Testing probability distributions using conditional samples

(when testers get to be picky)

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Plan of the talk

Property testing

Given a big, hidden "object" X one can only access by local, expensive inspections (e.g., oracle queries), and a property $\mathcal{P}$, the goal is to check in sublinear number of inspections if (a) X has the property or (b) X is "far" from all objects having the property.[1]

[1] wrt to some specified metric, and parameter $\varepsilon > 0$ given to the tester.

Background and motivation

What is distribution testing?

Property testing

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Testing distributions (standard model)

X is an unknown probability distribution D over some N-element set; the testing algorithm has blackbox sample access to D.

[1] wrt to some specified metric, and parameter $\varepsilon > 0$ given to the tester.

Distribution testing (1)

In more details.

Distance criterion: total variation distance ($\propto L_1$ distance)

$$d_{\mathrm{TV}}(D_1, D_2) \stackrel{\text{def}}{=} \frac{1}{2}\|D_1 - D_2\|_1 = \frac{1}{2}\sum_{i\in[N]} |D_1(i) - D_2(i)|.$$

Definition (Testing algorithm)

Let $\mathcal{P}$ be a property of distributions over $[N]$, and ORACLE_D be some type of oracle which provides access to D. A $q(\varepsilon, N)$-query ORACLE testing algorithm for $\mathcal{P}$ is an algorithm T which, given ε, N as input parameters and oracle access to an ORACLE_D oracle, and for any distribution D over $[N]$, makes at most $q(\varepsilon, N)$ calls to ORACLE_D, and:

- if $D \in \mathcal{P}$ then, w.p. at least $2/3$, T outputs ACCEPT;
- if $d_{\mathrm{TV}}(D, \mathcal{P}) \geq \varepsilon$ then, w.p. at least $2/3$, T outputs REJECT.

Distribution testing (2)

Comments

A few remarks

- tester is randomized;

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- "gray" area for $d_{\mathrm{TV}}(D, \mathcal{P}) \in (0, \varepsilon)$;
- $2/3$ is completely arbitrary;
- extends to several oracles and distributions;

Distribution testing (2)

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A few remarks

- tester is randomized;
- "gray" area for $d_{\mathrm{TV}}(D, \mathcal{P}) \in (0, \varepsilon)$;
- $2/3$ is completely arbitrary;
- extends to several oracles and distributions;
- our measure is the sample complexity (*not* the running time).

Distribution testing (3)

Concrete example: testing uniformity

Property $\mathcal{P}$ ("being $\mathcal{U}$, the uniform distribution over $[N]$") $\Leftrightarrow$ set $\mathcal{S}_\mathcal{P}$ of distributions with this property ($\mathcal{S}_\mathcal{P} = \{\mathcal{U}\}$)
Distance to $\mathcal{P}$:

$$\mathrm{d_{TV}}(D, \mathcal{S}_\mathcal{P}) = \min_{D' \in \mathcal{S}_\mathcal{P}} \mathrm{d_{TV}}(D, D') \underset{\text{here}}{=} \mathrm{d_{TV}}(D, \mathcal{U})$$

General outline

1. Draw a bunch of samples from D;
2. "Process" them, for instance by counting the number of points drawn more than once *(collisions)*;
3. Compare the result to what one would expect from the uniform distribution $\mathcal{U}$;
4. Reject if it differs too much; accept otherwise.

Well, it's more or less settled.

Fact

In the standard sampling model, most (natural) properties are "hard" to test; that is, require a strong dependence on N (at least $\Omega(\sqrt{N})$).

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Example

Testing *uniformity* has $\Theta(\sqrt{N}/\varepsilon^2)$ sample complexity [GR00, BFR$^+$10, Pan08], *equivalence to a known distribution* $\tilde{\Theta}(\sqrt{N}/\varepsilon^2)$ [BFF$^+$01, Pan08]; *equivalence of two unknown distributions* $\Omega(N^{2/3})$ [BFR$^+$10, Val11] (and essentially matching upperbound)...

Our model

More power to the tester

In a lot of natural applications, the tester has more control over the "experiment" it is running – e.g., by tuning the conditions or the settings to influence the outcome, effectively restricting its range. *This is not captured by the* SAMP *model*; to mend this, we consider a new model where the testing algorithm can ask for a specific range of outcomes, and get a draw *conditioned on it being in that domain.*

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Definition (COND oracle)

Fix a distribution D over $[N]$. A COND oracle for D, denoted COND_D, is defined as follows: The oracle is given as input a *query set* $S \subseteq [N]$ that has $D(S) > 0$, and returns an element $i \in S$, where the probability that element i is returned is $D_S(i) = D(i)/D(S)$, independently of all previous calls to the oracle.

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Question

Do COND oracles enable more efficient testing algorithms than SAMP oracles? And what does it reveal about testing distributions?

Our results

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Our results

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Do COND oracles enable more efficient testing algorithms than SAMP oracles? Yes, they do.

Our results

Comparison of the COND and SAMP models on several testing problems

Problem	Our results		Standard model
Is D uniform?	COND_D	$\Omega\left(\frac{1}{\varepsilon^2}\right)$	$\Theta\left(\frac{\sqrt{N}}{\varepsilon^2}\right)$ [GR00, BFR+10, Pan08]
	PCOND_D	$\tilde{O}\left(\frac{1}{\varepsilon^2}\right)$	
	ICOND_D	$\tilde{O}\left(\frac{\log^3 N}{\varepsilon^3}\right)$ $\Omega\left(\frac{\log N}{\log\log N}\right)$	
Is $D = D^*$?	COND_D	$\tilde{O}\left(\frac{1}{\varepsilon^4}\right)$	$\tilde{\Theta}\left(\frac{\sqrt{N}}{\varepsilon^2}\right)$ [BFF+01, Pan08]
	PCOND_D	$\tilde{O}\left(\frac{\log^4 N}{\varepsilon^4}\right)$ $\Omega\left(\sqrt{\frac{\log N}{\log\log N}}\right)$	
Are D_1, D_2 equivalent?	COND_{D_1,D_2}	$\tilde{O}\left(\frac{\log^5 N}{\varepsilon^4}\right)$	$\tilde{O}\left(\frac{N^{2/3}}{\varepsilon^{8/3}}\right)$ [BFR+10]
	PCOND_{D_1,D_2}	$\tilde{O}\left(\frac{\log^6 N}{\varepsilon^{21}}\right)$	$\Omega\left(N^{2/3}\right)$ [BFR+10, Val11]
How far is D from $\mathcal{U}$?	PCOND_D	$\tilde{O}\left(\frac{1}{\varepsilon^{20}}\right)$	$O\left(\frac{1}{\varepsilon^2}\frac{N}{\log N}\right)$ [VV11, VV10b] $\Omega\left(\frac{N}{\log N}\right)$ [VV11, VV10a]

Table: The upper bounds for the first 3 problems are for testing the property,while the last one involves estimating the total variation distance to uniformity to within an additive $\pm\varepsilon$.

Rest of the talk

Plan for rest of talk:

- testing uniformity: an upper bound (with pairwise queries)
- testing uniformity: a lower bound
- introducing tools: Estimate-Neighborhood and Approx-Eval
- testing uniformity, again: a (glimpse at) interval queries.

Why bother with N?

Theorem (Testing Uniformity with PCOND)

There exists a $\tilde{O}(1/\varepsilon^2)$-query PCOND_D *tester for uniformity, i.e. it accepts w.p. at least* $2/3$ *if* $D = \mathcal{U}$ *and rejects w.p. at least* $2/3$ *if* $d_{\mathrm{TV}}(D,\mathcal{U}) \geq \varepsilon$.

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High-level idea

Intuitively, if D is ε-far from uniform, it must have (a) a lot of points "very light"; and (b) a lot of weight on points "very heavy". Sampling $O(1/\varepsilon)$ points both uniformly and according to D, we obtain whp both light and heavy ones; and use PCOND to compare them.

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Not good enough ($O(1/\varepsilon^4)$ queries) $\rightsquigarrow$ refine this approach to get $\tilde{O}(1/\varepsilon^2)$.

Testing Uniformity (2)

Getting our hands dirty.

Algorithm 1: PCOND$_D$-TEST-UNIFORM

Set $t = \log(\frac{4}{\varepsilon}) + 1$.
Select $q = \Theta(1)$ points $i_1, \dots, i_q$ uniformly {Reference points}
for $j = 1$ to t **do**
  Call the SAMP$_D$ oracle $s_j = \Theta\left(2^j t\right)$ times to obtain points $h_1, \dots, h_{s_j}$ distributed according to D {Try to get a heavy point}
  Draw s_j points $\ell_1, \dots, \ell_{s_j}$ uniformly from $[N]$ {Try to get a light point}
  for all pairs $(x, y) = (i_r, h_{r'})$ and $(x, y) = (i_r, \ell_{r'})$ **do**
    Call COMPARE$_D(\{x\}, \{y\}, \Theta(\varepsilon 2^j), 2, \exp^{-\Theta(t)})$.
    if it does not return a value in $[1 - 2^{j-5}\frac{\varepsilon}{4}, 1 + 2^{j-5}\frac{\varepsilon}{4}]$ **then**
      output REJECT (and exit).
    end if
  end for
end for
Output ACCEPT

Testing Uniformity (3)

Proof (Outline).

Sample complexity by the setting of t, q and the calls to COMPARE

Completeness unless COMPARE fails to output a correct value, no rejection

Soundness Suppose D is ε-far from $\mathcal{U}$; refinement of the previous approach by bucketing low and high points:

$$H_j \stackrel{\text{def}}{=} \left\{ h \;\middle|\; \left(1+2^{j-1}\frac{\varepsilon}{4}\right)\frac{1}{N} \leq D(h) < \left(1+2^{j}\frac{\varepsilon}{4}\right)\frac{1}{N} \right\}$$

$$L_j \stackrel{\text{def}}{=} \left\{ \ell \;\middle|\; \left(1-2^{j}\frac{\varepsilon}{4}\right)\frac{1}{N} < D(\ell) \leq \left(1-2^{j-1}\frac{\varepsilon}{4}\right)\frac{1}{N} \right\}$$

for $j \in [t-1]$, with also H_0, L_0, H_t, L_t to cover everything; each loop iteration on l.3 "focuses" on a particular bucket.

$+$ Chernoff and union bounds. □

Testing Uniformity – Lower Bound (1)

Theorem (Testing Uniformity with COND)

Any COND_D *algorithm for testing whether* $D = \mathcal{U}$ *versus* $d_{\mathrm{TV}}(D, \mathcal{U}) \geq \varepsilon$ *must make* $\Omega(1/\varepsilon^2)$ *queries.*

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Remark

As PCOND is a restriction of COND, the previous upper bound was essentially optimal.

Testing Uniformity – Lower Bound (2)

High-level idea.

Reduce it to the problem of distinguishing between a fair and a biased coin, by defining a "no-instance" D_{no} s.t.

1. D_{no} is ε-far from $\mathcal{U}$;
2. any q-query tester $\mathcal{A}$ which distinguishes D_{no} from $\mathcal{U}$ can be turned into a tester $\mathcal{A}'$ distinguishing between (1) a sequence of q fair coin tosses and (2) a sequence of q (4ε)-biased coin tosses.

However, it is known that distinguishing between these two scenarios requires $\Omega(1/\varepsilon^2)$ coin tosses. □

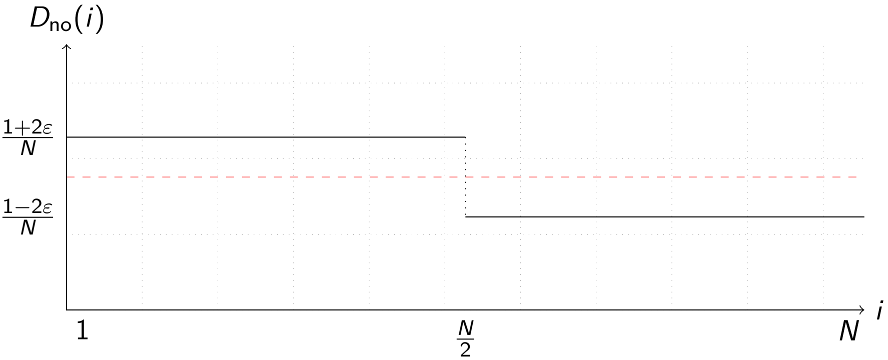

Figure: The no-instance D_{no}.

To run $\mathcal{A}$ from $\mathcal{A}'$, we must simulate COND_D (D either $\mathcal{U}$ or D_{no}) to provide the former with samples, given the corresponding coin tosses.

At step $1 \leq t \leq q$, $\mathcal{A}$ chooses to query $S \subset [N]$ (according to the $(t-1)$ previous answers it got from the simulation). $\mathcal{A}'$ behaves as follows:

- sets $S_0 \stackrel{\mathrm{def}}{=} S \cap [1, \frac{N}{2}]$, $S_1 \stackrel{\mathrm{def}}{=} S \cap [\frac{N}{2}+1, N]$;
- gets bit b_t, and draws $\sigma \sim \begin{cases} \mathrm{Bern}(u_t) & \text{if } b_t = 1 \\ \mathrm{Bern}(v_t) & \text{o.w.} \end{cases}$ (†)
- draws s u.a.r. from S_σ;
- gives (S, s) to $\mathcal{A}$.

(†) for a right choice of u_t, v_t depending on $|S_0|$, $|S_1|$, ε

Testing Uniformity – Summary

- $\Theta\left(\sqrt{N}/\varepsilon^2\right)$ with SAMP: counting collisions [GR00, BFR$^+$10, Pan08]
- $\tilde{O}(1/\varepsilon^2)$ with PCOND: comparing random pairs of points
- $\Omega(1/\varepsilon^2)$ with COND: reducing to fair vs. biased coin

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Remark

Testing with ICOND will require a logarithmic dependence on N.

Building tools (1)

- COMPARE
  Low-level procedure: compares the relative weight of sets X, Y, given some accuracy parameter η.
- ESTIMATE-NEIGHBORHOOD
  On input a point $i \in [N]$ and parameter γ, estimates the weight under D of the γ-neighborhood of i – that is, points with probability mass within a factor $(1+\gamma)$ of $D(i)$.
- APPROX-EVAL
  Given $i \in [N]$ and accuracy parameter η, returns an approximation of $D(i)$ – succeeds whp for most points i.

Building tools (2)

"Comparison is the death of joy." – Mark Twain.

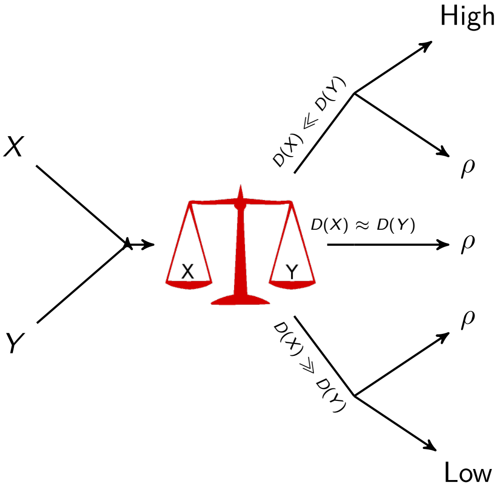
The low-level tool COMPARE

Given as input two disjoint subsets X, Y, parameters $\eta \in (0, 1]$, $K \geq 1$, and $\delta \in (0, 1/2]$, and COND access to D, the procedure COMPARE either outputs a value $\rho > 0$, High or Low, s.t:

- If $D(X)/K \leq D(Y) \leq K \cdot D(X)$ then w.p. $1 - \delta$ it outputs a value $\rho \in [1 - \eta, 1 + \eta]D(Y)/D(X)$;
- If $D(Y) > K \cdot D(X)$ then w.p. $1 - \delta$ it outputs either High or a value $\rho \in [1 - \eta, 1 + \eta]D(Y)/D(X)$;
- If $D(Y) < D(X)/K$ then w.p. $1 - \delta$ it outputs either Low or a value $\rho \in [1 - \eta, 1 + \eta]D(Y)/D(X)$.

COMPARE performs $O\left(\frac{K \log(1/\delta)}{\eta^2}\right)$ COND queries on $X \cup Y$.

Building tools (3)



Building tools (4)

Definition (γ-Neighborhood)

$$U_\gamma(x) \stackrel{\text{def}}{=} \left\{ y \in [N] : \frac{1}{1+\gamma} D(x) \leq D(y) \leq (1+\gamma) D(x) \right\}, \qquad \gamma \in [0,1]$$

Building tools (4)

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Goal

Given a point $x \in [N]$ and a parameter γ, get an approximation of $D(U_\gamma(x))$ – i.e., "how much weight does D put on points like x?"

Building tools (5)

The (slightly) higher-level subroutine Estimate-Neighborhood

Given as input a point x, parameters $\gamma, \beta, \eta, \delta \in (0, 1/2]$ and PCOND_D access, the procedure Estimate-Neighborhood outputs a pair $(\hat{w}, \alpha) \in [0, 1] \times (\gamma, 2\gamma)$ such that, for θ small:

1. If $D(U_\alpha(x)) \geq \beta$, then w.p. $1 - \delta$ we have $\hat{w} \in [1 - \eta, 1 + \eta] \cdot D(U_\alpha(x))$, and $D(U_{\alpha+\theta}(x) \setminus U_\alpha(x)) \leq \eta\beta/16$;
2. If $D(U_\alpha(x)) < \beta$, then w.p. $1 - \delta$ we have $\hat{w} \leq (1 + \eta) \cdot \beta$, and $D(U_{\alpha+\theta}(x) \setminus U_\alpha(x)) \leq \eta\beta/16$.

Estimate-Neighborhood performs $\tilde{O}\left(\frac{\log(1/\delta)}{\gamma^2\eta^4\beta^3\delta^2}\right)$ queries.

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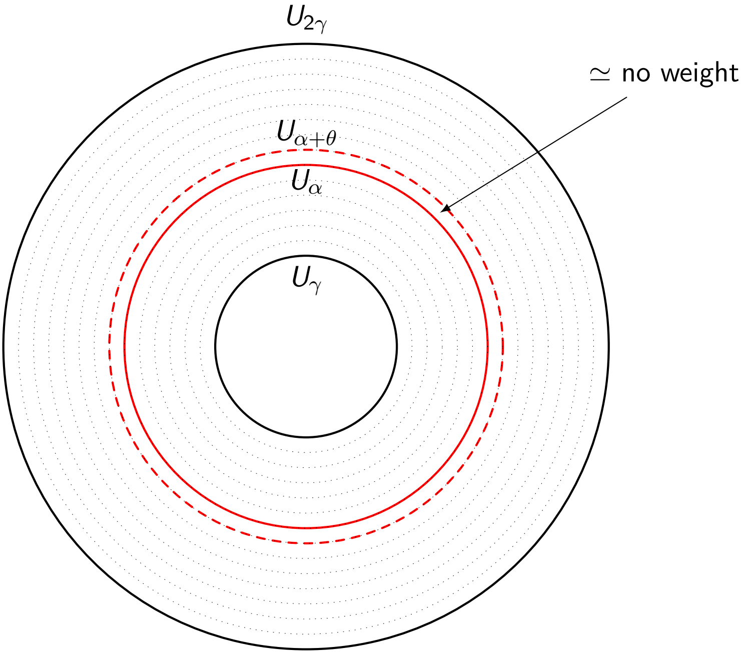
ESTIMATE-NEIGHBORHOOD performs $\tilde{O}\left(\frac{\log(1/\delta)}{\gamma^2\eta^4\beta^3\delta^2}\right)$ queries.

Remark

Does not estimate exactly $D(U_\gamma(x))$.


$U_{2\gamma}$
$\simeq$ no weight
$U_{\alpha+\theta}$
U_α
U_γ

Building tools (6)

EVAL oracle

A δ-EVAL_D simulator for D is a randomized procedure ORACLE such that w.p. $1-\delta$ the output of ORACLE on input $i^* \in [N]$ is $D(i^*)$.

Building tools (6)

(Approximate) EVAL oracle

An (ε, δ)-approximate EVAL_D simulator for D is a randomized procedure ORACLE such that w.p. $1 - \delta$ the output of ORACLE on input $i^* \in [N]$ is a value $\alpha \in [0, 1]$ such that $\alpha \in [1 - \varepsilon, 1 + \varepsilon]D(i^*)$.

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An (ε, δ)-approximate EVAL_D simulator for D is a randomized procedure ORACLE s.t for each ε, there is a fixed set $S^{(\varepsilon)} \subsetneq [N]$ with $D(S^{(\varepsilon)}) < \varepsilon$ for which the following holds. For all $i^* \in [N]$, ORACLE(i^*) is either a value $\alpha \in [0, 1]$ or Unknown, and furthermore:

(i) If $i^* \notin S^{(\varepsilon)}$ then w.p. $1 - \delta$ the output of ORACLE on input i^* is a value $\alpha \in [0, 1]$ such that $\alpha \in [1 - \varepsilon, 1 + \varepsilon]D(i^*)$;

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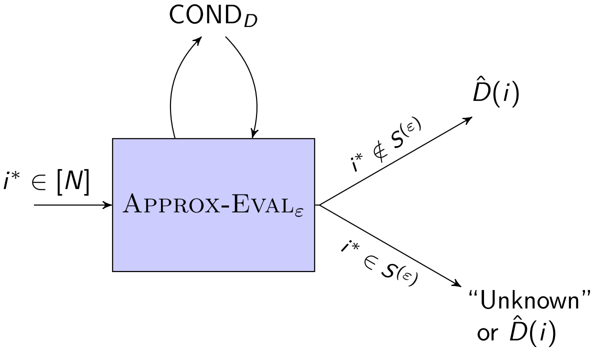
The high-level blackbox Approx-Eval

There is an algorithm Approx-Eval which uses $\tilde{O}\left(\frac{(\log N)^5 \cdot (\log(1/\delta))^2}{\varepsilon^3}\right)$ calls to COND_D, and is an (ε, δ)-approximate EVAL_D simulator.


COND$_D$
$i^* \in [N]$
APPROX-EVAL$_\varepsilon$
$i^* \notin S^{(\varepsilon)}$
$\hat{D}(i)$
$i^* \in S^{(\varepsilon)}$
"Unknown"
or $\hat{D}(i)$

Applications

Testing equivalence of two unknown distributions D_1, D_2

Blackbox access to D_1 *and* D_2 (two oracles); distinguish $D_1 = D_2$ vs. $\mathrm{d_{TV}}(D_1, D_2) \geq \varepsilon$.

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In the language of property testing: $\mathcal{S}_{\mathcal{P}} = \{\ (D, D) \mid D \text{ distribution }\}$, with metric over pairs of distributions $\mathrm{d}((D, D'), (P, P')) \stackrel{\text{def}}{=} \mathrm{d_{TV}}(D, P) + \mathrm{d_{TV}}(D', P')$.

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Two different approaches:

1. with PCOND and Estimate-Neighborhood – finding "representatives" points for both distributions;
2. with COND and Approx-Eval – adapting an EVAL algorithm from [RS09].

Other uses: estimating distance to uniformity (Estimate-Neighborhood), testing monotonicity[3] (Approx-Eval)...

[3](extension of the original results)

Testing Uniformity with ICOND

Main message

ICOND algorithms are weaker than PCOND ones for this: while $\text{poly}(\log N, 1/\varepsilon)$ queries are enough, $\tilde{\Omega}(\log N)$ are necessary.

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Overview

- **Upper bound** sort of binary descent on random points (custom-tailored version of APPROX-EVAL), to spot deviations from $1/N$;
- **Lower bound** family of "no-instances" + LB against non-adaptive + hybrid argument to get LB against adaptive.

Upper Bound

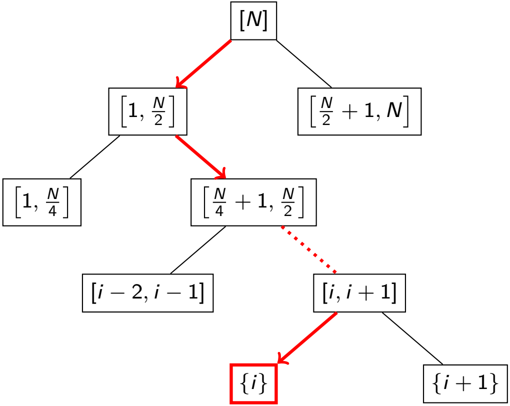

Figure: Idea of the "binary descent" on i: get an estimate of $D(i)$ by multiplying estimates at each branching, each time rejecting if ratio between weight of two subintervals is far from $\frac{1}{2}$. Repeat for $\Theta(1/\varepsilon)$ points drawn from D.

Conclusion

- new model for studying probability distributions
- arises naturally in a number of settings
- allows significantly more query-efficient algorithms

- generalizing to other structured domains? (e.g., the Boolean hypercube $\{0,1\}^n$)
- what about distribution learning in this framework
- more properties? (entropy, independence, monotonicity†...)

The end.

Thank you.

The full version of this work is available online (arXiv:1211.2664).

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