaptivity.

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Theorem (Hierarchy Theorem I)

For every $n \in \mathbb{N}$ and $0 \leq k \leq n^{0.33}$ there is a property $\mathcal{P}_{n,k}$ of strings over $\mathbb{F}_n$ such that:

(i) there exists a k-round-adaptive tester for $\mathcal{P}_{n,k}$ with query complexity $\tilde{O}(k)$, yet

(ii) any $(k-1)$-round-adaptive tester for $\mathcal{P}_{n,k}$ must make $\tilde{\Omega}(n/k^2)$ queries.

CAN WE HAVE SOMETHING A BIT LESS CONTRIVED?

It's only natural.

Yes, that also happens for actual things people care about.

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Yes, that also happens for actual things people care about.

Theorem (Hierarchy Theorem II)

Let $k \in \mathbb{N}$ be a constant. Then,

(i) there exists a k-round-adaptive tester with query complexity $O(1/\varepsilon)$ for $(2k+1)$-cycle freeness in the bounded-degree graph model; yet

(ii) any $(k-1)$-round-adaptive tester for $(2k+1)$-cycle freeness in the bounded-degree graph model must make $\Omega(\sqrt{n})$ queries, where n is the number of vertices in the graph.

OUTLINE OF THE PROOF

Main Idea

Getting a hierarchy theorem directly for property testing seems hard; but we know how to get one easily in the decision tree complexity model. Can we lift it to property testing?

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Getting a hierarchy theorem directly for property testing seems hard; but we know how to get one easily in the decision tree complexity model. Can we lift it to property testing?

Function f hard to compute in k rounds (but easy in $k+1$)

$$\Updownarrow$$

Property C_f hard to test in k rounds (but easy in $k+1$)

Fix any $\alpha > 0$. Let $C\colon \mathbb{F}_n^n \to \mathbb{F}_n^m$ be a code with constant relative distance $\delta(C) > 0$, with

- linearity: $\forall i \in [m]$, there is $a^{(i)} \in \mathbb{F}_n^n$ s.t. $C(x)_i = \langle a^{(i)}, x \rangle$ for all x;
- rate: $m \leq n^{1+\alpha}$;
- testability: C is a one-sided LTC* with non-adaptive tester;
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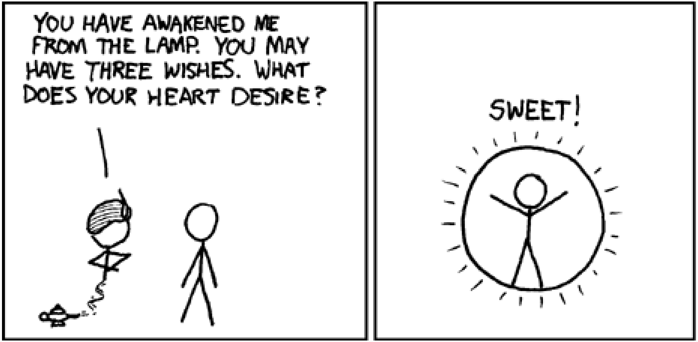
Theorem ([GGK15])

These things exist.*


YOU HAVE AWAKENED ME FROM THE LAMP. YOU MAY HAVE THREE WISHES. WHAT DOES YOUR HEART DESIRE?
SWEET!

For any $f\colon \mathbb{F}_n^n \to \{0,1\}$, consider the subset of codewords

$$\mathcal{C}_f := C(f^{-1}(1)) = \{\ C(x)\ :\ x \in \mathbb{F}_n^n,\ f(x) = 1\ \} \subseteq \mathcal{C}$$

Lemma. (LDT $\rightsquigarrow$ PT)

k-round-adaptive tester for $\mathcal{C}_f$ with query complexity q implies k-round-adaptive LDT* algorithm for f with query complexity q.

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k-round-adaptive DT algorithm for f with query complexity q implies k-round-adaptive tester for $\mathcal{C}_f$ with query complexity $\tilde{O}(q)$.

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Transference lemmas

Putting it together

Apply the above for f being the k-iterated address function $f_k : \mathbb{F}_n^n \to \{0,1\}$.

Lemma

For every $0 \leq k \leq \tilde{O}(n^{1/3})$, no k-round-adaptive LDT algorithm can compute f_{k+1} with $o(n/(k^2 \log n))$ queries..

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Proof.

Reduction to communication complexity,* lower bound of [NW93] on the "pointer-following" problem. □

OTHER RESULTS

THE END IS NIGH

- Can we swap the quantifiers in the theorems? ($\forall k \exists \mathcal{P}_k \rightsquigarrow \exists \mathcal{P} \forall k$)

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- Can we prove that for t-linearity?
- Can we simulate k rounds with ℓ rounds?
- Other applications of the transference lemmas?

CONCLUSION

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- **Codes are great!**

Thank you

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