Data Structures and Algorithms

Session 24. Earth Day, 2009

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Announcements

- Homework 6 due before last class: May 4th
- Final Review May 4th
- Exam Wednesday May 13th 1:10-4:00 PM, 633
  - cumulative, closed-book/notes

Review

- O(M log* N) running time for M unions/finds
  - Counted cost of each find by two kinds of "pennies": American/Canadian
  - Basic intuition: Canadian when node is in middle of rank group, American when node is between groups
- Comparison Sort lower bound vs. Radix Sort

Today's Plan

- Radix Sort specifics
- Comparison sorting algorithm characteristics
- Algorithms: Selection Sort, Insertion Sort, Shellsort, Heapsort, Mergesort, Quicksort

Radix Sort with Least Significant Digit

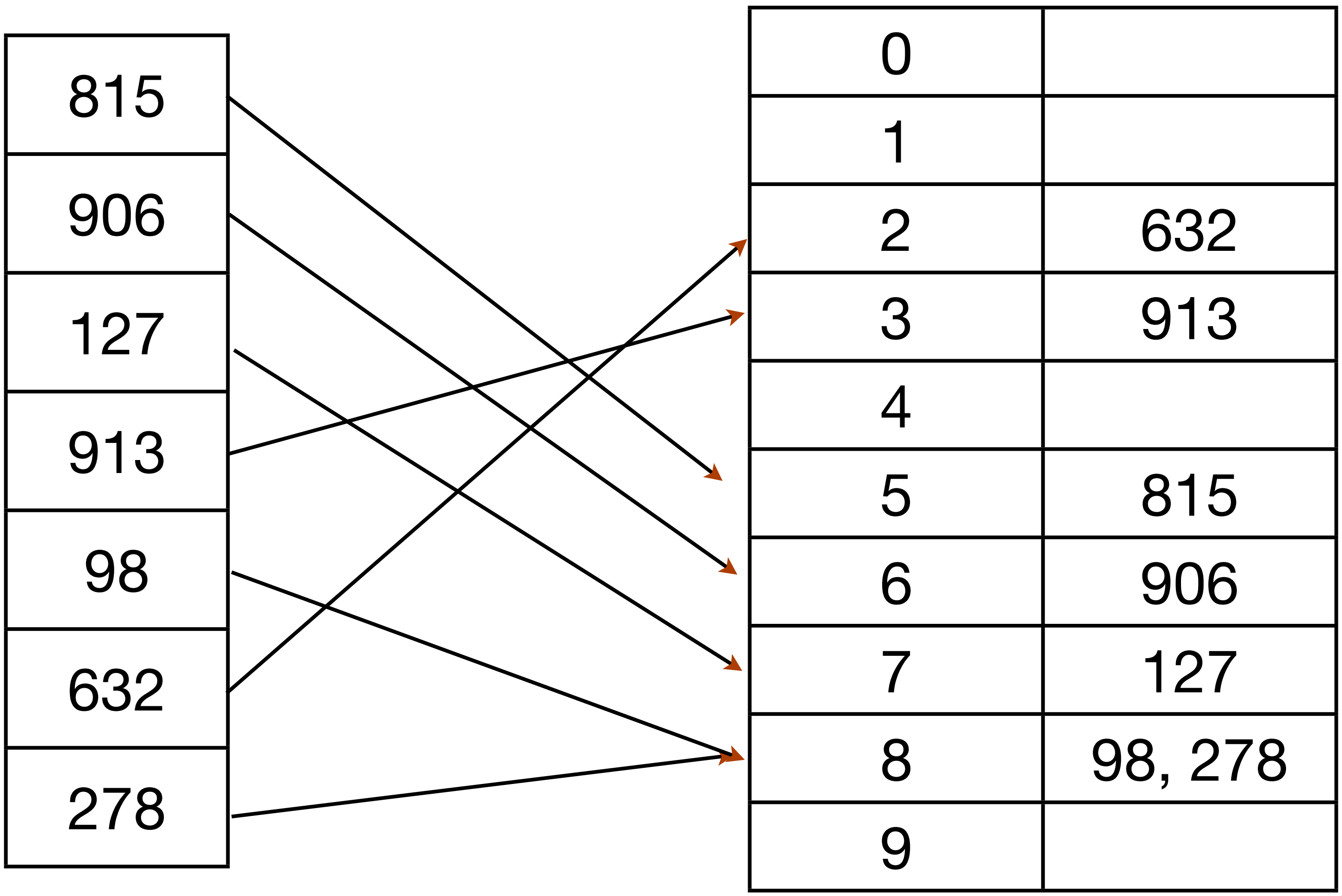
- CountingSort according to the least significant digit
- Repeat: CountingSort according to the next least significant digit
- Each step must be **stable**
- Running time: **O(Nk)** for maximum of **k** digits
- Space: **O(N+b)** for base-**b** number system*

Radix Sort Example

815
906
127
913
98
632
278

0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

Radix Sort Example



Radix Sort Example

632
913
815
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98
278

0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

Radix Sort Example

632
913
815
906
127
98
278

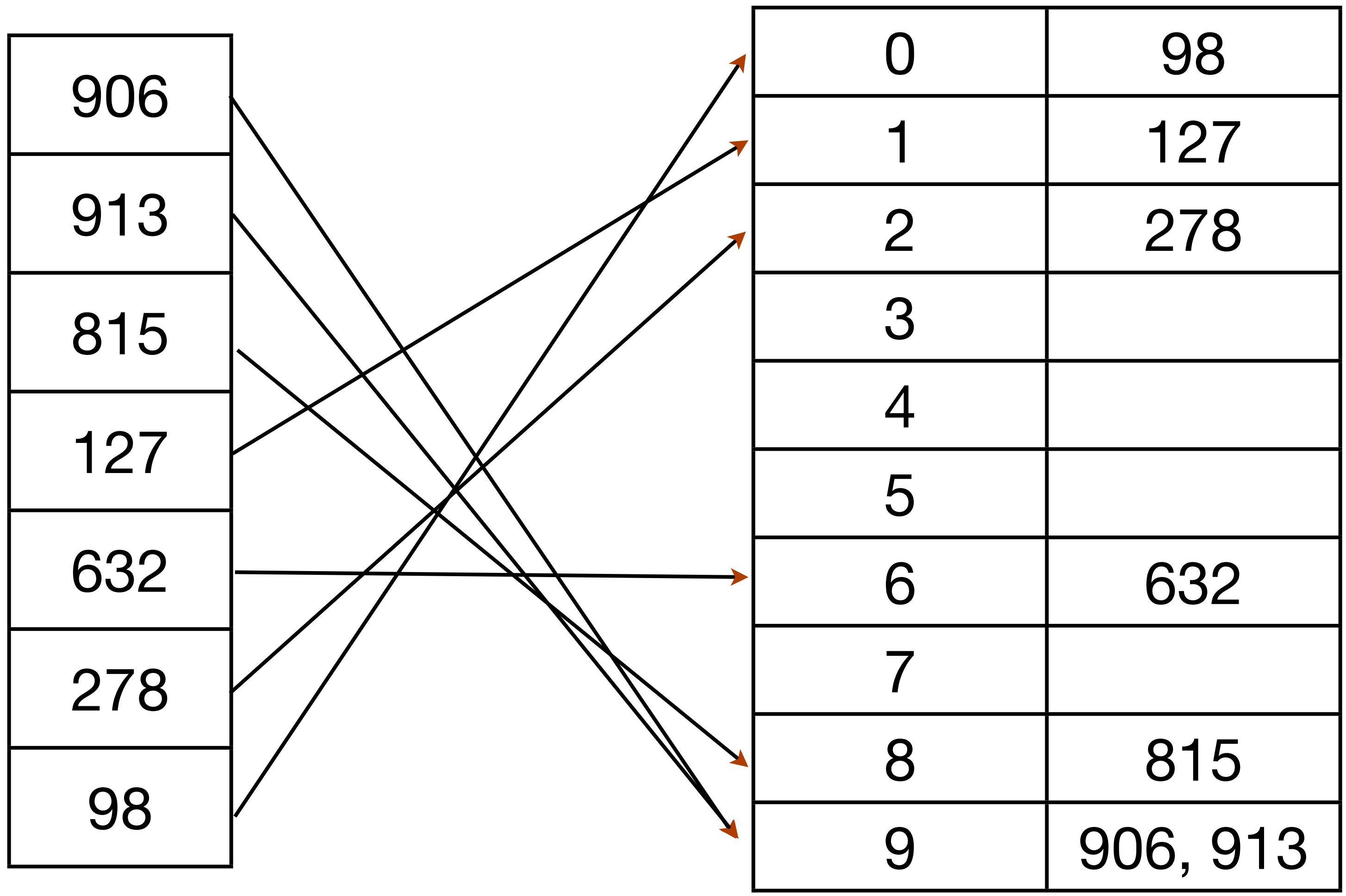
0	906
1	913, 815
2	127
3	632
4	
5	
6	
7	278
8	
9	98

Radix Sort Example

906
913
815
127
632
278
98

0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

Radix Sort Example



Analysis

- For maximum of **k** digits (in whatever base), we need **k** passes through the array, **O(Nk)**
- For base-**b** number system, we need **b** queues, which will end up containing **N** elements total, so O(**N+b**) space
- Stable because if elements are the same, they keep being enqueued and dequeued in the same order

Comparison Sorts

- Of course, Radix Sort only works well for sorting keys representable as digital numbers
- In general, we must often use comparison sorts
- We have proven an $\Omega(N \log N)$ lower bound for running time
- But algorithms also have other desirable characteristics

Sorting Algorithm Characteristics

- Worst case running time
- Worst case space usage (can it run in place?)
- Stability
- Average running time/space
- (simplicity)

Selection Sort

- Swap least unsorted element with first unsorted element
- Unstable
- Running time $O(N^2)$
- In place O(1) space
- Algorithm Animation

Insertion Sort

- Assume first **p** elements are sorted. Insert **(p+1)**'th element into appropriate location.
  - Save **A[p+1]** in temporary variable **t**, shift sorted elements greater than **t**, and insert **t**
- Stable
- Running time $O(N^2)$
- In place **O(1)** space

Insertion Sort Analysis

- When the sorted segment is **i** elements, we may need up to **i** shifts to insert the next element

$$\sum_{i=2}^{N} i = N(N-1)/2 - 1 = O(N^2)$$

- Stable because elements are visited in order and equal elements are inserted after its equals
- Algorithm Animation

Shellsort

- Essentially splits the array into subarrays and runs Insertion Sort on the subarrays
- Uses an increasing sequence, $h_1, \ldots, h_t$, such that $h_1 = 1$.
- At phase **k**, all elements h_k apart are sorted; the array is called h_k-sorted
- for every **i**, $A[i] \leq A[i + h_k]$

Shell Sort Correctness

- Efficiency of algorithm depends on that elements sorted at earlier stages remain sorted in later stages

- Unstable. Example: 2-sort the following: [5 5 1]

Increment Sequences

- Shell suggested the sequence $h_t = \lfloor N/2 \rfloor$ and $h_k = \lfloor h_{k+1}/2 \rfloor$, which was suboptimal
- A better sequence is $h_k = 2^k - 1$
- Shellsort using better sequence is proven $\Theta(N^{3/2})$
- Often used for its simplicity and sub-quadratic time, even though **O(N log N)** algorithms exist
- <u>Animation</u>

Heapsort

- Build a **max** heap from the array: **O(N)**
- call deleteMax **N** times: **O(N log N)**
- **O(1)** space
- Simple if we abstract heaps
- Unstable
- Animation

Mergesort

- Quintessential divide-and-conquer example
- Mergesort each half of the array, merge the results
- Merge by iterating through both halves, compare the current elements, copy lesser of the two into output array
- Animation

Mergesort Recurrence

- Merge operation is costs **O(N)**
- **T(N) = 2 T(N/2) + N**
- We solved this recurrence for the recursive solutions to the homework 1 theory problem

$$= \sum_{i=0}^{\log N} 2^i c \frac{N}{2^i}$$

$$= \sum_{i=0}^{\log N} cN = cN \log N$$

Quicksort

- Choose an element as the **pivot**
- Partition the array into elements greater than pivot and elements less than pivot
- Quicksort each partition

- Animation

Choosing a Pivot

- The worst case for Quicksort is when the partitions are of size zero and **N-1**
- Ideally, the pivot is the median, so each partition is about half
- If your input is random, you can choose the first element, but this is very bad for presorted input!
- Choosing randomly works, but a better method is…

Median-of-Three

- Choose three entries, use the median as pivot
- If we choose randomly, **2/N** probability of worst case pivots
- Median-of-three gives **0** probability of worst case, tiny probability of 2nd-worst case. (Approx. $2/N^3$)
- Randomness less important, so choosing (first, middle, last) works reasonably well

Partitioning the Array

- Once pivot is chosen, swap pivot to end of array. Start counters **i**=1 and **j**=**N-1**
- Intuition: **i** will look at less-than partition, **j** will look at greater-than partition
- Increment **i** and decrement **j** until we find elements that don't belong (**A[i]** > **pivot** or **A[j]** < **pivot**)
- Swap (**A[i]**, **A[j]**), continue increment/decrements
- When **i** and **j** touch, swap pivot with **A[j]**

Quicksort Worst Case

- Running time recurrence includes the cost of partitioning, then the cost of 2 quicksorts
- We don't know the size of the partitions, so let **i** be the size of the first partition
- **T(N) = T(i)+T(N-i-1) + N**
- Worst case is **T(N) = T(N-1) + N**

Quicksort Average Case

- We'll average over all partition sizes:

$$T(N) = \frac{2}{N-1}\sum_{i=0}^{N-1} T(i) + N$$

$$NT(N) = 2\sum_{i=0}^{N-1} T(i) + N^2$$

$$(N-1)T(N-1) = 2\sum_{i=0}^{N-2} T(i) + (N-1)^2$$

Quicksort Average Case

$$NT(N) = 2\sum_{i=0}^{N-1} T(i) + N^2$$

$$(N-1)T(N-1) = 2\sum_{i=0}^{N-2} T(i) + (N-1)^2$$

$$\begin{aligned} NT(N) - (N-1)T(N-1) \quad &= \quad 2\left[\sum_{i=0}^{N-1} T(i) - \sum_{i=0}^{N-2} T(i)\right] \\ &\quad\ + N^2 - (N-1)^2 \end{aligned}$$

Quicksort Average Case

$$
\begin{aligned}
NT(N) - (N-1)T(N-1) &= 2\left[\sum_{i=0}^{N-1} T(i) - \sum_{i=0}^{N-2} T(i)\right] \\
&\quad + N^2 - (N-1)^2 \\
NT(N) - (N-1)T(N-1) &= 2T(N-1) + 2N - 1
\end{aligned}
$$

$$NT(N) = (N+1)T(N-1) + 2N$$

$$\frac{T(N)}{N+1} = \frac{T(N-1)}{N} + \frac{2}{N+1}$$

Quicksort Average Case

$$\frac{T(N)}{N+1} = \frac{T(N-1)}{N} + \frac{2}{N+1}$$

$$\frac{T(N-2)}{N-1} = \frac{T(N-3)}{N-2} + \frac{2}{N-1}$$

$$\frac{T(2)}{3} = \frac{T(1)}{2} + \frac{2}{3}$$

$$\frac{T(N)}{N+1} = \frac{T(1)}{2} + 2\sum_{i=3}^{N+1} \frac{1}{i}$$

$$\frac{T(N)}{N+1} = O(\log N)$$

$$T(N) = O(N \log N)$$

Quicksort Properties

- Unstable
- Average time O(N log N)
- Worst case time $O(N^2)$
- Space O(log N)/$O(N^2)$ because we need to store the pivots

Summary

	Worst Case Time	Average Time	Space	Stable?
Selection	$O(N^2)$	$O(N^2)$	$O(1)$	No
Insertion	$O(N^2)$	$O(N^2)$	$O(1)$	Yes
Shell	$O(N^{3/2})$	?	$O(1)$	No
Heap	$O(N \log N)$	$O(N \log N)$	$O(1)$	No
Merge	$O(N \log N)$	$O(N \log N)$	$O(N)/O(1)$	Yes/No
Quick	$O(N^2)$	$O(N \log N)$	$O(\log N)$	No

Reading

- http://www.sorting-algorithms.com/
- Weiss Chapter 7