# TESTING CLASSES OF DISTRIBUTIONS

General Approaches to Particular Problems

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# “DISTRIBUTION TESTING?”

# WHY?

Property testing of probability distributions:

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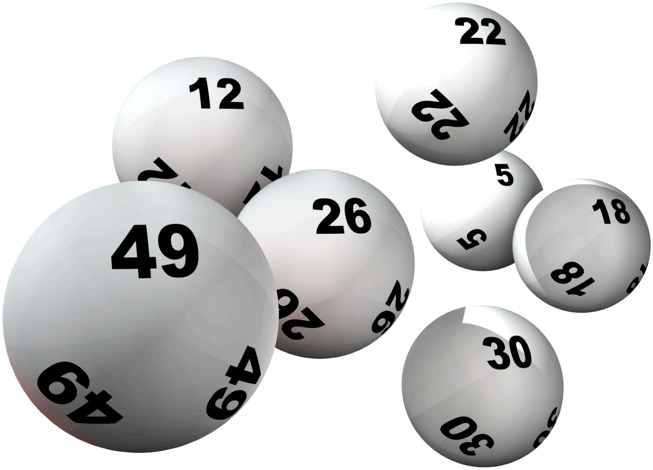
- Big Dataset: too big
- Expensive access: pricey data
- “Model selection”: many options

Need to infer information – one bit – from the data: fast, or with very few samples.


12
22
5
18
26
49
30

(Property) Distribution Testing:

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in an (egg)shell.

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Distance parameter $\varepsilon \in (0, 1]$

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...but almost none on general frameworks.*

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There exists a *generic* algorithm that can test membership to any class that satisfies some structural criterion. (Moreover, for many such $\mathcal{C}$ this algorithm has near-optimal sample complexity.)

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(works for tolerant testing too.)

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Monotonicity, unimodality, log-concavity, monotone hazard rate, independence. (Tight upper bounds!)

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- Only need to prove a structural, existential result about $\mathcal{C}$!
- Learning and testing (in $\ell_1$) are unrelated for distributions.
- Testing-by-learning was seemingly ruled out... [VV11]

# A UNIFIED APPROACH TO THINGS

## GENERIC UPPER BOUND

Say $\mathcal{C}$ is $(\gamma, L(\gamma))$-decomposable if any $D \in \mathcal{C}$ is well-approximated by some piecewise-constant distribution on L pieces $I_1, \ldots, I_L$:

1. $D(i) \in [(1-\gamma), (1+\gamma)] \cdot \frac{D(I)}{|I|}$ for all $i \in I$; or
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I.e., each $D \in \mathcal{C}$ is piecewise flat, in a strong $\ell_2$-like sense.

Then...

Any $(\gamma, L(\gamma))$-decomposable $\mathcal{C}$ can be tested by the same generic algorithm, with $\tilde{O}\left(\frac{\sqrt{L(\varepsilon)n}}{\varepsilon^3} + \frac{L(\varepsilon)}{\varepsilon^2}\right)$ samples.

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Algorithm inspired from [BKR04]:

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Algorithm inspired from [BKR04]: decompose, learn, check.

**Decompose:** Attempt to recursively partition $[n]$ into $L$ intervals where $\ell_2(D, U) \leq \varepsilon/|I|$ or $D(I)$ small – should succeed if $D \in \mathcal{C}$ (by decomposability).

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# GENERIC UPPER BOUND: OVERVIEW

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Preliminary step: restrict to effective support.

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## A few catches

Preliminary step: restrict to effective support. Also... efficiency.

## A few perks

Decomposability composes very well!

**Theorem**

Suppose $\mathcal{C}$ can be agnostically learned with sample complexity $q(\varepsilon, n)$ and contains a subclass $\mathcal{C}'$ that requires $t(\varepsilon, n) \gg q(\varepsilon, n)$ samples to be $\varepsilon$-tested. Then $\mathcal{C}$ requires $t(\varepsilon, n)$ samples to be $\varepsilon$-tested as well.

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Combined with [VV14] and learning results from the literature, immediately implies many new or previous lower bounds. (Taking $\mathcal{C}' = \{U\}$ or $\{\mathrm{Bin}(n, 1/2)\}$ often enough)

# TESTING-BY-LEARNING

# NO, WE CAN'T.

The usual argument for testing functions (or graphs)[1]:

1. Learn $f$ as if $f \in \mathcal{C}$, getting $\hat{f}$.
2. Check if $d(\hat{f}, \mathcal{C})$ is small.
3. Check if $d(\hat{f}, f)$ is small.

(Step 2 not even needed if the learning is proper.) If Step 1 is efficient, then so is the overall tester...

Testing is no harder than learning!

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but not for distributions. Step 3 is no longer easy for them! [VV11]

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# YES, WE CAN!

So we hit a wall...

1. Learn D (in $\ell_1$) as if $D \in \mathcal{C}$, getting $\hat{D}$.
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[ADK15]'s idea: not breaking the wall. The wall is fine.

# YES, WE CAN!

So we hit a wall...

1. Learn D (in $\chi^2$) as if $D \in \mathcal{C}$, getting $\hat{D}$.
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3. Check if $\chi^2(\hat{D}, D)$ is small (or $\ell_1(\hat{D}, D)$ is big). $O(\sqrt{n}/\varepsilon^2)$ samples

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## WHEN CAN WE?

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When does it apply? Need an efficient $\chi^2$ learner for $\mathcal{C}$.

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### Applications

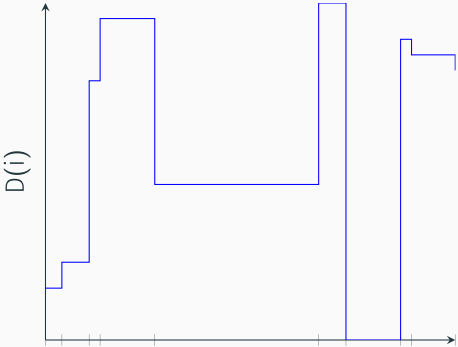
Monotonicity, log-concavity, unimodality*, MHR, independence…

### Perks and catches

It's optimal!* But efficiency, as before, requires work.

# AND NOW...TESTING FLAT THINGS.

"Technically n, morally k."



# k-HISTOGRAM DISTRIBUTIONS

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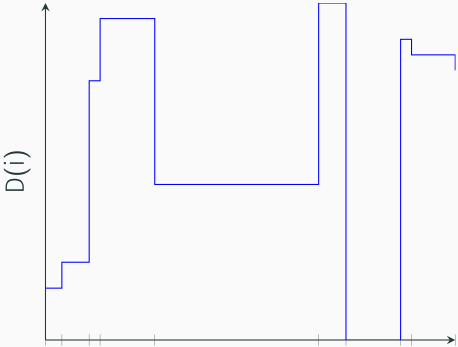

# How hard can it be to test that?

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**Theorem**

Testing k-histograms requires $\Omega\left(\frac{\sqrt{n}}{\varepsilon^2} + \frac{k}{\varepsilon \log k}\right)$ samples.

For $k \gg \sqrt{n}$, first "natural property" provably harder than uniformity.

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Apply the “testing-by-learning” technique of [ADK15].

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Do not actually "learn, then test." Implicitly learn in $\chi^2$, then use testing to refine the learning.

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(This is where the extra $\log k$ factor comes from.)

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Being a k-histogram is ~~not really~~ really not a symmetric property.

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Symmetrize it by applying a random permutation!

# LOWER BOUND (SECOND TERM)

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    - If $\operatorname{supp}(D) \leq k/2$, then D′ is a k-histogram with probability 1;
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## Upshot

Can use a tester for k-histograms to solve the support size estimation problem!

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3. Use a tester for k-histograms on the resulting distribution D′:
    - If $\text{supp}(D) \leq k/2$, then D′ is a k-histogram with probability 1;
    - If $\text{supp}(D) \geq 3k/2$, then D′ is not a $\ell$-histogram for any $\ell < 1.1k$ with probability 2/3; (and D′ $\Omega(1)$-far from any k-histogram)

### Upshot

Can use a tester for k-histograms to solve the support size estimation problem! But this requires $\tilde{\Omega}(k)$ samples.

# Questions?

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