3.1 (Theorems of Boolean algebra) Give the proofs of the following theorems:

(a) Theorem 2(a) and (b)
(b) Theorem 3(a) and (b)
(c) Theorem 4(a) and (b)
(d) Theorem 5(a) and (b)
(e) Theorem 6(a) and (b)

THEOREM 2(a): \(x + 1 = 1\).

Proof:
\[
x + 1 = 1 \cdot (x + 1) \quad \text{by identity (Ax. 2b)}
\]
\[
= (x + x')(x + 1) \quad \text{by complement (Ax. 5a)}
\]
\[
= x' + x' \cdot 1 \quad \text{by distributivity (Ax. 4b)}
\]
\[
= x' \quad \text{by identity (Ax. 2b)}
\]
\[
= 1 \quad \text{by complement (Ax. 5a)}
\]

THEOREM 2(b): \(x \cdot 0 = 0\) by duality.

THEOREM 4(a) Involution: \((x')' = x\).

Proof:
\[
(x')' = (x')' + 0 \quad \text{by identity (Ax. 2a)}
\]
\[
= (x')' + (xx') \quad \text{by complement (Ax. 5b)}
\]
\[
= ((x')' + x)((x')' + x') \quad \text{by distributivity (Ax. 4a)}
\]
\[
= ((x')' + x) \cdot 1 \quad \text{by complement (Ax. 5a)}
\]
\[
= (x')' + x \quad \text{by identity (Ax. 2b)}
\]
\[
= (x')' \cdot 1 + x \quad \text{by identity (Ax. 2b)}
\]
\[
= (x')'(x + x') + x \quad \text{by complement (Ax. 5a)}
\]
\[
= ((x')'x) + ((x')'x') + x \quad \text{by distributivity (Ax. 4b)}
\]
\[
= ((x')'x) + 0 + x \quad \text{by complement (Ax. 5b)}
\]
\[
= (x')'x + x \quad \text{by identity (Ax. 2a)}
\]
\[
= x \quad \text{by absorption (Th. 3)}
\]
3.2 (Theorems and proofs) Using truth tables, prove the validity of the following identities:

(a) \((xyz)' = x' + y' + z'\)
(b) \(xy'' + x'y = (xy + x'y')'\)
(c) \(xy + x'z + yz = xy + x'z\)
(d) \((x + y + z)' = x'y'z'\)

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<th>xy'</th>
<th>x'y'</th>
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<th>xy</th>
<th>x'y'</th>
<th>(xy + x'y')'</th>
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3.3 (Theorems and proofs) Prove algebraically these extensions of De Morgan’s theorem:

(a) \((xyz)' = x' + y' + z'\)
(b) \((x + y + z)' = x'y'z'\)

Proof:
\[
(x + y + z)' = ((x + y) + z)' \quad \text{by associativity}
\]
\[
= (x + y)' + z' \quad \text{by De Morgan’s (6a)}
\]
\[
= (x' + y')z' \quad \text{by De Morgan’s (6a)}
\]
\[
= x'y'z' \quad \text{by associativity}
\]

3.4 (Boolean functions) Derive truth tables for the following Boolean functions:

(a) \(F(x, y, z, w) = xz + yw + xz'\)
(b) \(F(x, y, z, w) = x'y'z + x'z'w' + xzw' + xy'w\)
(c) \(F(x, y, z) = (x + z)'\)
(d) \(F(x, y, z) = (x + z)'(x + y')\)
3.5 (Boolean functions) Derive the complements of the functions in Problem 3.4, using De Morgan’s Law.

(a) $(xz + yw + xz')' = (xz)'(yw)'(xz')' = (x' + z')(y' + w')(x' + z)$

(b) $(x + y + z')(x + z + w)(x' + z' + w)(x' + y + w')$

3.6 (Boolean algebra) Prove by algebraic manipulation that the following expressions are equivalent:

$z'x + xy = z'x + xy + xy$  \hspace{1cm} \text{indempotency}$
$= z'x(y + y') + xy(x + z') + xy$  \hspace{1cm} \text{identity}$
$= z'x + x'y'z + xz + xz' + xy$  \hspace{1cm} \text{distributivity}$
$= x'y'z + (x + x')yz + xy$  \hspace{1cm} \text{distributivity}$
$= x'y'z + yz + xy$  \hspace{1cm} \text{identity}$

3.7 (Canonical forms) Derive the sum-of-minterms and the product-of-maxterms canonical forms for the following Boolean functions:

(a) $F = x \oplus y \oplus z$
(b) $F = zw' + xyw' + xy'$

$z \oplus y \oplus z = (xy' + x'z')z' + (xy' + x'z'y)z$
$= xy'z' + x'z'z + ((x' + y)(x + y'))z$
$= xy'z' + x'z'z + z'y'z + xyz$

(a) $F(x, y, z) = \Sigma(1, 2, 4, 7)$
(b) $F(x, y, z) = \Pi(0, 3, 5, 6)$

$= (x + y + z)(z' + y' + z)(x' + y' + z')(x + y' + z')$
3.9 (Standard forms) For the Boolean functions specified in Problem 3.6, derive the sum-of-products and the product-of-sums standard forms, using the minimal number of operators.

(a) \( F = x \oplus y \oplus z \)
(b) \( F = zw' + zy'w' + xy'z \)

\[
F = x \oplus y \oplus z \\
\text{(a)} = x'y'z' + x'yz + xy'z + xyz' \text{ min. sum-of-products} \\
\text{(b)} = (z + y + z)(x' + y' + z)(x' + y + z')(x + y' + z') \text{ min. product-of-sums}
\]

3.10 (Algebraic manipulation) Minimize the number of operators in the following Boolean expressions:

(a) \( x'y' + xy + xy' \)
(b) \( (x + y)(x + y') \)
(c) \( x'y' + x'y + xz \)
(d) \( y'z' + x'y + x'z + yz' \)

(b) \( z \)
(c) \( z' + z \)

3.14 (Boolean implementations) Implement the XOR function by means of:

(a) NAND gates only
(b) NOR gates only
(c) AND, OR, and NOT gates

![XOR Circuit Diagram](diagram.png)