Communication with Imperfect Shared Randomness

(Joint work with Venkatesan Guruswami (CMU), Raghu Meka (?) and Madhu Sudan (MSR))

Who? Clément Canonne (Columbia University)

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Communication & Complexity

There is a world outside of n

Context

There is Alice, Bob, what they communicate *and what they don't have to*.

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The

$$f \colon \{0,1\}^n \times \{0,1\}^n \to \{0,1\}\ ,$$

they compute; the protocol

$$\Pi$$

they use; from which

$$D_x, D_y$$

their inputs come; what is blue and what red means.

Communication & Complexity

But context is not perfect. . .

Noise, misunderstandings, false assumptions

Context is almost never perfectly shared.

Communication & Complexity

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Communication & Complexity

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Context is almost never perfectly shared.

- My periwinkle is your orchid.
- the printer on the 5th floor of Columbia is not *exactly* the model my laptop has a driver for.
- what precisely is a "French baguette" around here?

Communication & Complexity

What about randomness?

Equality testing

Complexity?

I have $x \in \{0,1\}^n$, you have $y \in \{0,1\}^n$, are they equal?

- Deterministic det-cc$(\mathrm{EQ}) = \Theta(n)$
- Private randomness private-cc$(\mathrm{EQ}) = \Theta(\log n)$
- Shared randomness psr-cc$(\mathrm{EQ}) = O(1)$

(Recall Newman's Theorem:

$$\text{private-cc}(P) \leq \text{psr-cc}(P) + O(\log n).)$$

This work

Randomness *and* uncertainty

ISR (Imperfectly Shared Randomness)

What if the randomness ("context") was not perfectly in sync?

To compute $f(x, y)$:

- Alice: has access to $r \in \{\pm 1\}^*$, gets input $x \in \{0,1\}^n$
- Bob: has access to $s \in \{\pm 1\}^*$, gets input $y \in \{0,1\}^n$

w/ $r \sim_\rho s$: $\mathbb{E} r_i = \mathbb{E} s_i = 0$, $\mathbb{E} r_i s_i = \rho$, $(r_i, s_i) \perp\!\!\!\perp (r_j, s_j)$.

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Studied (independently) by [BGI14] (different focus: "referee model"; more general correlations).

ISR: general relations

For every P with $x, y \in \{0,1\}^n$ and $0 \leq \rho \leq \rho' \leq 1$,

$$\begin{aligned}\text{psr-cc}(P) &\leq \text{isr-cc}_{\rho'}(P) \leq \text{isr-cc}_{\rho}(P) \\ &\leq \text{private-cc}(P) \leq \text{psr-cc}(P) + O(\log n).\end{aligned}$$

(also true for one-way: $\text{psr-cc}^{\mathrm{ow}}, \text{isr-cc}_{\rho}^{\mathrm{ow}}, \text{private-cc}^{\mathrm{ow}}$)

$\rightsquigarrow$ but for many problems, $\log n$ is already huge.

Rest of the talk

1. A first example: the COMPRESSION problem
2. General upperbound on ISR in terms of PSR
3. Strong lower bound: Alice, Bob, Charlie and Dana.

First result: Compression

Compression with uncertain priors	Alice has P, gets $m \sim P$; Bob knows $Q \simeq P$, wants m.
Previous work	
$P = Q$	$H(P)$ (Huffman coding)
$P \simeq_\Delta Q$	$H(P) + 2\Delta$ [JKKS11] (w/ shared randomness)
$P \simeq_\Delta Q$	$O(H(P) + \Delta + \log\log N)$ [HS14] (deterministic)

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This work

For all $\epsilon > 0$,

$\text{isr-cc}_\rho^{\mathrm{ow}}(\textsc{Compress}_\Delta) \leq \frac{1+\epsilon}{1-h(\frac{1-\rho}{2})}(H(P) + 2\Delta + O(1))$

"natural protocol"

General upperbound

It's inner products all the way down!

Theorem $\forall \rho > 0$, $\exists c < \infty$ *such that* $\forall k$, *we have*

$$\text{PSR-CC}(k) \subseteq \text{ISR-CC}_{\rho}^{\mathrm{ow}}(c^k)\,.$$

Proof. (Outline)

- Define $\textsc{GapInnerProduct}$, "complete" for PSR-CC(k) (see strategies as X_R, $Y_R\{0,1\}^{2^k}$; use Newman's Theorem to bound # R's);
- Show there exists a (Gaussian-based) isr protocol for $\textsc{GapInnerProduct}$, with $O_\rho(4^k)$ bits of comm.

□

General upperbound

Can we do better?

For problems in $\mathsf{PSR\text{-}CC}^{\mathrm{ow}}(k)$? $\mathsf{PSR\text{-}CC}^{\mathrm{ow}}(k) \subseteq \mathsf{ISR\text{-}CC}^{\mathrm{ow}}_{\rho}(c^{o(k)})$?

For $\mathsf{ISR\text{-}CC}_{\rho}$? $\mathsf{PSR\text{-}CC}(\omega(k)) \subseteq \mathsf{ISR\text{-}CC}_{\rho}(c^{k})$?

General upperbound

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Answer No.

Strong converse: lower bound

It's as good as it gets.

Theorem $\forall k$, $\exists P = (P_n)_{n\in\mathbb{N}}$ *s.t.* $\text{psr-cc}^{\text{ow}}(P) \leq k$, *yet* $\forall \rho < 1$

$$\text{isr-cc}_\rho(P) = 2^{\Omega_\rho(k)}\,.$$

Proof. (High-level)

- Define SparseGapInnerProduct, relaxation of GapInnerProduct.
- Show it has as $O(\log q)$-bit one-way psr protocol (Alice uses the shared randomness to send *one* coordinate to Bob)
- isr lower bound: argue that for any (fixed)* strategy of Alice and Bob using less that $\sqrt{q}$ bits, either (a) something impossible happens in the Boolean world, or (b) something impossible happens in the Gaussian world.

□

Strong converse: lower bound

Two-pronged impossibility, first prong.

Case (a)

The strategies $(f_r, g_s)_{r,s}$ have common high-influence variable *(recall the one-way psr protocol)*.
But then, two players Charlie and Dana can* leverage this strategies to win an *agreement distillation* game:

Definition (Agreement distillation)

Charlie and Dana have no inputs. Their goal is to output w_C and w_D satisfying:

$$\Pr[\, w_C = w_D \,] \geq \gamma;$$
$$H_\infty(w_C), H_\infty(w_D) \geq \kappa.$$

But this requires $\Omega(\kappa) - \log(1/\gamma)$ bits of communication (via [BM10, Theorem 1]).

Strong converse: lower bound

Two-pronged impossibility, second prong.

Case (b) $f_r\colon \{0,1\}^n \to K_A \subset [0,1]^{2^k}$, $g_s\colon \{0,1\}^n \to K_B \subset [0,1]^{2^k}$ have no common high-influence variable.
We then show that this implies $k = 2^{\Omega(\sqrt{q})}$, by using an *Invariance Principle* (in the spirit of [Mos10]) to "go to the Gaussian world": if f, g are low-degree polynomials with no common influential variable, then

$$\mathbb{E}_{(x,y)\sim N^{\otimes n}}\left[\langle f(x), g(y)\rangle\right] \simeq \mathbb{E}_{(X,Y)\sim \mathcal{G}^{\otimes n}}\left[\langle F(X), G(Y)\rangle\right]$$

and Charlie and Dana can use this solve (yet another) problem, the Gaussian Inner Product ($\textsc{GaussianCorrelation}_\xi$).

But... a reduction to $\textsc{Disjointness}$ shows that (even with psr) this requires $\Omega 1/\xi$ bits of communication. □

Conclusions

Summary

- Dealing with more realistic situations: Alice, Bob, and what they do not know about each other;
- comes into play when n is **huge** (Newman's Theorem becomes loose);
- show general and tight relations and reductions in this model, with both upper and lower bounds.
- a new invariance theorem, and use in comm. complexity.

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- a new invariance theorem, and use in comm. complexity.

What about...

- more general forms of correlations?
- cases where even randomness is expensive? (minimize its use)
- *one-sided error*?

Thank you.

(Questions?)

Theorem (Our Invariance Principle)

Fix any two parameters $p_1, p_2 \in (-1,1)$. *For all* $\varepsilon \in (0,1]$, $\ell \in \mathbb{N}$, $\theta_0 \in [0,1)$, *and closed convex sets* $K_1, K_2 \subseteq [0,1]^\ell$ *there exist* $\tau > 0$ *and mappings*

$$T_1 \colon \{f \colon \{+1,-1\}^n \to K_1\} \to \{F \colon \mathbb{R}^n \to K_1\}$$
$$T_2 \colon \{g \colon \{+1,-1\}^n \to K_2\} \to \{G \colon \mathbb{R}^n \to K_2\}$$

such that for all $\theta \in [-\theta_0, \theta_0]$, *if* f, g *satisfy*

$$\max_{i \in [n]} \min \left(\max_{j \in [\ell]} \mathrm{Inf}_i(d) f_j, \max_{j \in [\ell]} \mathrm{Inf}_i(d) g_j \right) \le \tau$$

then, for $F = T_1(f)$ *and* $G = T_2(g)$, *we have*
where $N = N_{p_1,p_2,\theta}$ *and* $\mathcal{G}$ *is the Gaussian distribution which matches the first and second-order moments of* N.

Theorem (Invariance Theorem of [GHM$^+$11])

Let (Ω, μ) be a finite prob. space with each prob. at least $\alpha \le 1/2$. Let $b = |\Omega|$ and $\mathcal{L} = \{\chi_0 = 1, \chi_1, \chi_2, \dots, \chi_{b-1}\}$ be a basis for r.v.'s over Ω. Let $\Upsilon = \{\xi_0 = 1, \xi_1, \dots, \xi_{b-1}\}$ be an ensemble of real-valued Gaussian r.v.'s with 1st and 2nd moments matching those of the χ_i's; and $h = (h_1, h_2, \dots, h_t)\colon \Omega^n \to \mathbb{R}^t$ s.t.

$$\mathrm{Inf}_i(h_\ell) \le \tau, \qquad \mathrm{Var}(h_\ell) \le 1$$

for all $i \in [n]$ and $\ell \in [t]$. For $\eta \in (0,1)$, let H_ℓ ($\ell \in [t]$) be the multilinear polynomial associated with $T_{1-\eta} h_\ell$ w.r.t. $\mathcal{L}$. If $\Psi\colon \mathbb{R}^t \to \mathbb{R}$ is Λ-Lipschitz (w.r.t. the L_2-norm), then

$$\left|\mathbb{E}\Big[\Psi\big(H_1(\mathcal{L}^n), \cdots, H_t(\mathcal{L}^n)\big)\Big] - \mathbb{E}\Big[\Psi\big(H_1(\Upsilon^n), \cdots, H_t(\Upsilon^n)\big)\Big]\right| \le C(t) \cdot \Lambda \cdot \tau^{\frac{\eta}{18} \log \frac{1}{\alpha}} = o_\tau(1)$$

for some constant $C = C(t)$.

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