Data Structures and Algorithms

Session 14. March 9, 2009

Instructor: Bert Huang

http://www.cs.columbia.edu/~bert/courses/3137

Announcements

- Homework 3 is due
  - Solutions 1 hour after class
- Course Evaluation
- Midterm Exam March 11th

Review

- Clarification about isomorphism
- buildHeap example
- HeapSort and HeapSelect

Math Background: Exponents

$$
\begin{aligned}
X^A X^B &= X^{A+B} \\
\frac{X^A}{X^B} &= X^{A-B} \\
\left(X^A\right)^B &= X^{AB} \\
X^N + X^N &= 2X^N \neq X^{2N} \\
2^N + 2^N &= 2^{N+1}
\end{aligned}
$$

Math Background: Logarithms

$$X^A = B \text{ iff } \log_X B = A$$

$$\begin{aligned} \log_A B &= \frac{\log_C B}{\log_C A};\ A, B, C > 0, A \neq 1 \\ \log AB &= \log A + \log B;\ \ A, B > 0 \end{aligned}$$

Math Background: Series

$$\sum_{i=0}^{N} 2^i = 2^{N+1} - 1$$

$$\sum_{i=0}^{N} A^i = \frac{A^{N+1} - 1}{A - 1}$$

$$\sum_{i=1}^{N} i = \frac{N(N+1)}{2} \approx \frac{N^2}{2}$$

$$\sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6} \approx \frac{N^3}{3}$$

Big-Oh Notation

- We adopt special notation to define **upper bounds** and **lower bounds** on functions
- In CS, usually the functions we are bounding are running times, memory requirements.
- We will refer to the running time as *T(N)*

Definitions

- For *N* greater than some constant, we have the following definitions:

$$
\begin{aligned}
T(N) &= O(f(N)) \leftarrow T(N) \leq cf(N) \\
T(N) &= \Omega(g(N)) \leftarrow T(N) \geq cf(N) \\
T(N) &= \Theta(h(N)) \leftarrow \begin{matrix} T(N) = O(h(N)), \\ T(N) = \Omega(h(N)) \end{matrix}
\end{aligned}
$$

- There exists some constant c such that cf(N) bounds T(N)

Definitions

- Alternately, O(f(N)) can be thought of as meaning

$$T(N) = O(f(N)) \leftarrow \lim_{N\to\infty} f(N) \geq \lim_{N\to\infty} T(N)$$

- Big-Oh notation is also referred to as **asymptotic** analysis, for this reason.

Comparing Growth Rates

$$T_1(N) = O(f(N)) \text{ and } T_2(N) = O(g(N))$$

then

$$(a) \qquad T_1(N) + T_2(N) = O(f(N) + g(N))$$

$$(b) \qquad T_1(N)T_2(N) = O(f(N)g(N))$$

- If you have to, use l'Hôpital's rule

$$\lim_{N \to \infty} f(N)/g(N) = \lim_{N \to \infty} f'(N)/g'(N)$$

Abstract Data Types

- Defined by:
  - What information it stores
  - How the information is organized
  - How the information can be accessed
- Doesn’t specify **implementation**

Tradeoffs

	insert	remove	lookup	index
ArrayList	**O(N)**	**O(N)**	**O(N)**	**O(1)**
LinkedList	**O(1)**	**O(1)**	**O(N)**	**O(N)**
Stack/Queue	**O(1)**	**O(1)**	N/A	N/A
BST	**O(d)=O(N)**	**O(d)=O(N)**	**O(d)=O(N)**	N/A
AVL	**O(log N)**	**O(log N)**	**O(log N)**	N/A

- There may not be free lunch, but sometimes there's a cheaper lunch

Abstract Data Type: Lists

- An ordered series of objects
- Each object has a previous and next
  - Except *first* has no previous, *last* has no next
- We can insert an object to a list (at location *k*)
- We can remove an object from a list
- We can read an object from a list (location *k*)

Array Implementation of Lists

- 1st Hurdle: arrays have sizes
  - Create bigger array when we run out of space, copy old array to big array
- 2nd Hurdle: Inserting object anywhere but the end
  - Shift all entries forward one. O(N)
- Get kth and insertion to end constant time O(1)

Linked Lists vs. Array Lists

- Linked Lists
  - No additional penalty on size
  - Insert/remove O(1)*
  - get kth costs O(N)*
  - Need some extra memory for links
- Array Lists
  - Need to estimate size/grow array
  - Insert/remove O(N)*
  - get kth costs O(1)
  - Arrays are compact in memory

Stack Definition

- Essentially a very restricted List
- Two (main) operations:
  - Push(AnyType x)
  - Pop(AnyType x)
- Analogy – Cafeteria Trays, PEZ

Stack Implementations

- Linked List:
  - Push(x) <-> add(x,0)
  - Pop(x) <-> remove(0)
- Array:
  - Push(x) <-> Array[k++] = x
  - Pop(x) <-> return Array[--k]

Queues

- Stacks are **L**ast **I**n **F**irst **O**ut
- Queues are **F**irst **I**n **F**irst **O**ut, first-come first-served
- Operations: **enqueue** and **dequeue**

Queue Implementation

- Linked List
  - add(x,0) to enqueue, remove(N-1) to dequeue
- Array List won't work well!
  - add(x,0) is expensive
  - Solution: use a circular array

Circular Array Queue

- Don’t bother shifting after removing from array list
- Keep track of start and end of queue
- When run out of space, wrap around
  - modular arithmetic
- When array is full, increase size using list tactic

Trees

- Extension of Linked List structure:
  - Each node connects to multiple nodes
- Examples include file systems, Java class hierarchies

Tree Terminology

- Just like Lists, **Trees** are collections of **nodes**
- Conceptualize trees upside down (like family trees)
  - the top node is the **root**
  - nodes are connected by **edges**
  - edges define **parent** and **child** nodes
  - nodes with no children are called **leaves**

More Tree Terminology

- Nodes that share the same parent are **siblings**
- A **path** is a sequence of nodes such that the next node in the sequence is a child of the previous
- a node’s **depth** is the length of the path from root
- the **height** of a tree is the maximum depth
- if a path exists between two nodes, one is an **ancestor** and the other is a **descendant**

Tree Traversals

- Suppose we want to print all the nodes in a tree
- What order should we visit the nodes?
  - **Preorder** - read the parent before its children
  - **Postorder** - read the parent after its children

Preorder vs. Postorder

* preorder(node x)

```
preorder(node x)
    print(x)
    for child : Children
        preorder(child)
```

* postorder(node x)

```
postorder(node x)
    for child : Children
        postorder(child)
    print(x)
```

Binary Trees

- Nodes can only have two children:
  - left child and right child
- Simplifies implementation and logic
- Provides new **inorder** traversal

Inorder Traversal

- Read left child, then parent, then right child
- Essentially scans *whole* tree from left to right
- inorder(node x)

```
    inorder(x.left)
    print(x)
    inorder(x.right)
```

Binary Tree Properties

- A binary tree is **full** if each node has 2 or 0 children
- A binary tree is **perfect** if it is full and each leaf is at the same depth
  - That depth is O(log N)

Search (Tree) ADT

- ADT that allows insertion, removal, and searching by **key**
  - A **key** is a value that can be compared
  - In Java, we use the **Comparable** interface
  - Comparison must obey transitive property
- Notice that the Search ADT doesn't use any index

Inserting into a BST

- **insert(x)** calls **insert(x,root)**
- Recursive concept:
  - insert(x,t)

```
if (x > t.key)
    insert(x, t.right)
elseif (x < t.key)
    insert(x, t.left)
```

- Actual code needs to manage links/null etc

Searching a BST

- **findMin(t)**
  if (**t.left == null**) return **t.key**
  else return **findMin(t.left)**

- **contains(x,t)**
  if (**t == null**) return **false**
  if (**x == t.key**) return **true**
  if (**x > t.key**), then return **contains(x, t.right)**
  if (**x < t.key**), then return **contains(x, t.left)**

Deleting from a BST

- Removing a leaf is easy, removing a node with one child is also easy
- Nodes with no grandchildren are easy
- Nodes with both children and grandchildren need more thought
    - Why can’t we replace the removed node with either of its children?

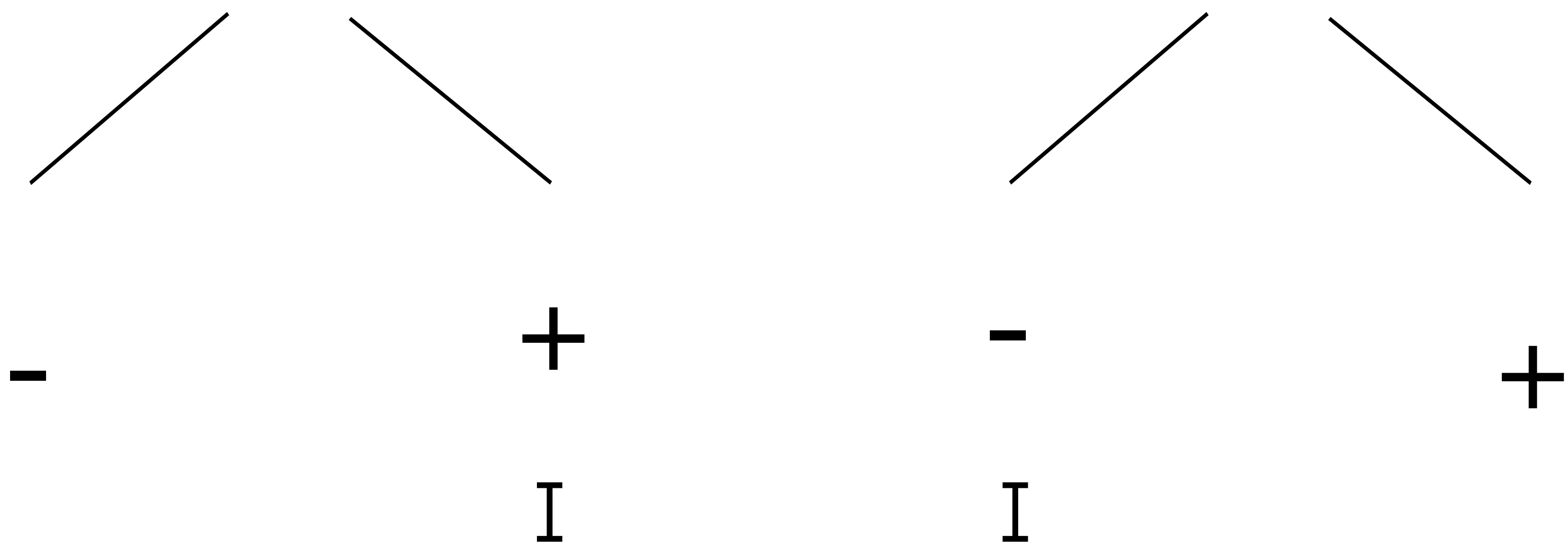
A Removal Strategy

- First, find node to be removed, **t**
- Replace with the smallest node from the right subtree
  - **a = findMin(t.right);**
    t.key = a.key;
- Then delete original smallest node in right subtree
  remove(a.key, t.right)

AVL Trees

- Motivation: want height of tree to be close to log N
- AVL Tree Property:
  For each node, all keys in its left subtree are less than the node's and all keys in its right subtree are greater. **Furthermore, the height of the left and right subtrees differ by at most 1**

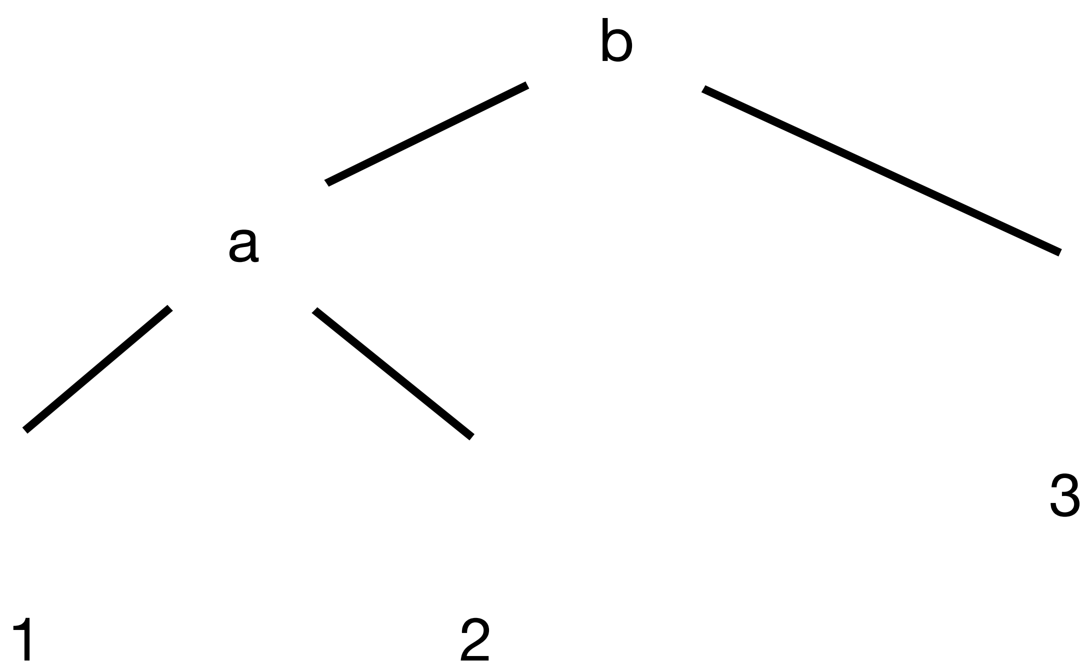
AVL Tree Visual



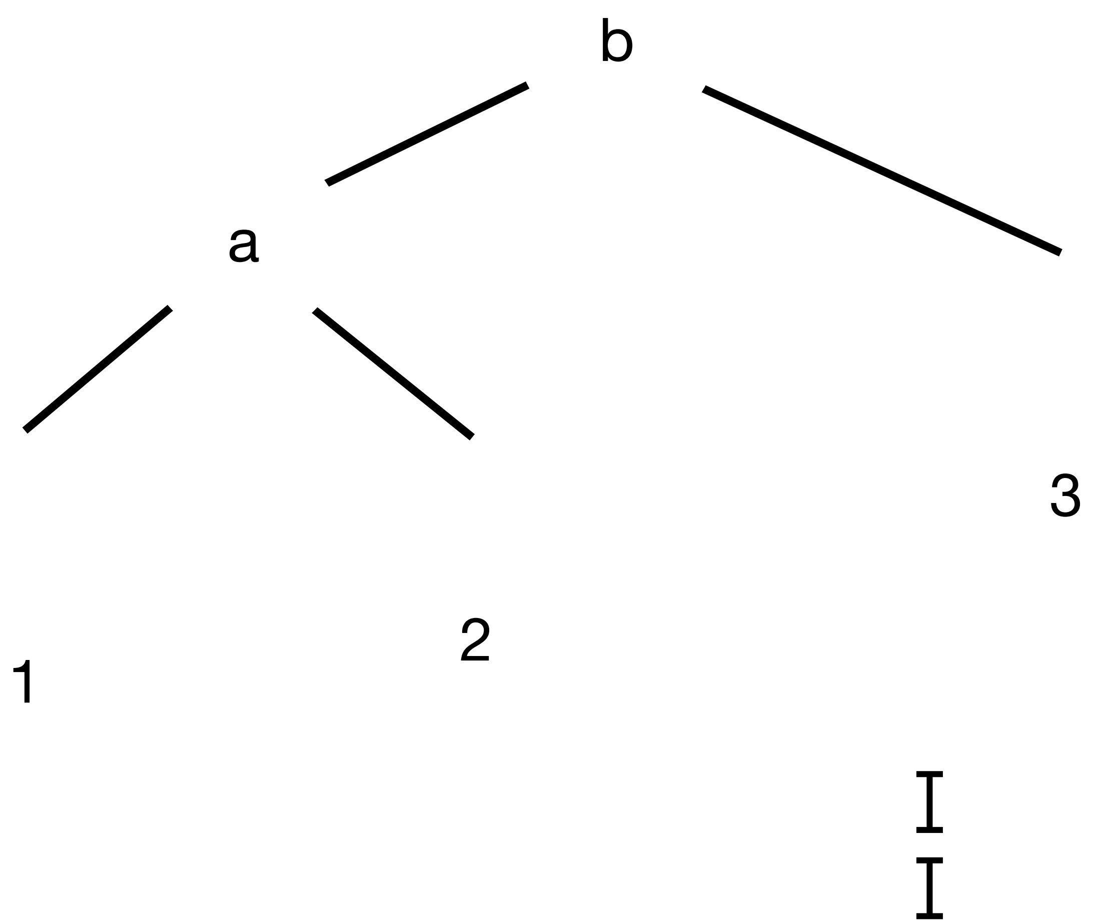
Tree Rotations

- To balance the tree after an insertion violates the AVL property,
    - rearrange the tree; make a new node the root.
    - This rearrangement is called a **rotation.**
    - There are 2 types of rotations.

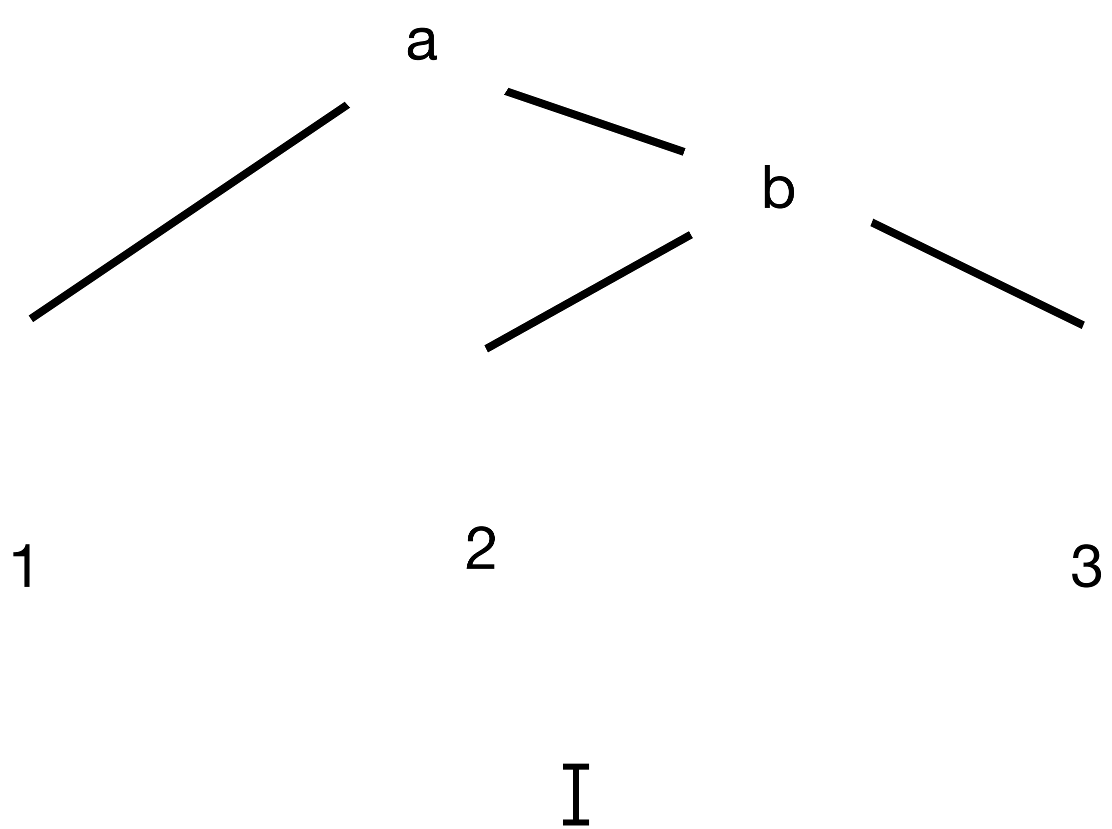
AVL Tree Visual: Before insert



AVL Tree Visual: After insert



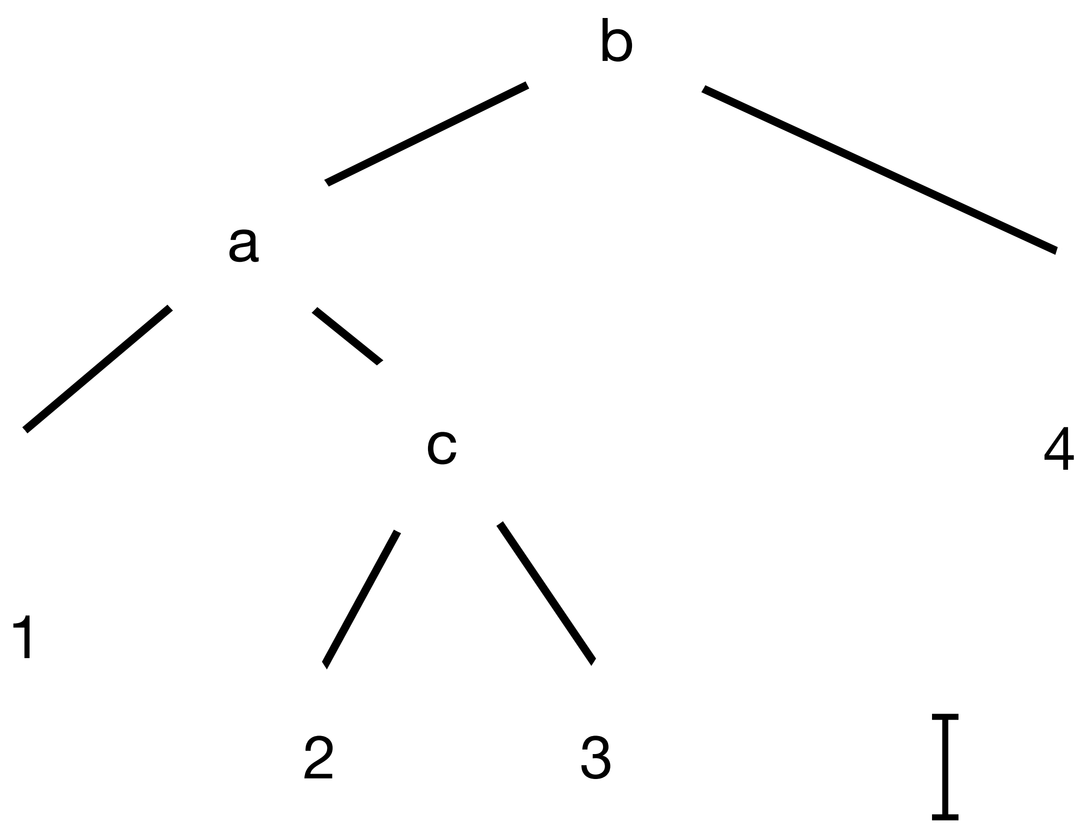
AVL Tree Visual: Single Rotation



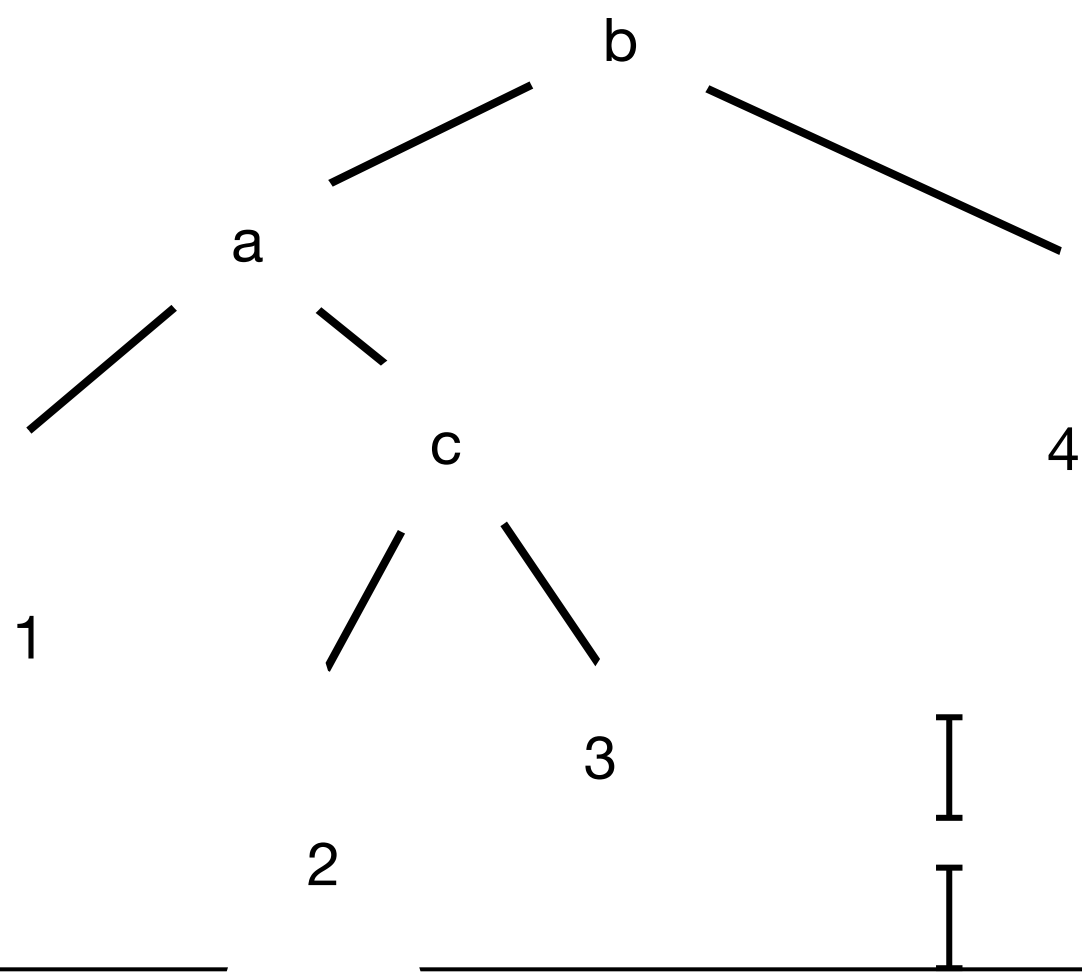
AVL Tree
Single Rotation

- Works when new node is added to outer subtree (left-left or right-right)
- What about inner subtrees? (left-right or right-left)

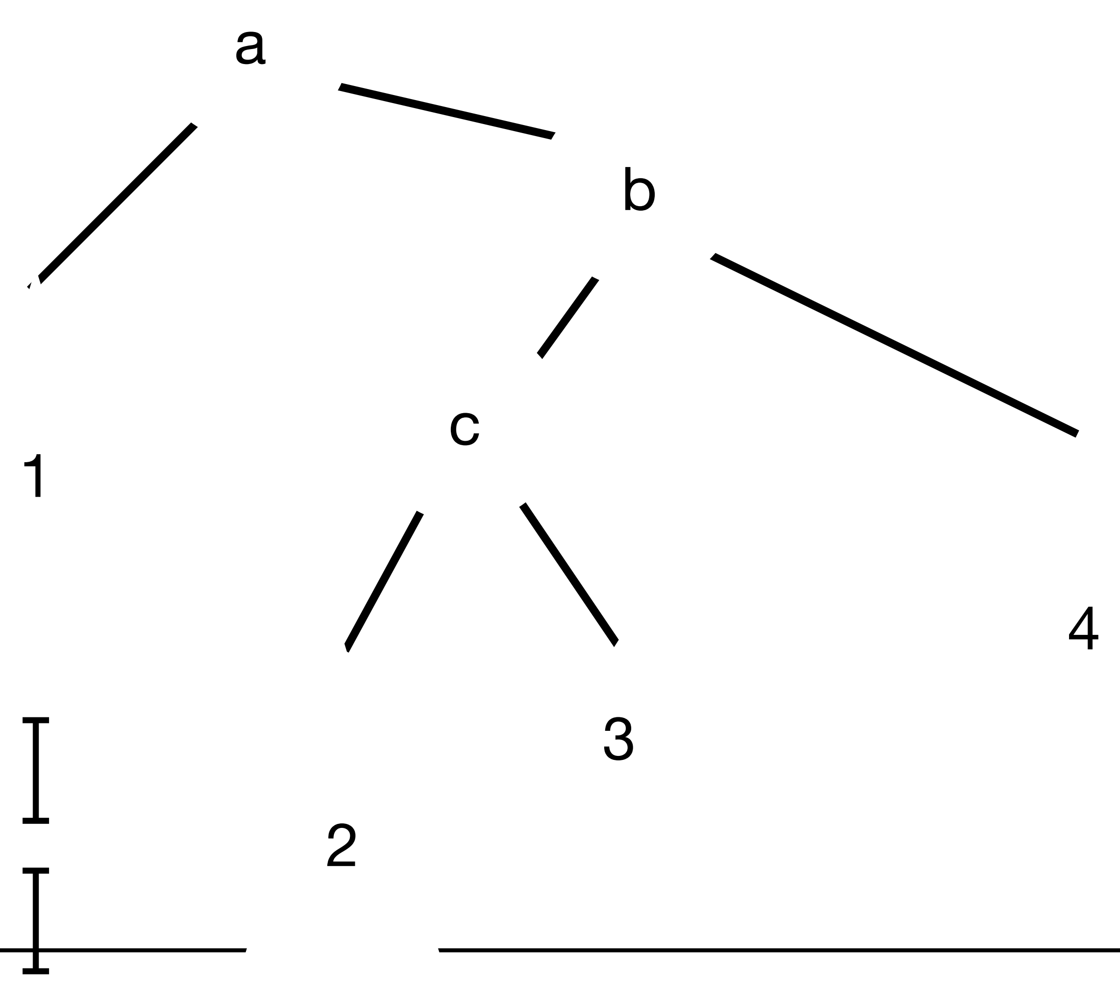
AVL Tree Visual: Before Insert 2



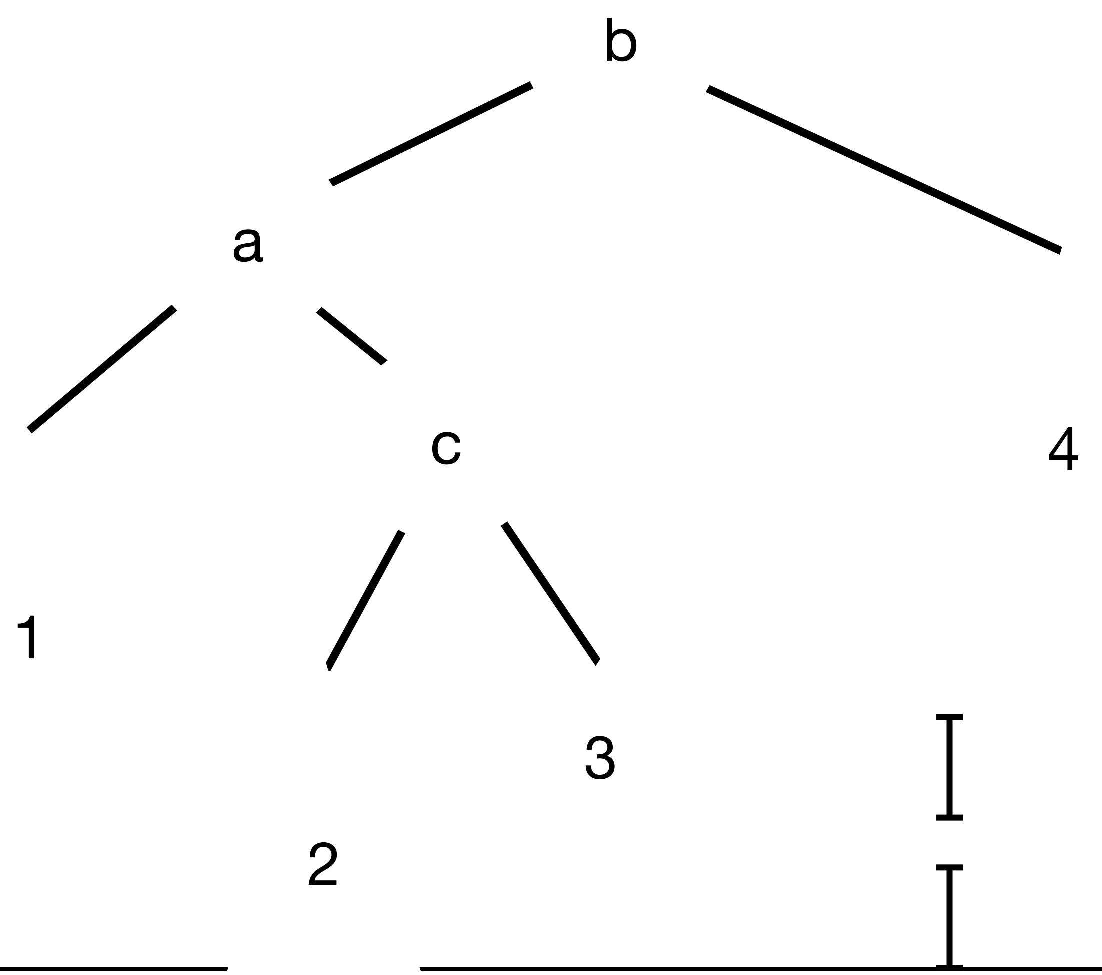
AVL Tree Visual: After Insert 2



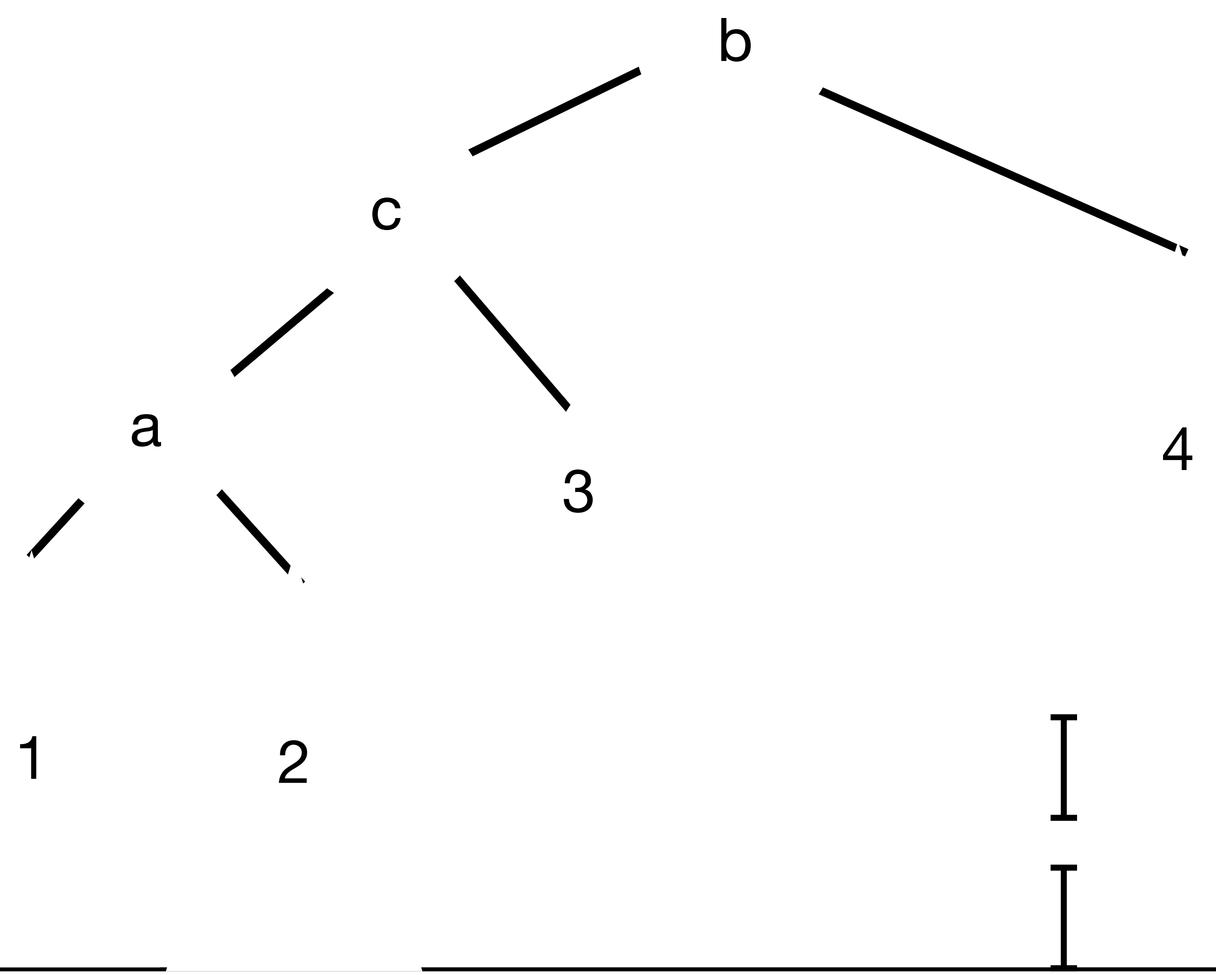
AVL Tree Visual: Single Rotation Fails



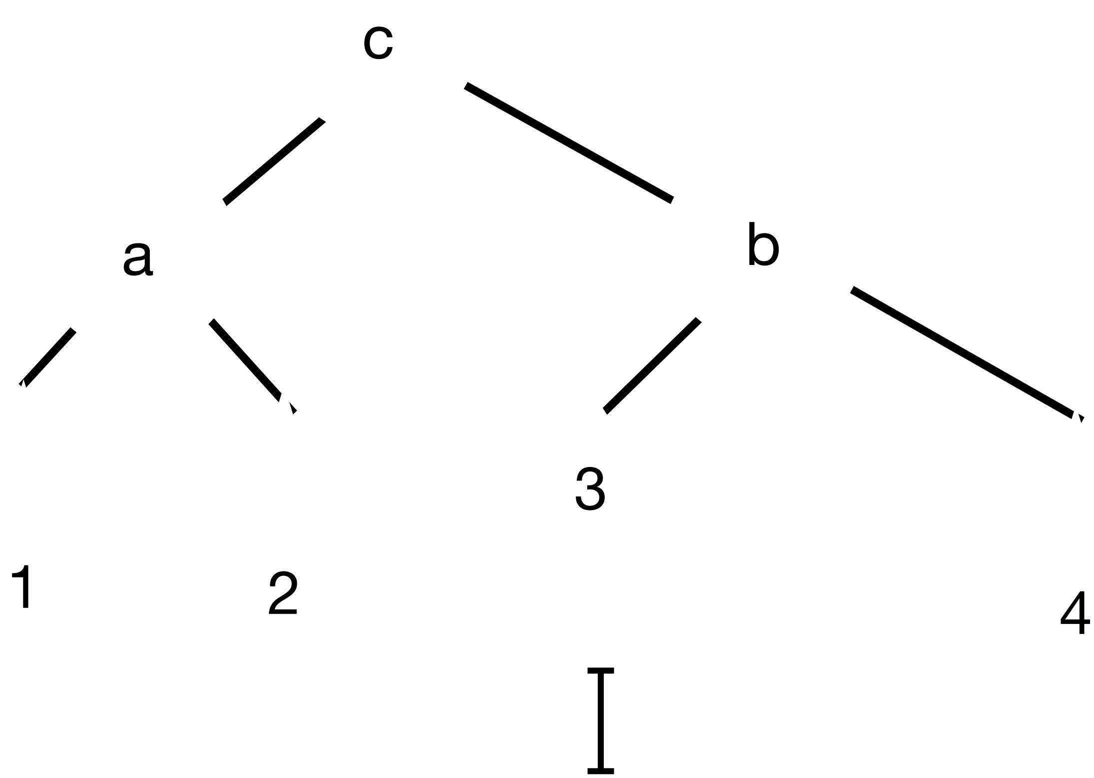
AVL Tree Visual: Double Rotation



AVL Tree Visual: Double Rotation



AVL Tree Visual: Double Rotation



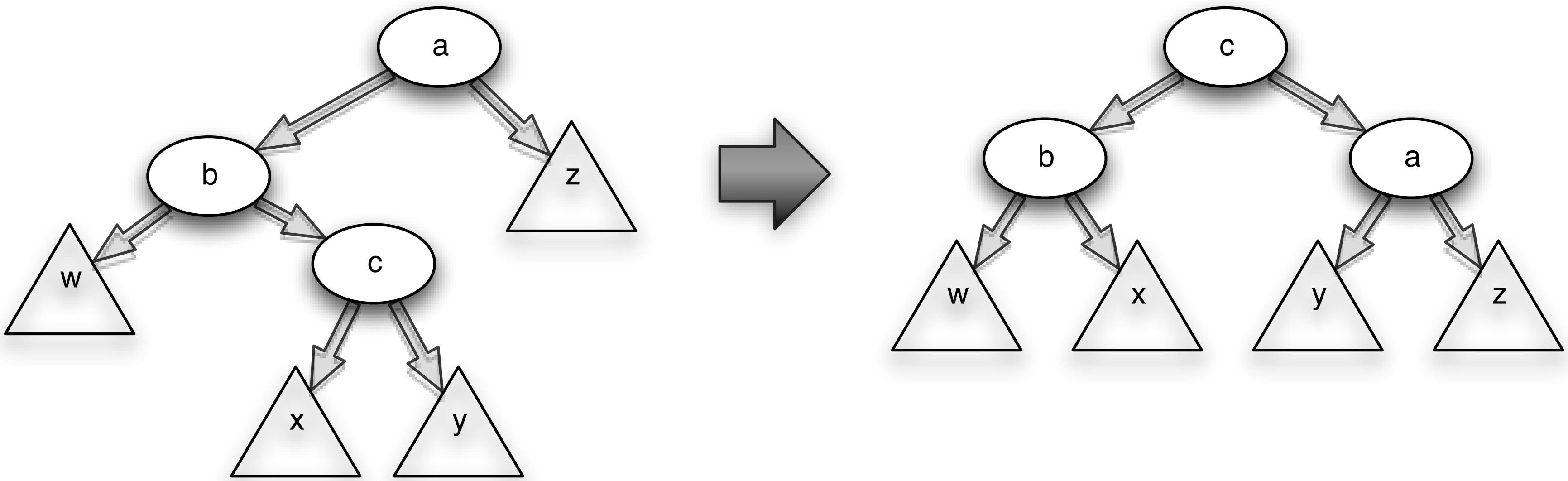
Splay Trees

- Like AVL trees, use the standard binary search tree property
- After any operation on a node, make that node the new root of the tree
    - Make the node the root by repeating one of two moves that make the tree more spread out

Easy cases

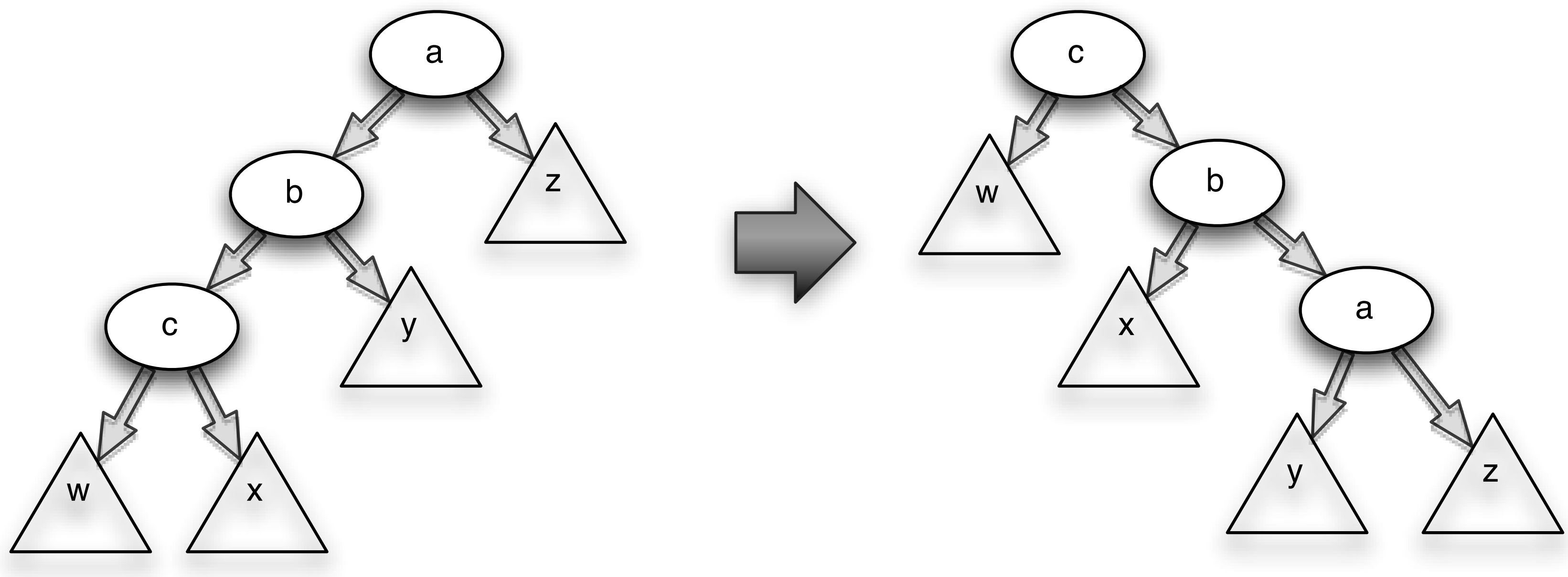
- If node is root, do nothing
- If node is child of root, do single AVL rotation
- Otherwise, node has a grandparent, and there are two cases

Case 1: zig-zag



- Use when the node is the right child of a left child (or left-right)
- Double rotate, just like AVL tree

Case 2: zig-zig



- Use when node is the right-right child (or left-left)
- Reverse the order of grandparent->parent->node
  - Make it node->parent->grandparent

Priority Queues

- New abstract data type Priority Queue:
  - Insert: add node with key
  - deleteMin: delete the node with smallest key
  - (increase/decrease priority)

Heap Implementation

- Priority queues are most commonly implemented using Binary Heaps
    - Binary tree with special properties
- Heap Structure Property: all nodes are full, (except possibly one at the bottom level)
- Heap Order Property: any node is smaller than its children

Array Implementation

- A full tree is regular: we can easily store in an array
  - Root at **A[1]**
  - Root's children at **A[2]**, **A[3]**
  - Node **i** has children at **2i** and **(2i+1)**
  - Parent at **floor(i/2)**
- No links necessary, so faster (in most languages)

Insert

- To insert key **X**, create a hole in bottom level
- **Percolate up**
  - Is hole's parent is less than **X**
    - If so, put **X** in hole, heap order satisfied
    - If not, swap hole and parent and repeat

DeleteMin

- Save root node, and delete, creating a hole
- Take the last element in the heap **X**
- **Percolate down**:
  - Check if X is less than hole’s children
    - if so, we’re done
    - if not, swap hole and smallest child and repeat

Building a Heap from an Array

- How do we construct a binary heap from an array?
- Simple solution: insert each entry one at a time
- Each insert is worst case **O(log N)**, so creating a heap in this way is **O(N log N)**
- Instead, we can jam the entries into a full binary tree and run **percolateDown** intelligently

buildHeap

- Start at deepest non-leaf node
  - in array, this is node N/2
- **percolateDown** on all nodes in reverse level-order
  - for i = N/2 to 1
    percolateDown(i)