Data Structures and Algorithms

Session 18. April 1, 2009

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Announcements

- Homework 4 due next class
  - Huffman compression must handle any characters in dictionary.txt
  - Spell checker can ignore case

Review

- Rehashing
- String hash function example
- Graphs
  - Terminology and properties
  - Implementation

Today's Plan

- Topological Sort
- Shortest Path
    - Unweighted version
    - Weighted version

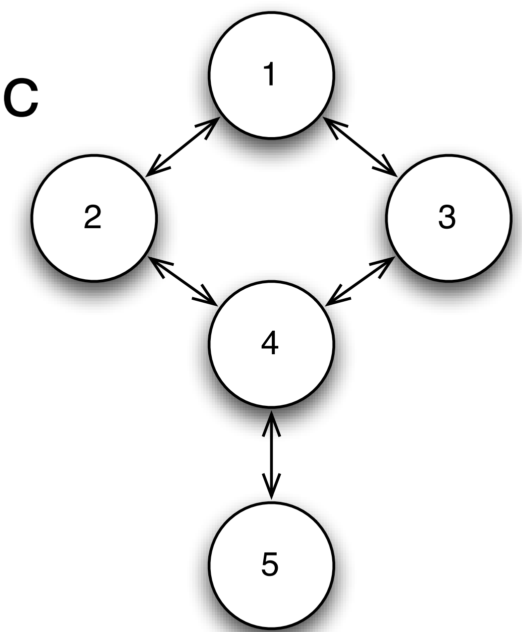
Implementation

- Option 1:
  - Store all nodes in an indexed list
  - Represent edges with **adjacency matrix**
- Option 2:
  - Explicitly store **adjacency lists**

Adjacency Matrices

- 2d-array **A** of boolean variables
- A[i][j] is true when node **i** is adjacent to node **j**
  - If graph is undirected, A is symmetric

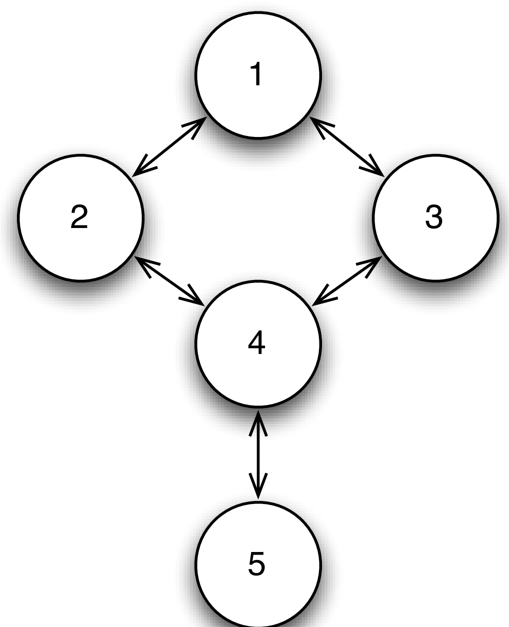
	1	2	3	4	5
1	0	**1**	**1**	0	0
2	**1**	0	0	**1**	0
3	**1**	0	0	**1**	0
4	0	**1**	**1**	0	**1**
5	0	0	0	**1**	0



Adjacency Lists

- Each node stores references to its neighbors

1	2	3	
2	1	4	
3	1	4	
4	2	3	5
5	4		



Math Notation for Graphs

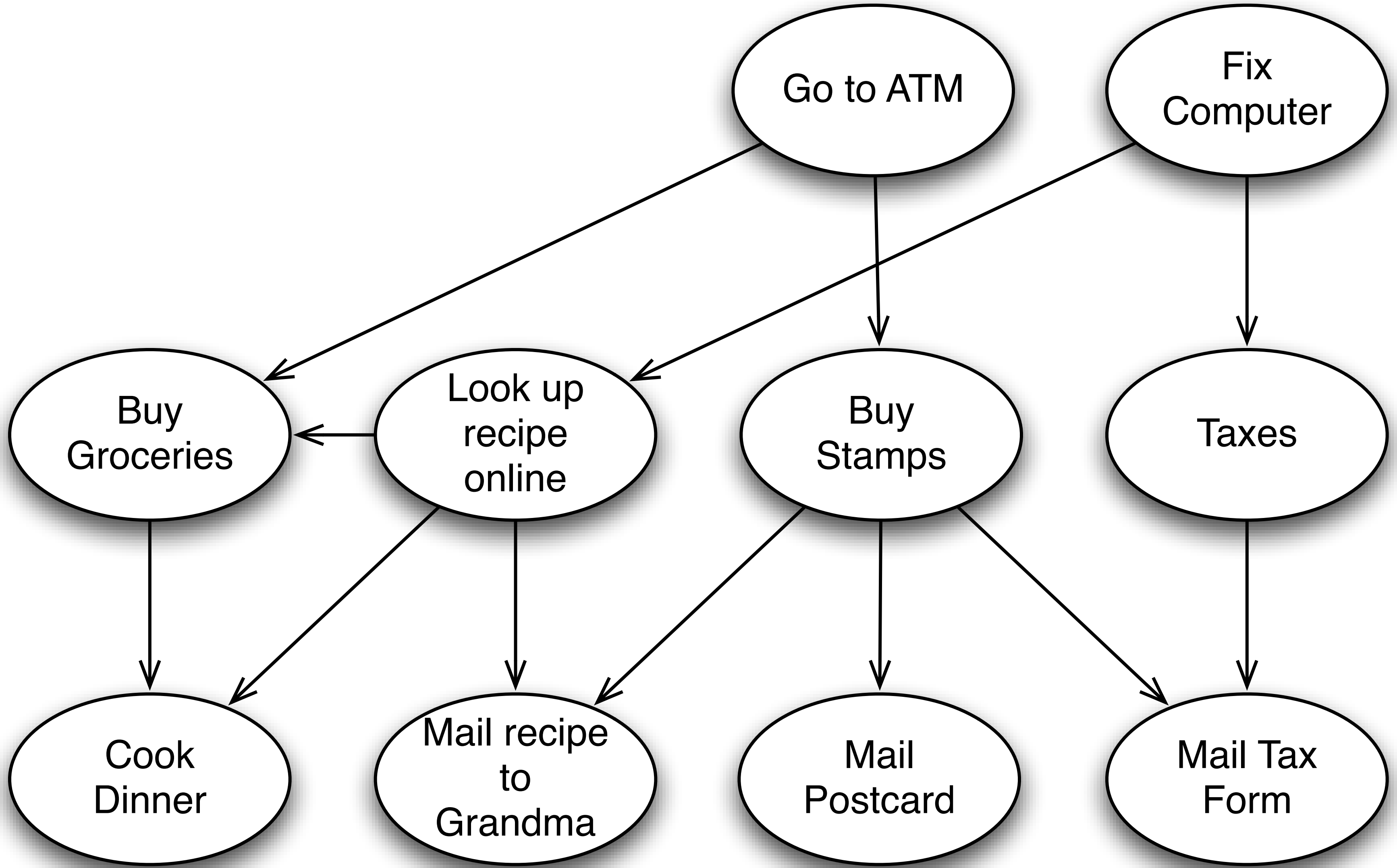
- Set Notation:
  - $v \in V$ (v is in V)
  - $U \cup V$ (union)
  - $U \cap V$ (intersection)
  - $U \subset V$
    (U is a subset of V)

- G = {V, E}
- G is the graph
- V is set of vertices
- E is set of edges
- $(v_i, v_j) \in E$
- |V| = N = size of V

Topological Sort

- Problem definition:
  - Given a directed acyclic graph G, order the nodes such that for each edge $(v_i, v_j) \in E$, v_i is before v_j in the ordering.
- e.g., scheduling errands when some tasks depend on other tasks being completed.

Topological Sort Ex.

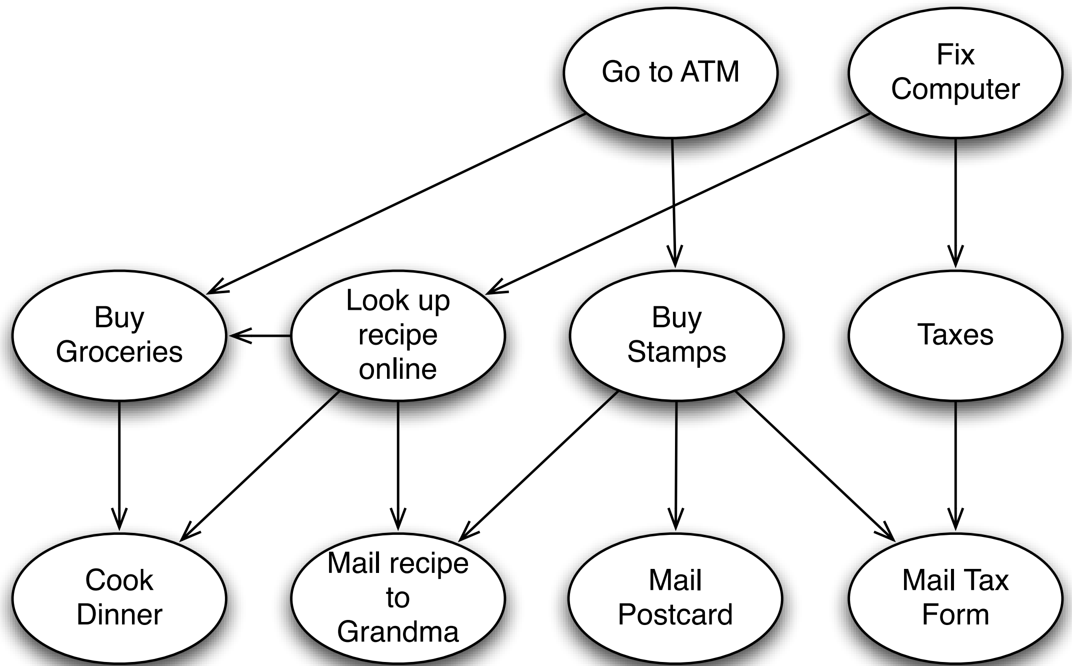


Topological Sort
Naïve Algorithm

- **Degree** means # of edges,
  indegree means # of incoming edges
- 1. Compute the **indegree** of all nodes
- 2. Print any node with indegree 0
- 3. Remove the node we just printed. Go to 1.
- Which nodes' indegrees change?

Topological Sort Better Algorithm

- 1. Compute all indegrees
- 2. Put all indegree 0 nodes into a Collection
- 3. Print and remove a node from Collection
- 4. Decrement indegrees of the node's neighbors.
- 5. If any neighbor has indegree 0, place in Collection. Go to 3.



ATM	comp	grocer-ies	recipe	stamps	taxes	cook	grand-ma	post-card	mail taxes
0	0	2	1	1	1	2	2	1	2

Topological Sort Running time

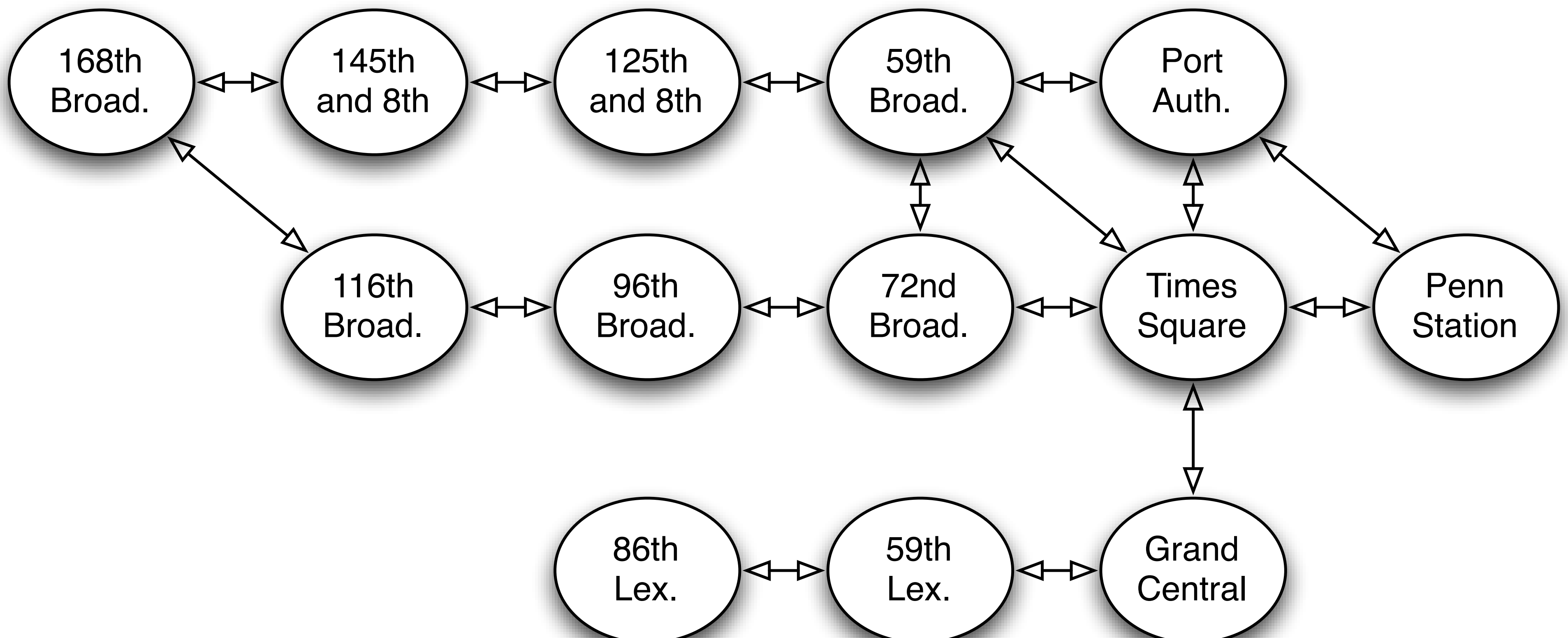
- Initial indegree computation: $O(|E|)$
  - Unless we update indegree as we build graph
- $|V|$ nodes must be enqueued/dequeued
- Dequeue requires operation for outgoing edges
- Each edge is used, but never repeated
- Total running time $O(|V| + |E|)$

Shortest Path

- Given **G = (V,E)**, and a node **s** $\in$ **V**, find the shortest (weighted) path from **s** to every other vertex in **G**.
- Motivating example: subway travel
  - Nodes are junctions, transfer locations
  - Edge weights are estimated time of travel

Approximate MTA Express Stop Subgraph

- A few inaccuracies (don't use this to plan any trips)

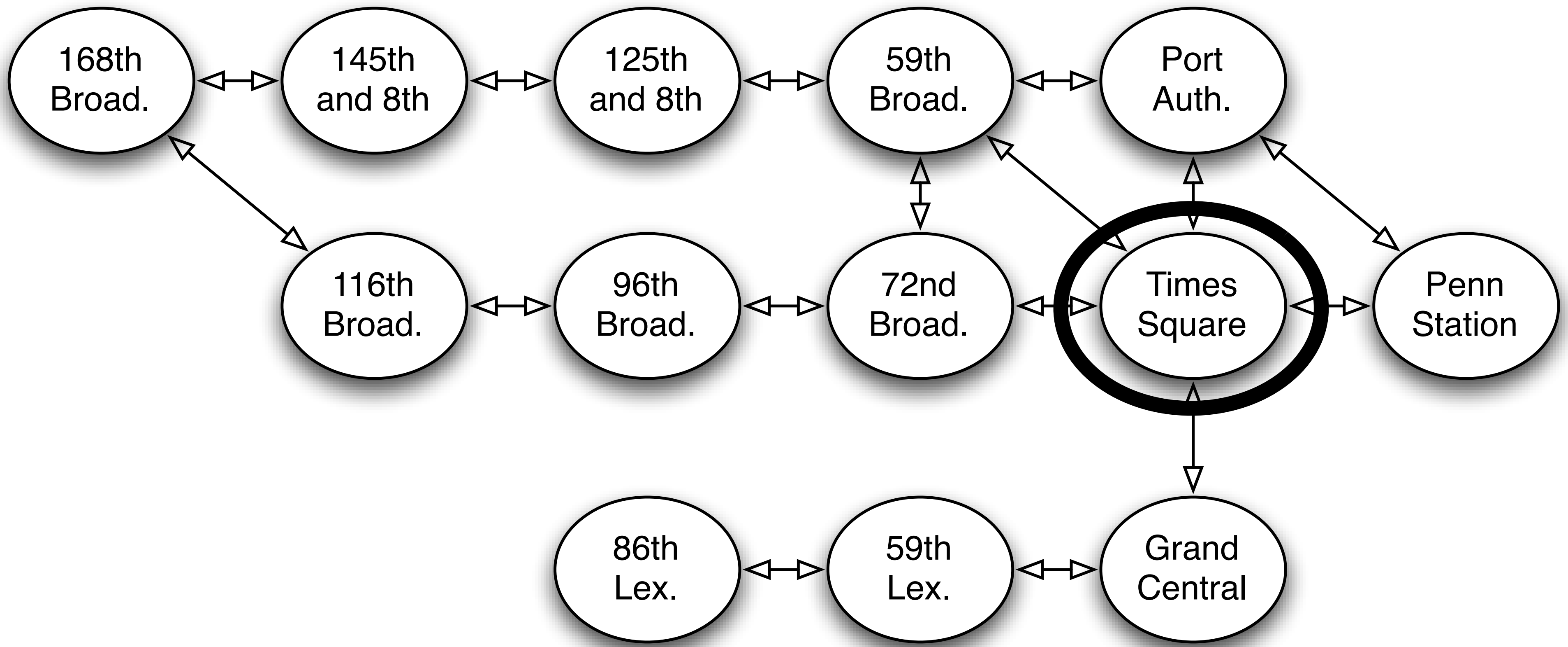


Breadth First Search

- Like a level-order traversal
- Find all adjacent nodes (level 1)
- Find *new* nodes adjacent to level 1 nodes (level 2)
- ... and so on
- We can implement this with a queue

Unweighted Shortest Path Algorithm

- Set node s' distance to 0 and enqueue s.
- Then repeat the following:
  - Dequeue node **v**. For unset neighbor **u**:
    - set neighbor **u**'s distance to **v**'s distance +1
    - mark that we reached **v** from **u**
    - enqueue **u**



	168^{th} Broad.	145^{th} Broad.	125^{th} 8th	59^{th} Broad.	Port Auth.	116^{th} Broad.	96^{th} Broad.	72^{nd} Broad.	Times Sq.	Penn St.	86^{th} Lex.	59^{th} Lex.	Grand Centr.
dist									0				
prev									source				

Weighted Shortest Path

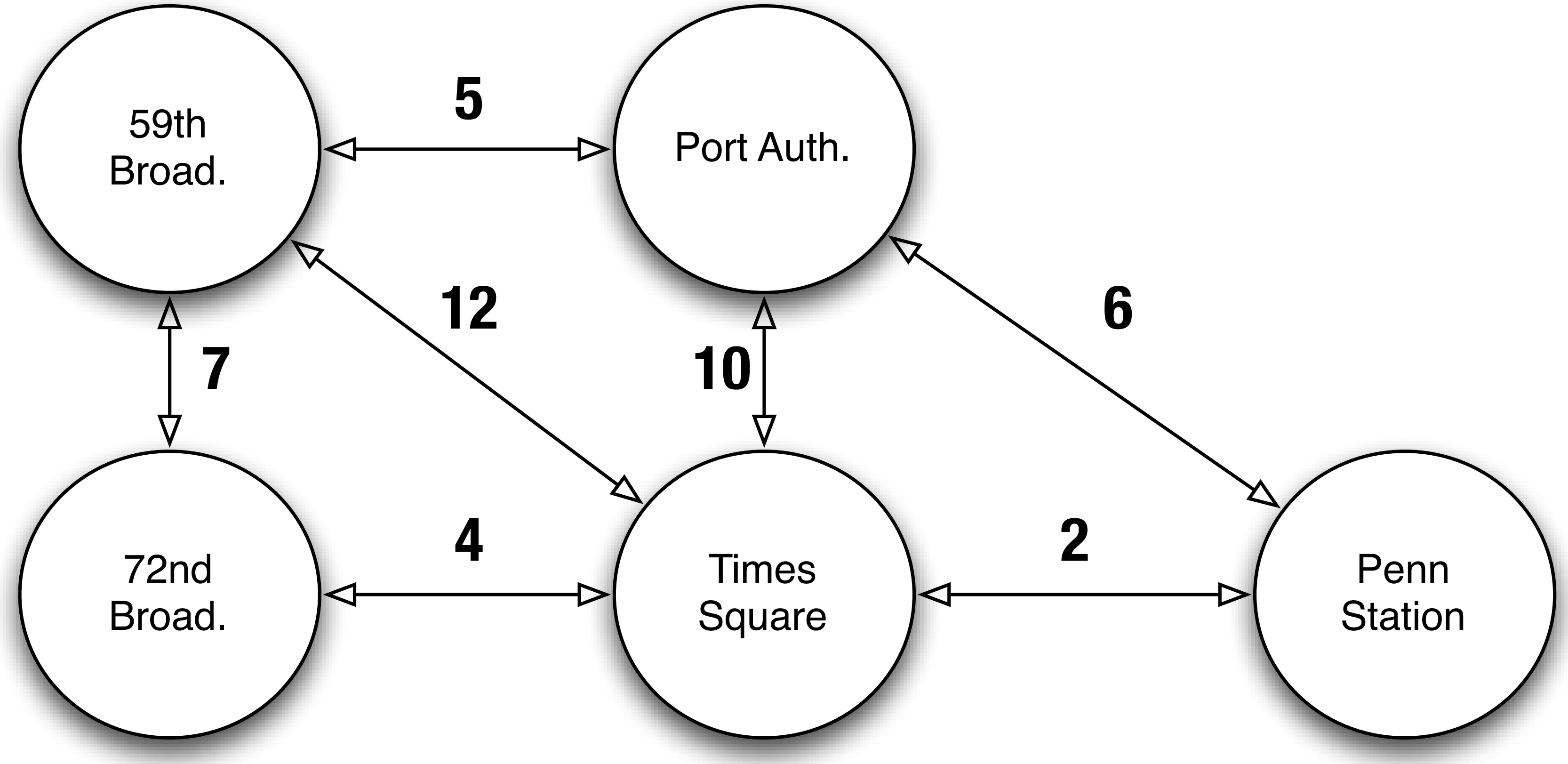
- The problem becomes more difficult when edges have different weights
- Weights represent different costs on using that edge
- Standard algorithm is **Dijkstra's Algorithm**

Dijkstra’s Algorithm

- Keep distance overestimates **D(v)** for each node **v** (all non-source nodes are initially infinite)
- 1. Choose node **v** with smallest *unknown* distance
- 2. Declare that **v**’s shortest distance is *known*
- 3. Update distance estimates for neighbors

Updating Distances

- For each of **v**'s neighbors, **w**,
- if min(**D(v)+ weight(v,w)**, **D(w)**)
  - i.e., update **D(w)** if the path going through **v** is cheaper than the best path so far to **w**



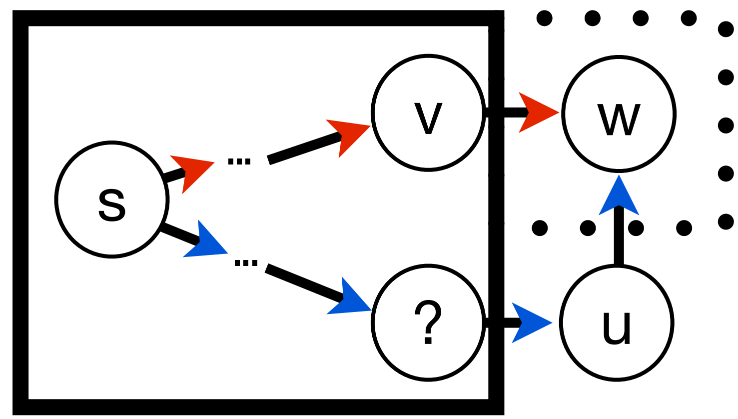
59th Broad.	Port Auth.	72nd Broad	Times Sq.	Penn St.
inf	inf	inf	inf	0
?	?	?	?	home

Dijkstra's Algorithm Analysis

- First, convince ourselves that the algorithm works.
- At each stage, we have a set of nodes whose shortest paths we know
- In the base case, the set is the source node.
- Inductive step: if we have a correct set, is greedily adding the shortest neighbor correct?

Proof by Contradiction (Sketch)

- Contradiction: Dijkstra's finds a shortest path to node **w** through **v**, but there exists an even shorter path
- This shorter path must pass from inside our known set to outside.
- Call the 1st node in cheaper path outside our set **u**



- The path to **u** must be shorter than the path to **w**
  - But then we would have chosen **u** instead

Computational Cost

- Keep a priority queue of all unknown nodes
- Each stage requires a **deleteMin**, and then some **decreaseKey**s (the # of neighbors of node)
- We call **decreaseKey** once per edge, we call **deleteMin** once per vertex
- Both operations are $O(\log |V|)$
- Total cost: $O(|E| \log |V| + |V| \log |V|) = O(|E| \log |V|)$

Reading

- Weiss Section 9.3 (today's material)
- Weiss Section 9.4 (Monday's material)