ARE FEW BINS ENOUGH?

Testing k-histogram distributions

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“DISTRIBUTION TESTING?”

WHY?

Property testing of probability distributions:

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Property testing of probability distributions: sublinear, Property testing of probability distributions: sublinear, approximate,

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- “Model selection”: many options

Need to infer information – one bit – from the data: fast, or with very few samples.


12
22
5
18
26
49
30

HOW?

(Property) Distribution Testing:

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in an (egg)shell.

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Must decide:

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(and be correct on any D with probability at least 2/3)

AND?

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- and more [CDGR16, ADK15, DK16]...

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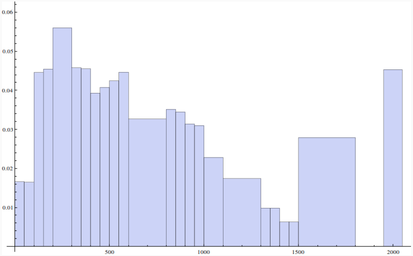
"Distributions succinctly represented" (k $\ll$ n)



TESTING-BY-LEARNING

NO, WE CAN'T.

The usual argument for testing functions (or graphs)[1]:

1. Learn f as if $f \in \mathcal{C}$, getting $\hat{f}$.
2. Check if $d(\hat{f}, \mathcal{C})$ is small.
3. Check if $d(\hat{f}, f)$ is small.

(Step 2 not even needed if the learning is proper.) If Step 1 is efficient, then so is the overall tester...

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So we hit a wall...

1. Learn D (in ℓ_1) as if $D \in \mathcal{C}$, getting $\hat{D}$.
2. Check if $\ell_1(\hat{D}, \mathcal{C})$ is small.
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YES, WE CAN!

So we hit a wall...

1. Learn D (in χ^2) as if $D \in \mathcal{C}$, getting $\hat{D}$.
2. Check if $\ell_1(\hat{D}, \mathcal{C})$ is small.
3. Check if $\chi^2(\hat{D}, D)$ is small (or $\ell_1(\hat{D}, D)$ is big). $O(\sqrt{n}/\varepsilon^2)$ samples

[ADK15]'s idea: not breaking the wall. The wall is fine.

WHEN CAN WE?

When does it apply?

When does it apply? Need an efficient χ^2 learner for $\mathcal{C}$.

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Applications

Monotonicity, log-concavity, unimodality*, MHR, independence…

Perks and catches

It's optimal!* But efficiency may require work.

AND NOW...TESTING FLAT THINGS.

“Technically n, morally k.”

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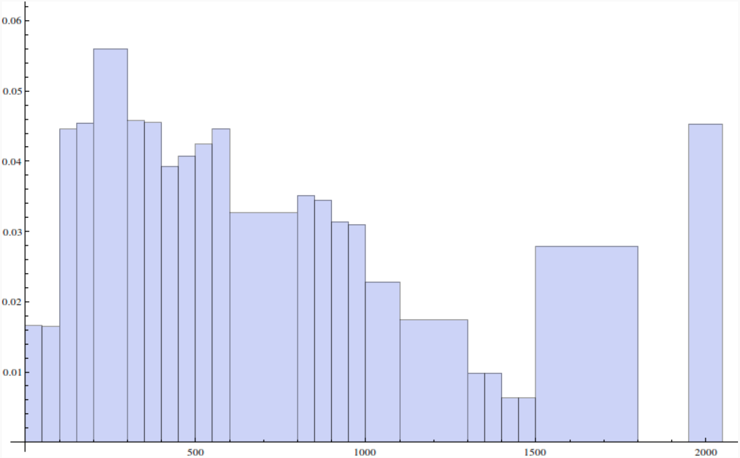

How hard can it be to test that?

Previously:

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For $k \gg \sqrt{n}$, first "natural property" provably harder than uniformity.

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Do not actually "learn, then test." Implicitly learn in χ^2, then use testing to refine the learning.

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(This is where the extra $\log k$ factor comes from.)

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A solution

Symmetrize it by applying a random permutation!

lower bound (second term)

Distribution D on [2k]: support size $\leq k/2$ or $\geq 3k/2$?

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Can use a tester for k-histograms to solve the support size estimation problem!

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Upshot

Can use a tester for k-histograms to solve the support size estimation problem! But this requires $\tilde{\Omega}(k)$ samples.

Questions?

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