Data Structures in Java

Lecture 21: Introduction to NP-Completeness

12/9/2015

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Algorithms and Problem Solving

- Purpose of algorithms: find solutions to problems.
- Data Structures provide ways of organizing data such that problems can be solved more efficiently
  - Examples: Hashmaps provide constant time access by key, Heaps provide a cheap way to explore different possibilities in order…
- When confronted with a new problem, how do we:
  - **Get an idea of how difficult it is?**
  - Develop an algorithm to solve it?

Problem Difficulty

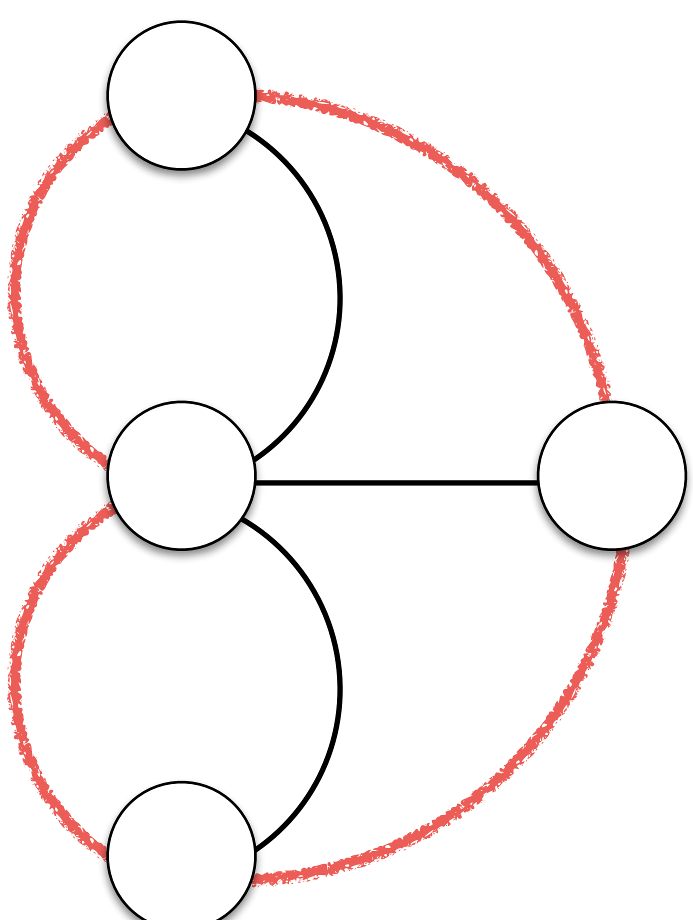
- We can think of the *difficulty of a problem* in terms of *the best algorithm we can find to solve the problem.*
  - Most problems we discussed so far have linear time solutions O(N), or slightly more than linear O(N log N).
  - We often considered anything worse than $O(N^2)$ to be a **bad** solution.
  - For some problems we don't know efficient algorithms.
- What is the best algorithm we can hope for, for a given problem?
  (for instance, $\Omega(N \log N)$ for comparison based sorting).

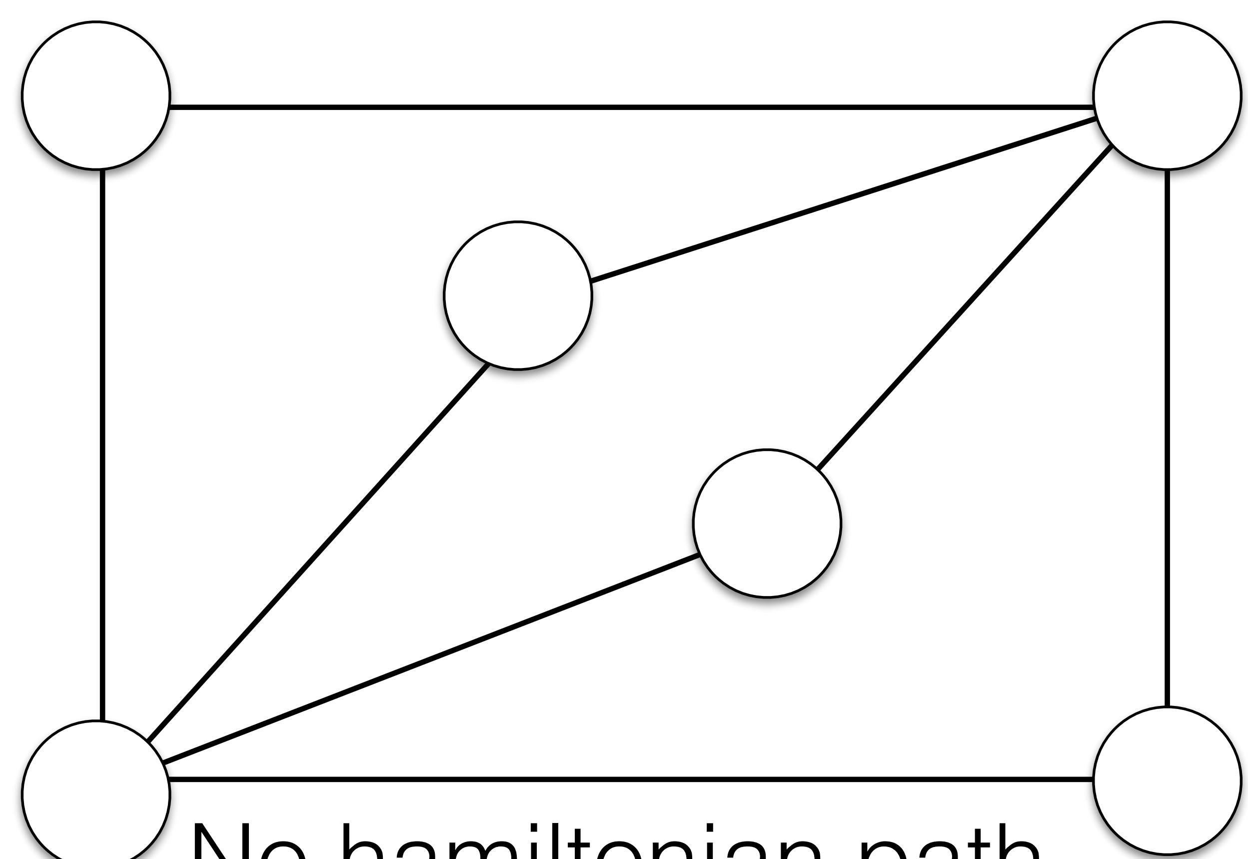
Polynomial and Exponential Time

- Two common classes of running time for algorithms:
  - Polynomial: $O(N^k)$ for some constant k.
  - Exponential: $O(2^{N^k})$ for some constant k

Hamiltonian Cycle

- A Hamiltonian Path is a path through an undirected graph that visits *every vertex* exactly once (except that the first and last vertex may be the same).
- A Hamiltonian Cycle is a Hamiltonian Path that starts and ends in the same node.

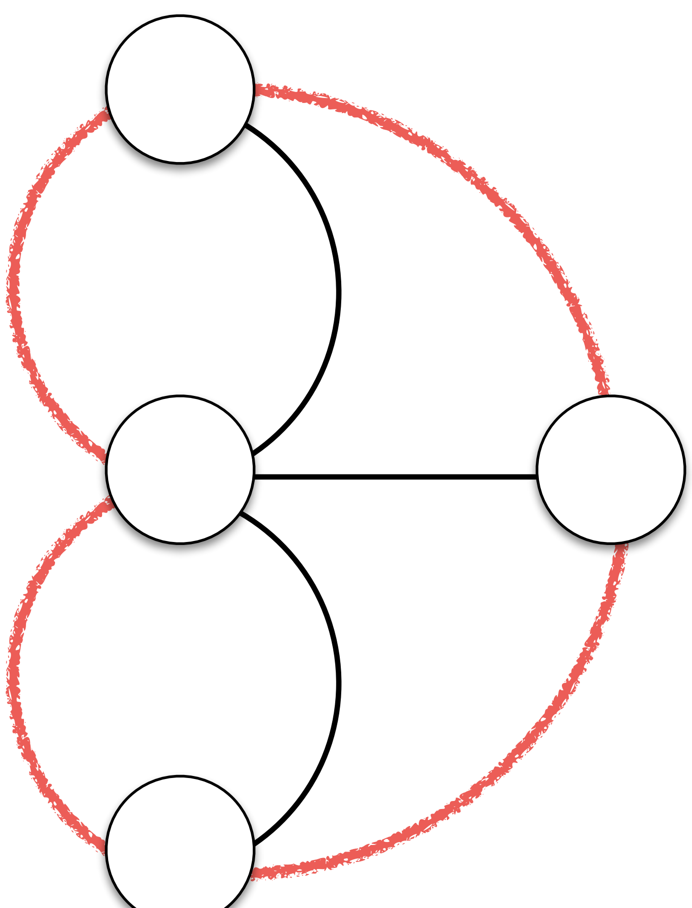


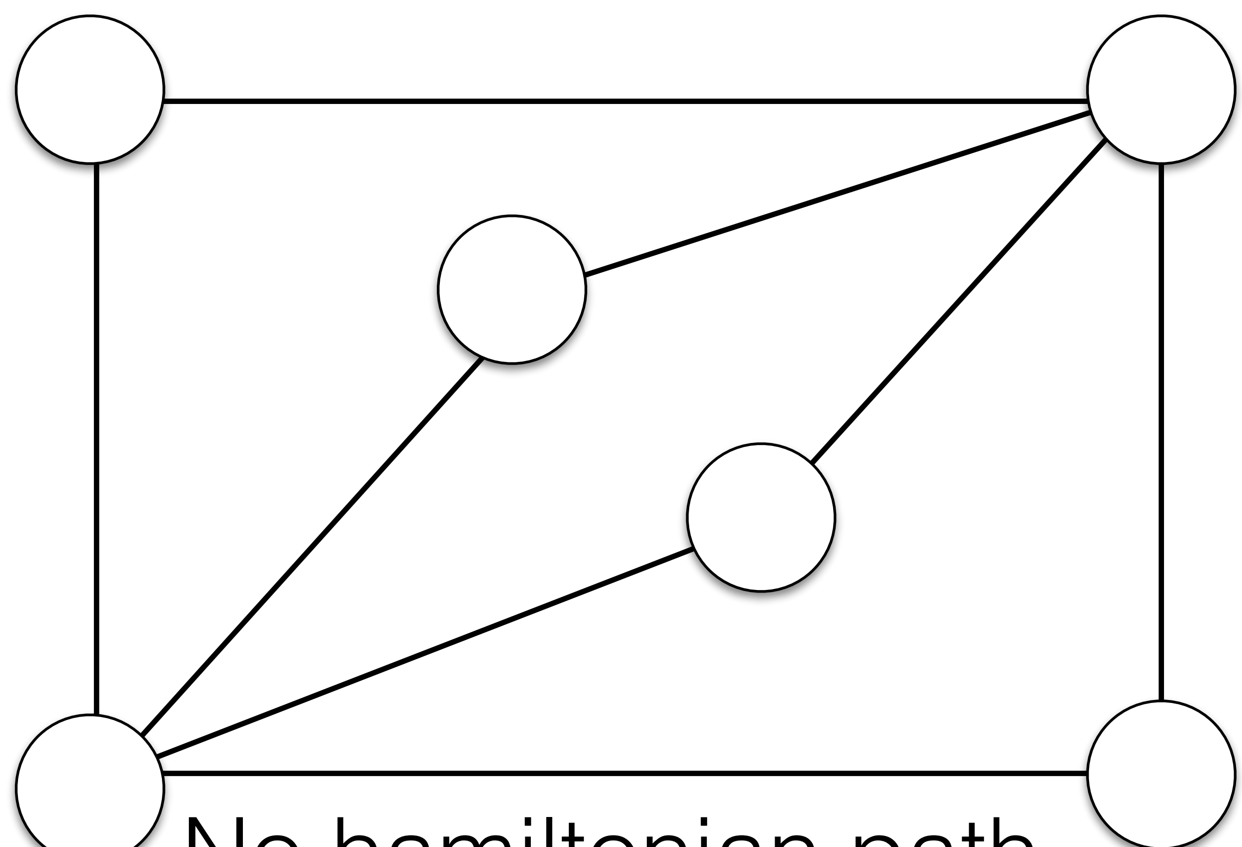


No hamiltonian path.

Hamiltonian Cycle

- Can check if a graph contains an Euler Cycle in linear time.
- Surprisingly, checking if a graph contains a Hamiltonian Path/Cycle is much harder!
- No polynomial time solution (i.e. $O(N^k)$) is known.

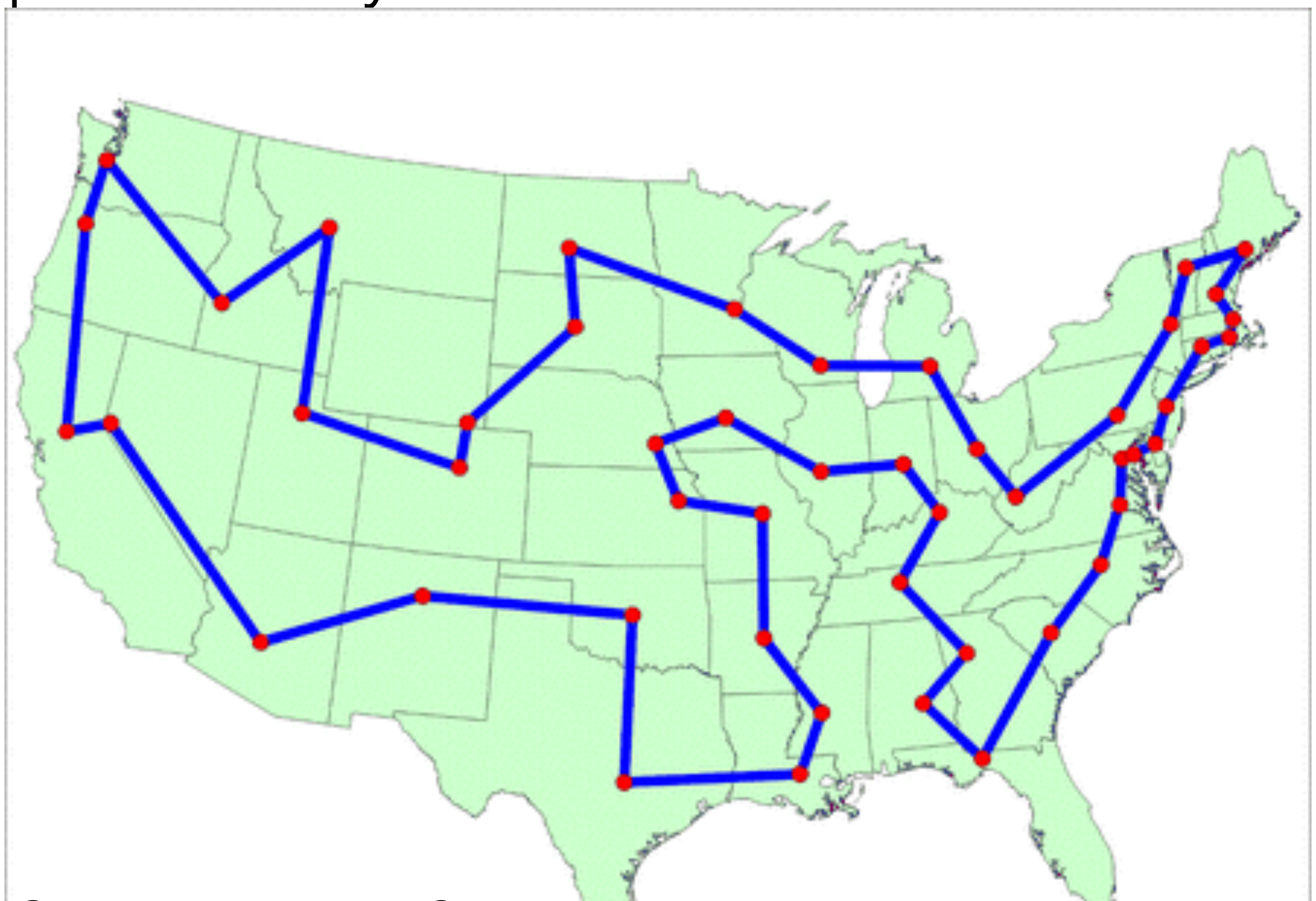




No hamiltonian path.

Traveling Salesman Problem (TSP)

Given a *complete*, undirected graph G = (V,E), find the shortest possible cycle that visits all vertices.



Optimal Traveling Salesman Tour through all 48 continental state capitals.

Source: SAP

TSP - How many tours are there?

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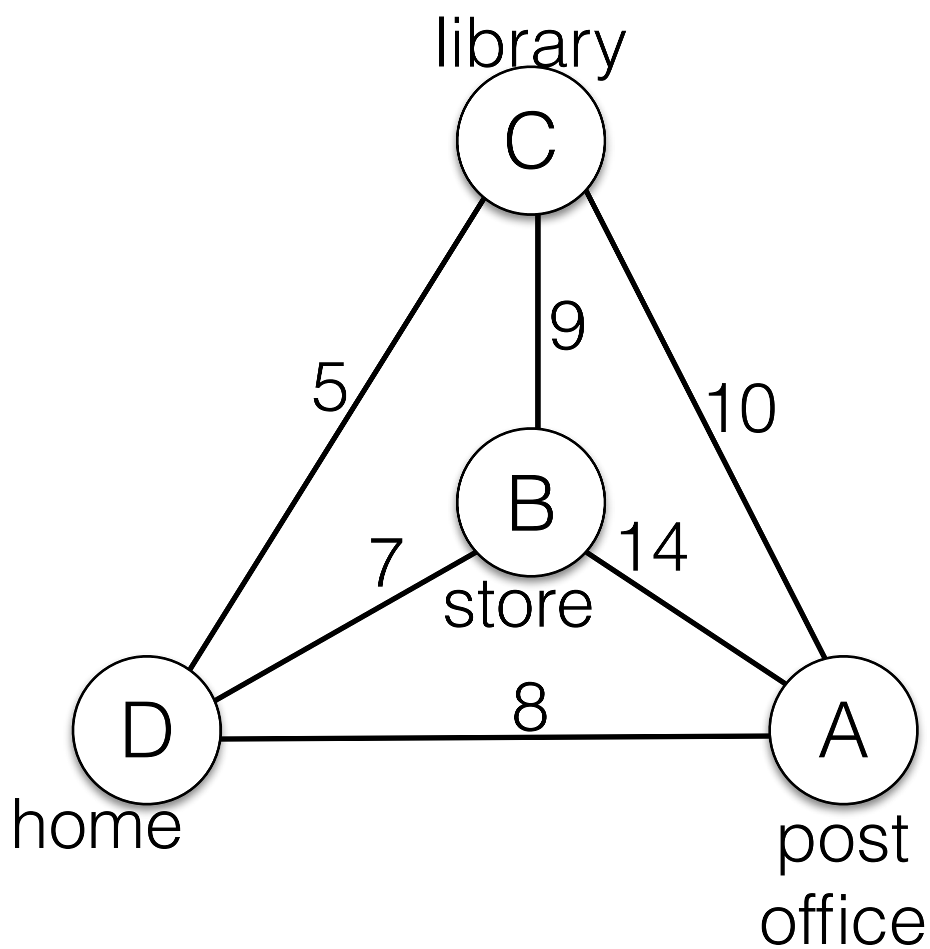

We can visit the vertices of the graph in ANY order.

How many possibilities are there?

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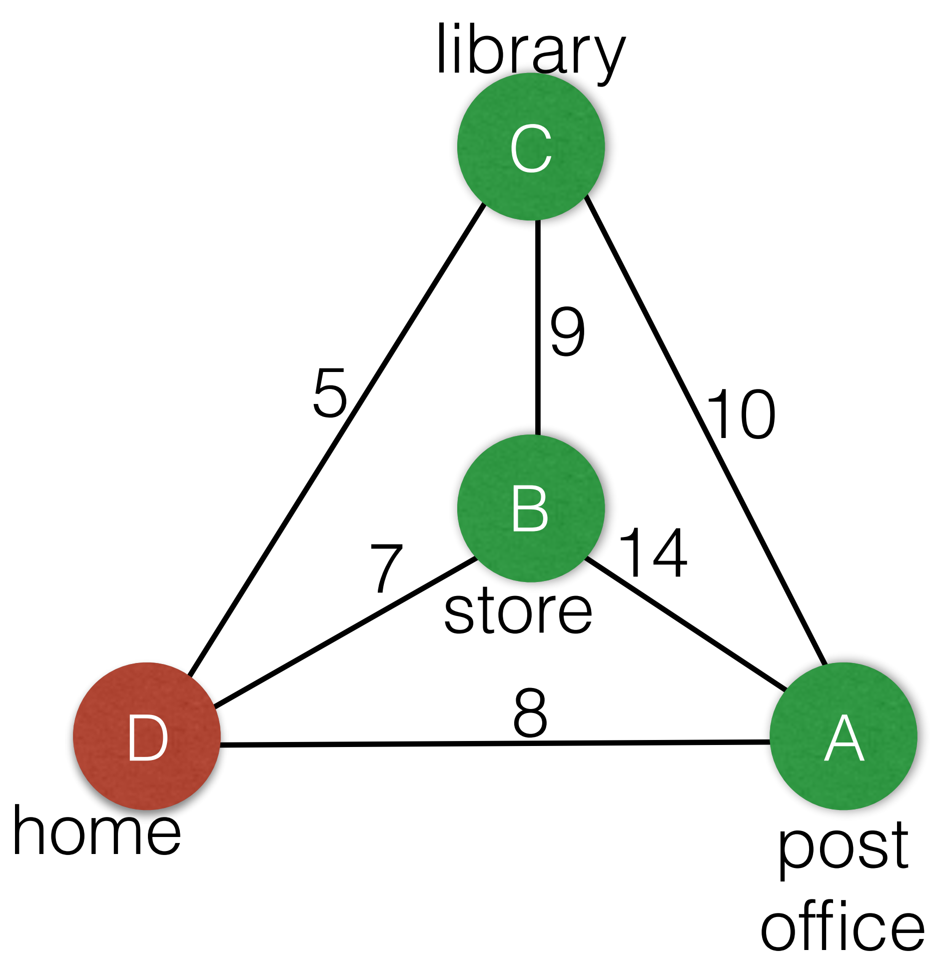

We start at D.
Because the graph is complete,
we can go to any of the other N-1 nodes.

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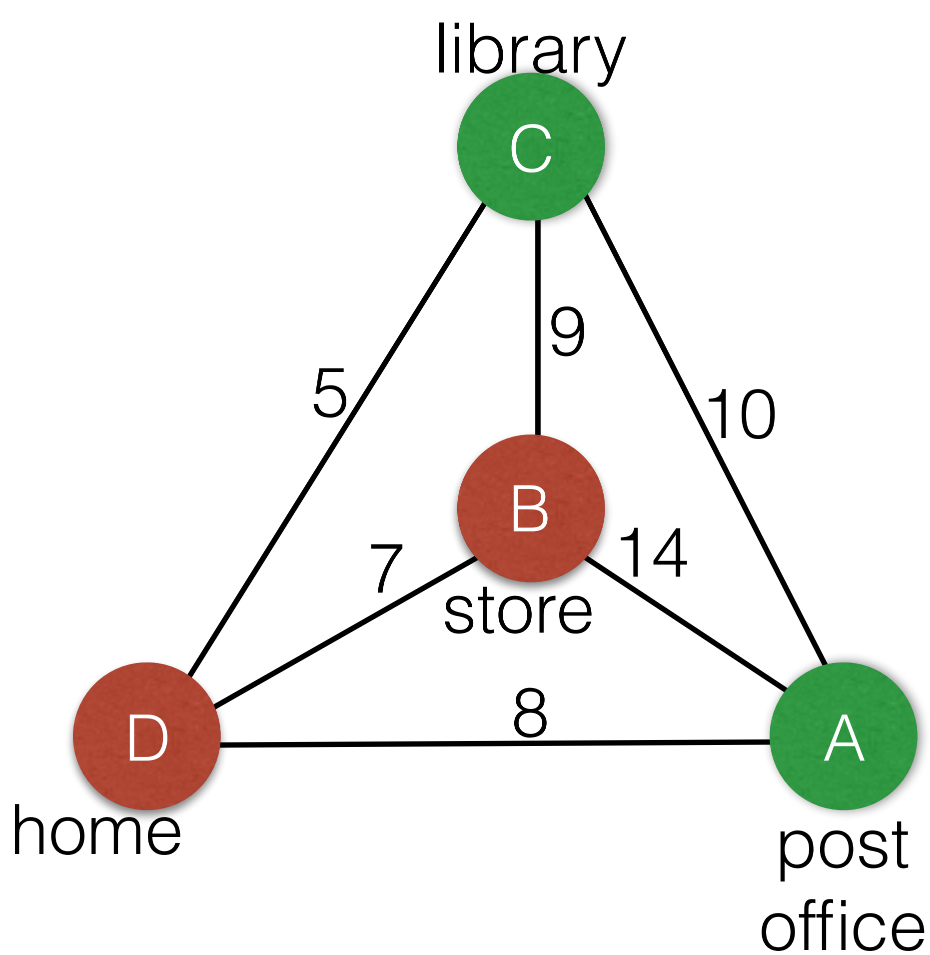

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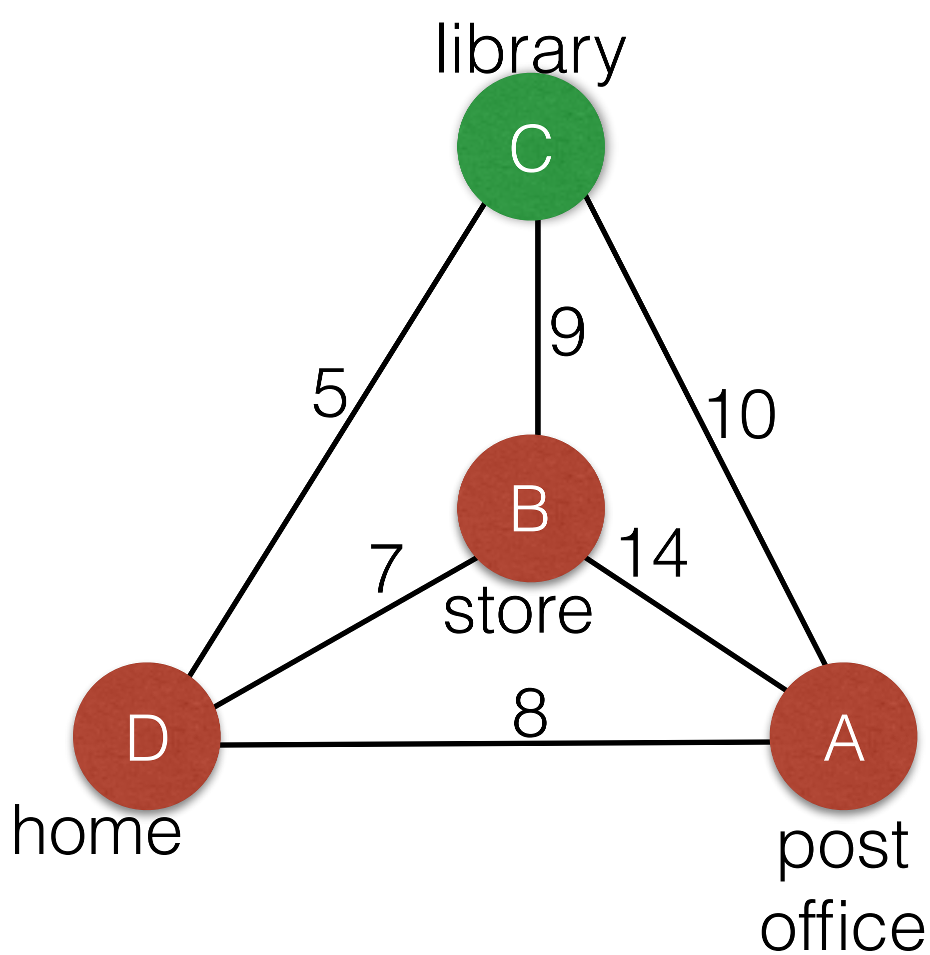

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Once we decide for a node, we can go to N-3 remaining nodes.

…

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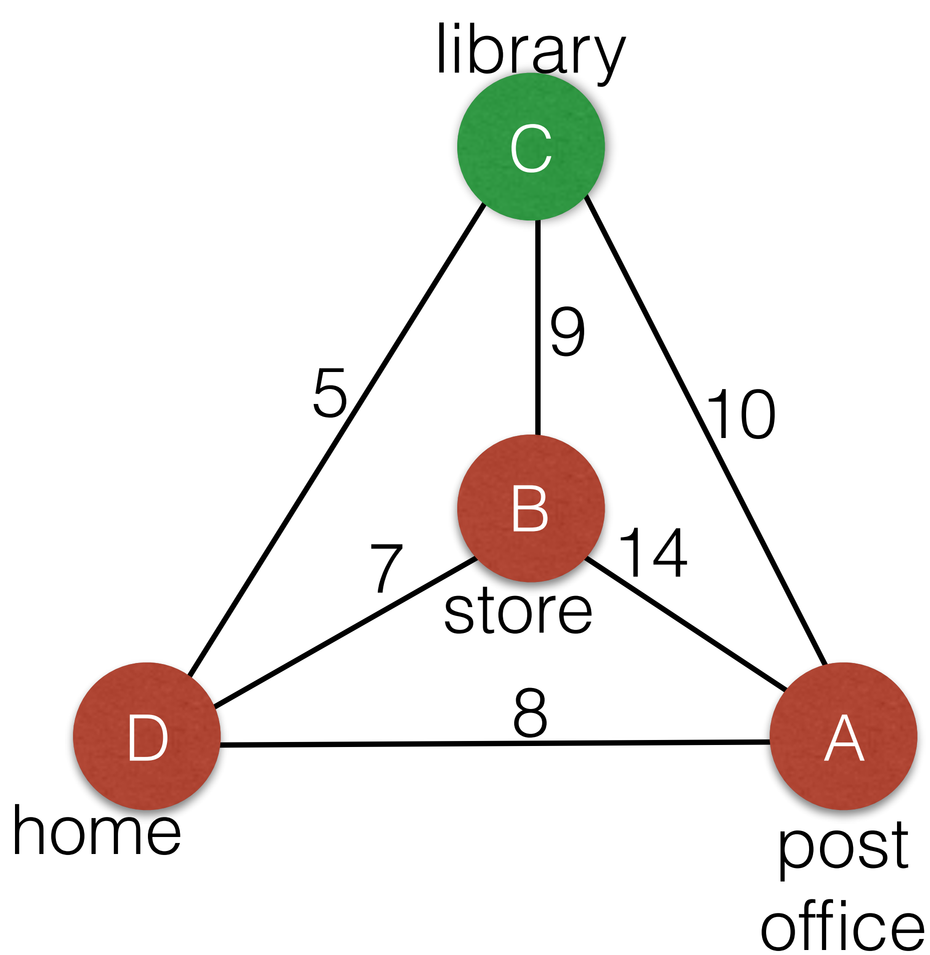

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… $(N-1) \cdot (N-2) \cdots (1) = (N-1)!$

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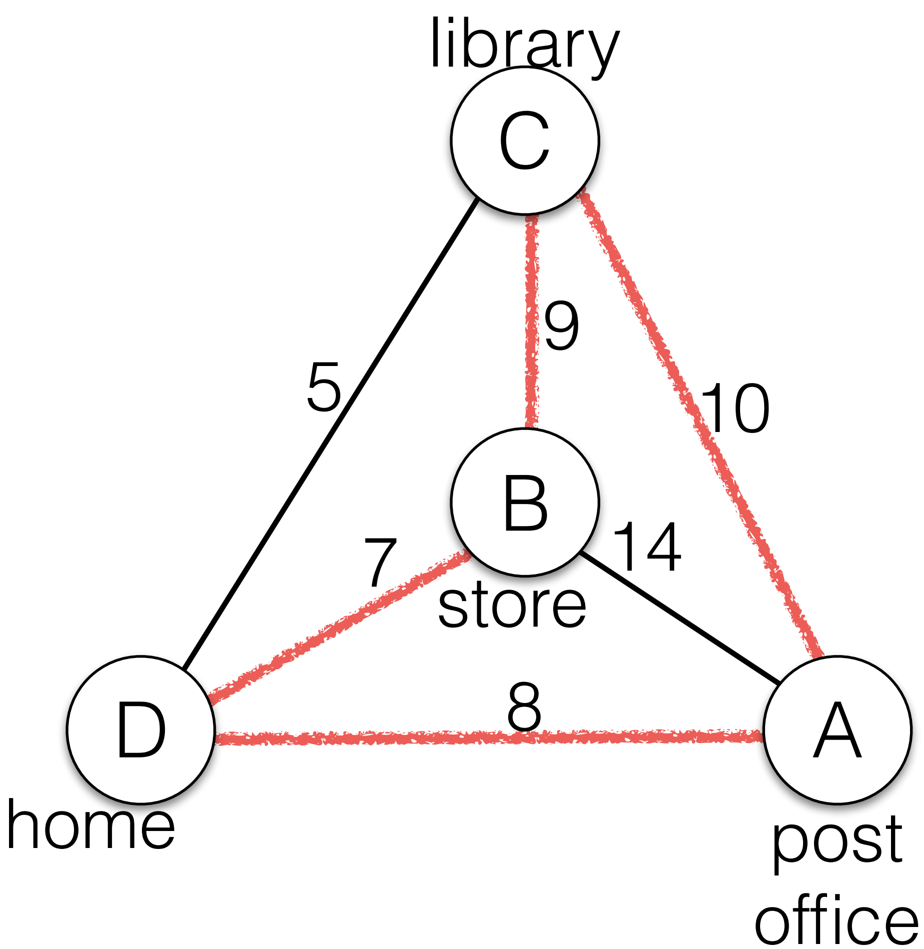

D A C B = D B C A

There are

$$(N-1)\cdot(N-2)\cdots(1) = (N-1)!$$

possibilities, but we can traverse complete tours in either direction.

There are $\dfrac{(N-1)!}{2}$ complete tours.

TSP - Brute Force Approach

Try all possible $\frac{(N-1)!}{2}$ tours and return the shortest one.

Obviously this algorithm runs in $O(N!)$

TSP - Brute Force Approach

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Better algorithm:

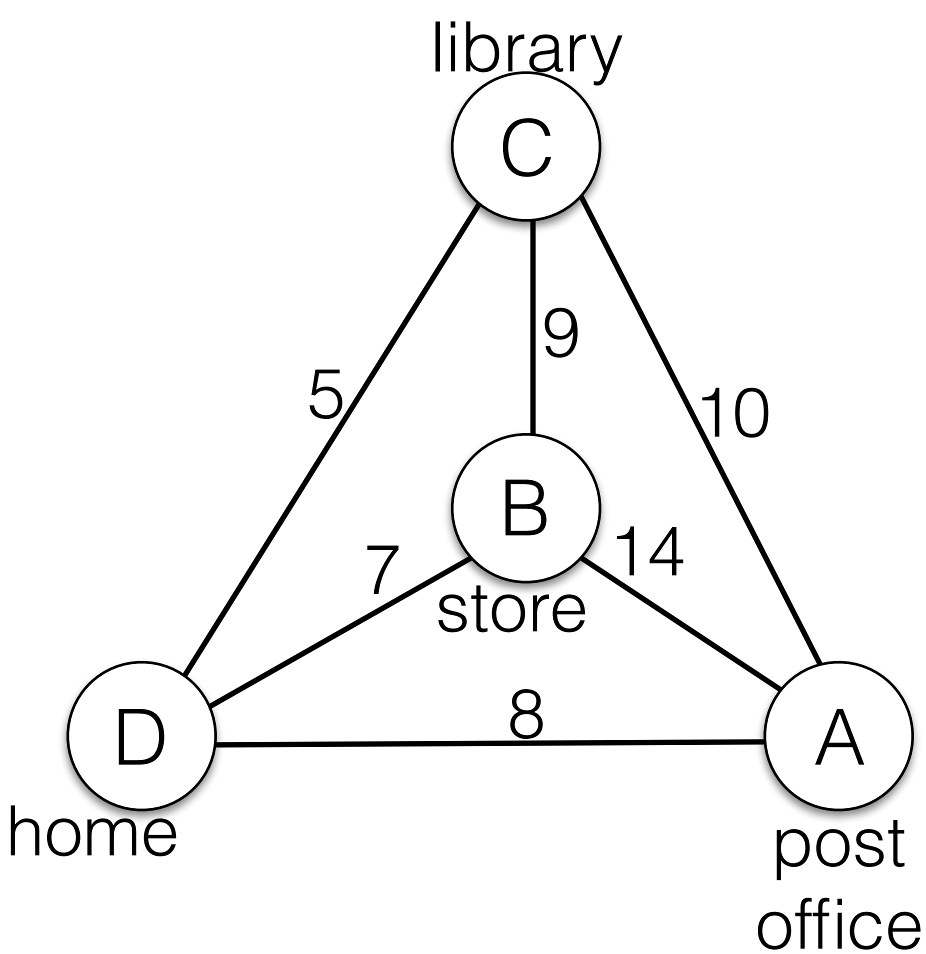
Dynamic Programming algorithm byHeld-Karp (1962)

$$O(2^N N^2)$$

No polynomial time algorithm is known!

TSP - Greedy Approximation

How about a greedy approximation?



Start with node D, always follow the lowest edge until all vertices have been visited.

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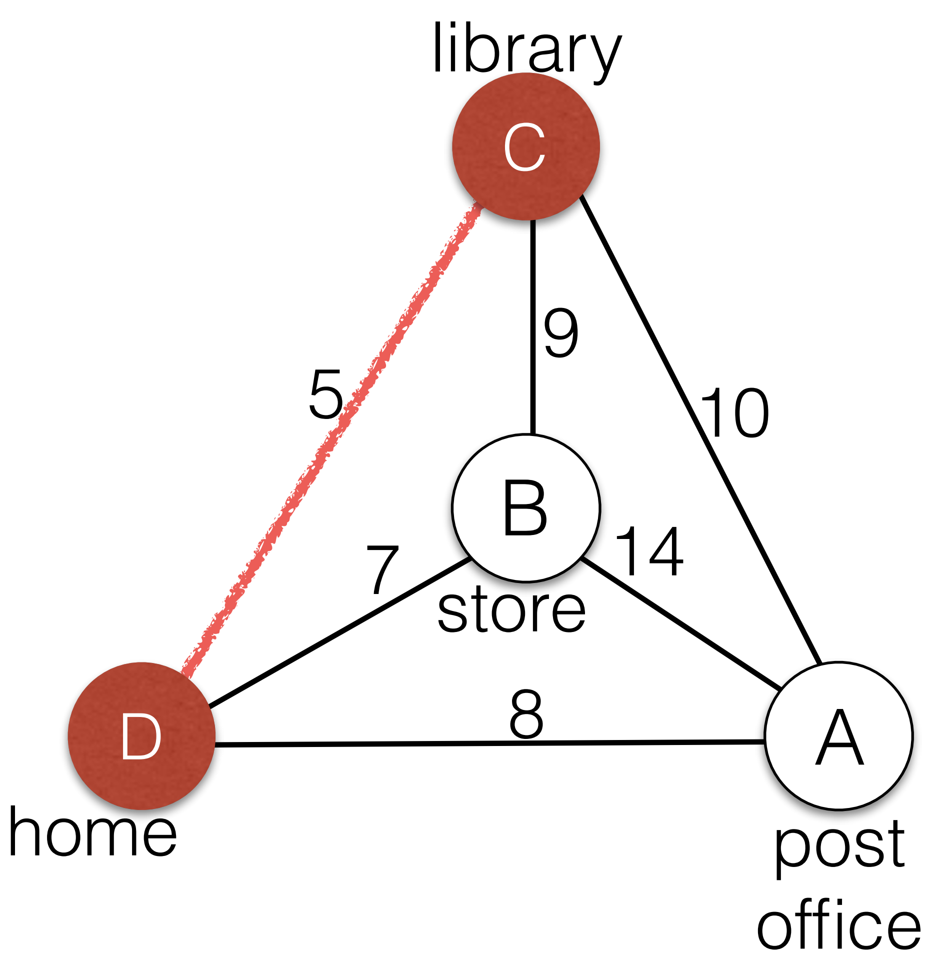

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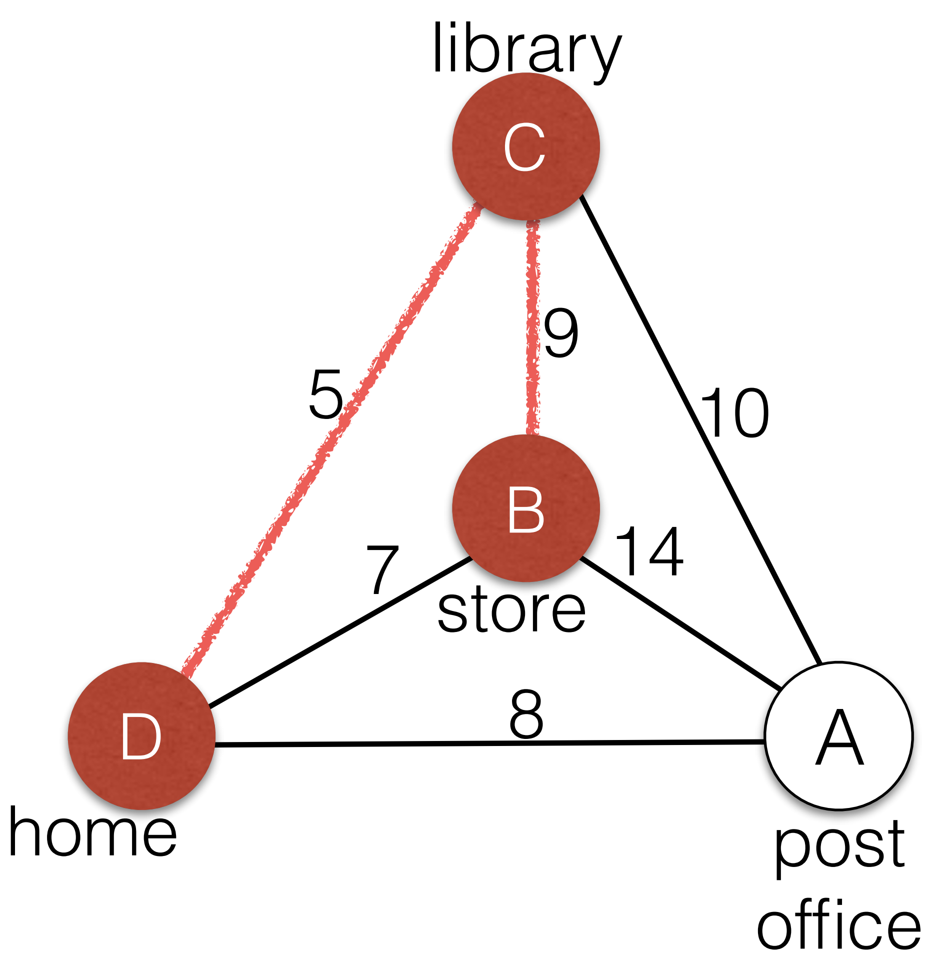

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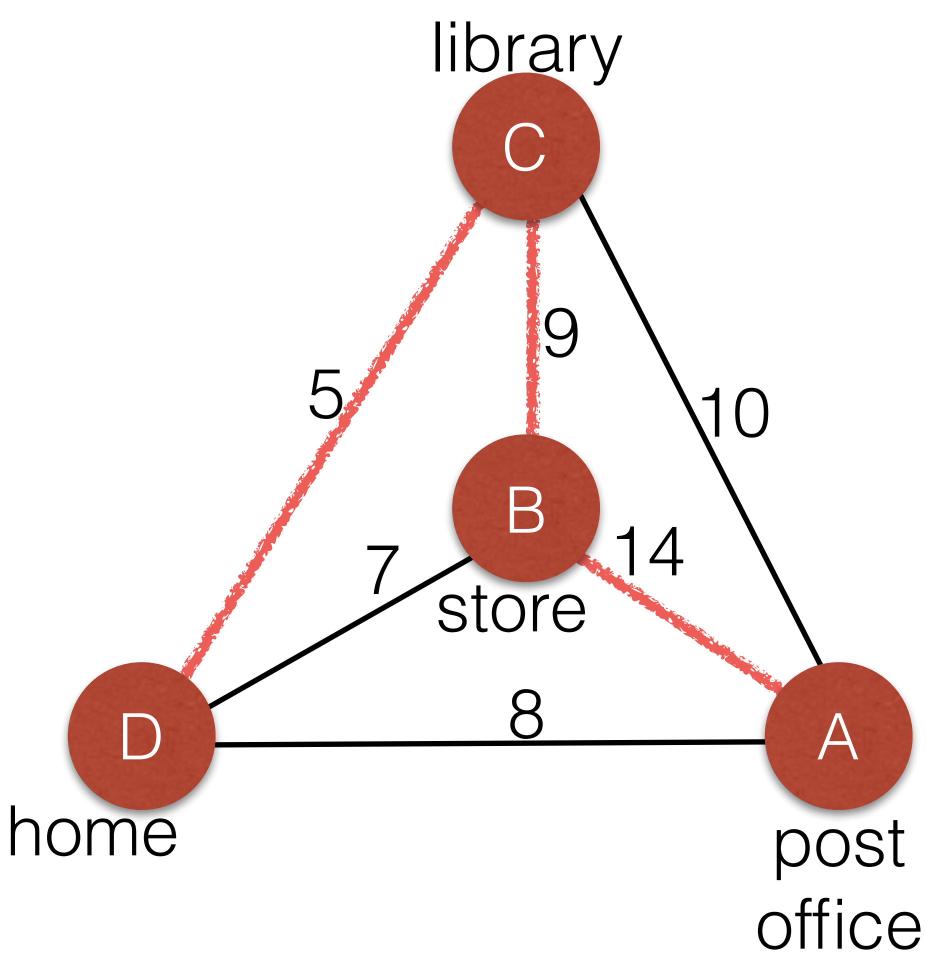

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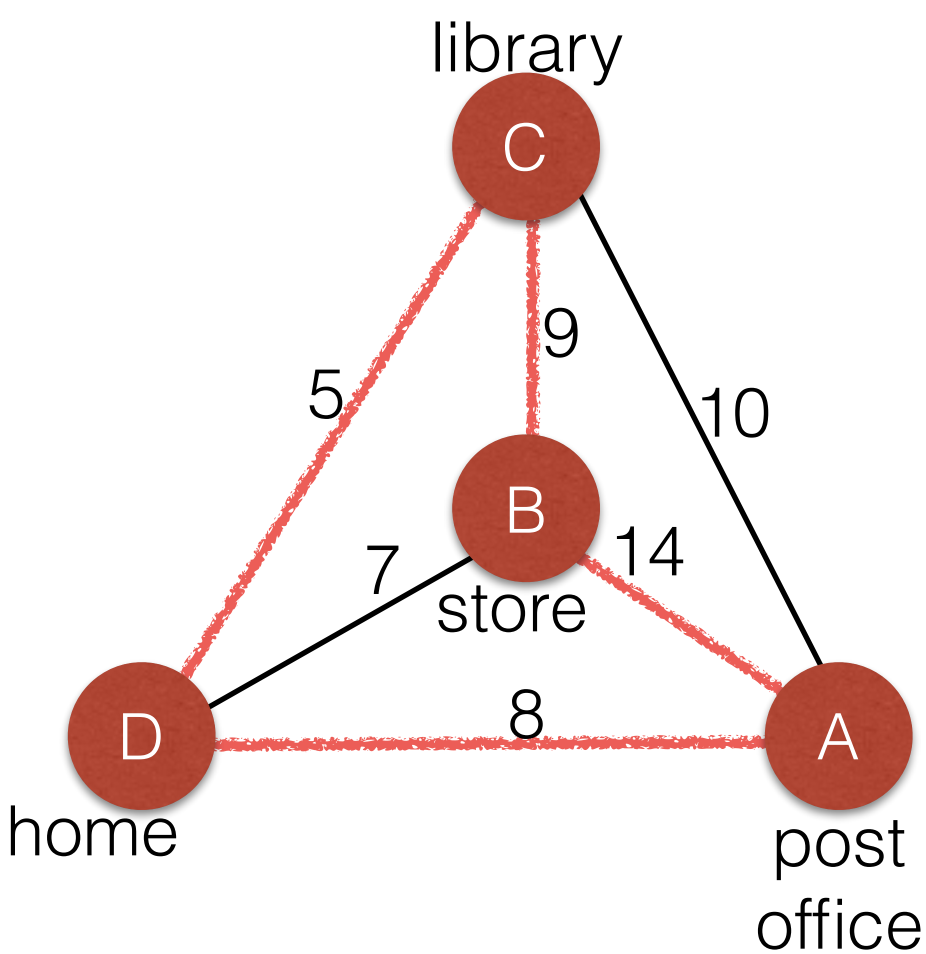

cost = 36

Start with node D, always follow the lowest edge until all vertices have been visited.

Unfortunately, this is not guaranteed to find an optimal solution.

TSP - Greedy Approximation

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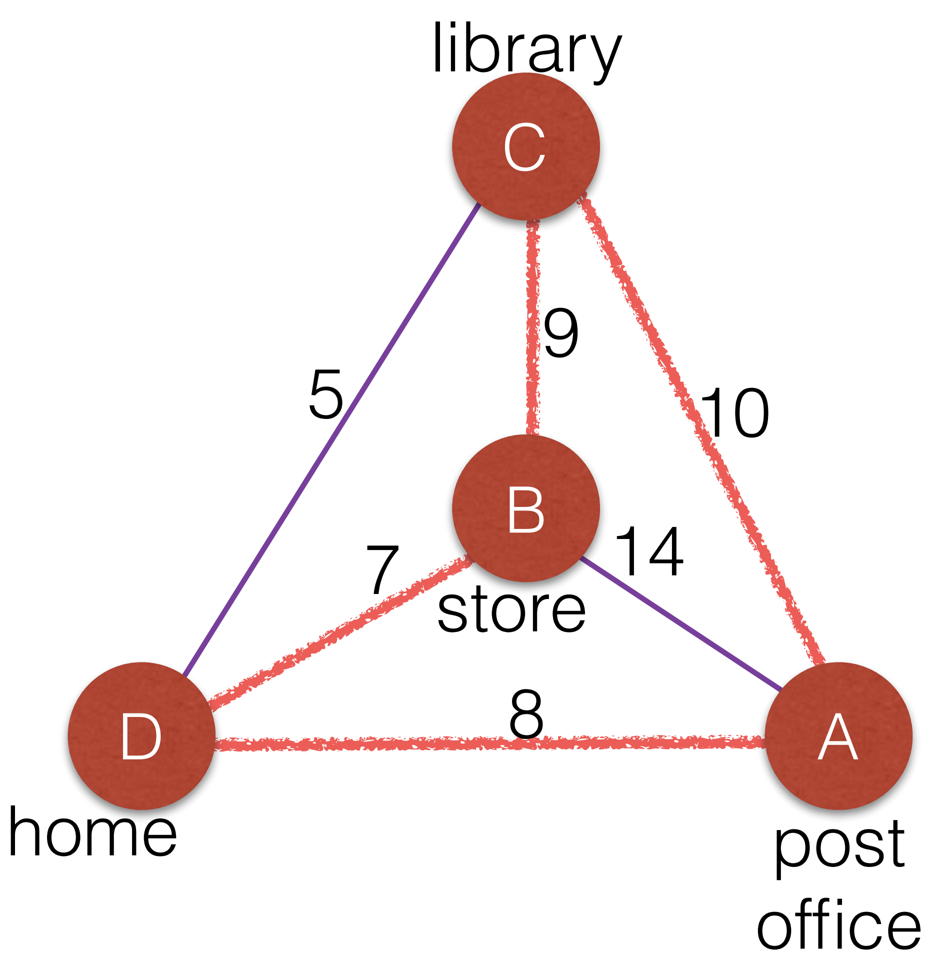


cost = 34

Start with node D, always follow the lowest edge until all vertices have been visited.

Unfortunately, this is not guaranteed to find an optimal solution.

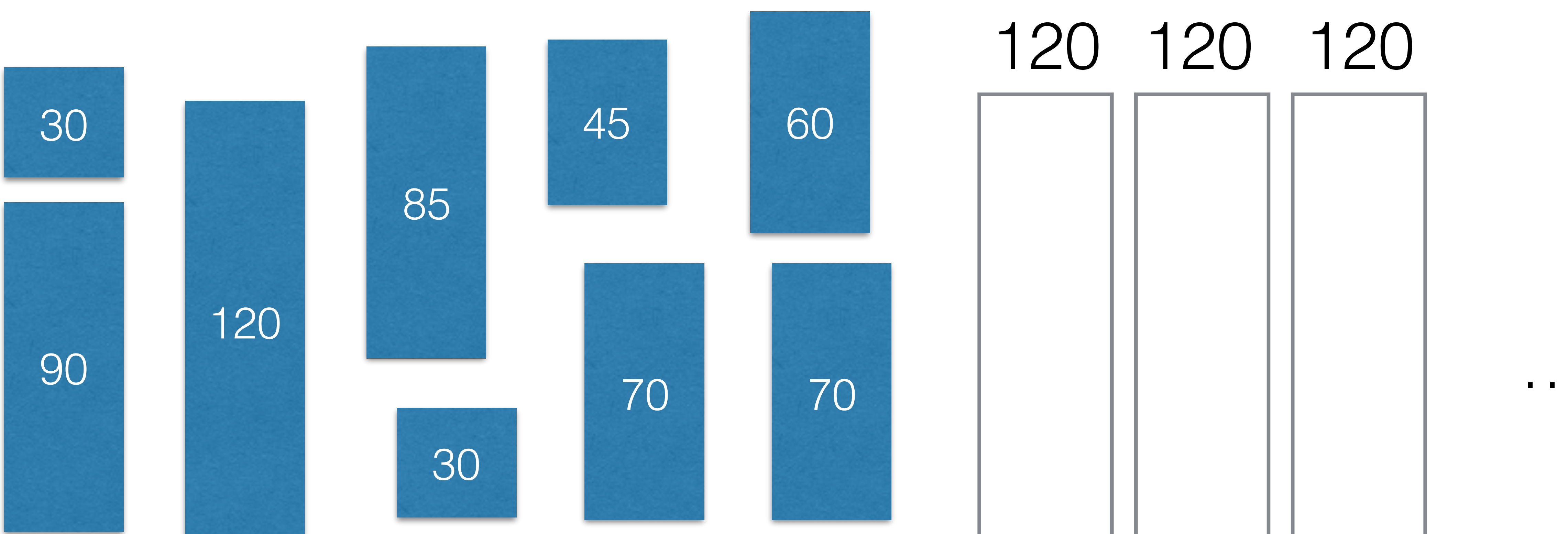
Combinatorial Optimization

- Many of the graph problems we discussed are *combinatorial optimization* problems.
  - Select the “best” structure from a set of output structures subject to some constraints.
  - Examples:
    - Select the minimum spanning tree from the set of all spanning trees.
    - Select the lowest-cost traveling salesman tour from the set of possible tours through a complete graph.

Bin Packing Problem

- You have:
  - N items of sizes $s_1, \ldots, s_N$
  - Any number of bins of fixed size V.

- Goal: pack the items into bins such that the number of bins needed is minimized. The sum of the item sizes in each bin must not exceed V.



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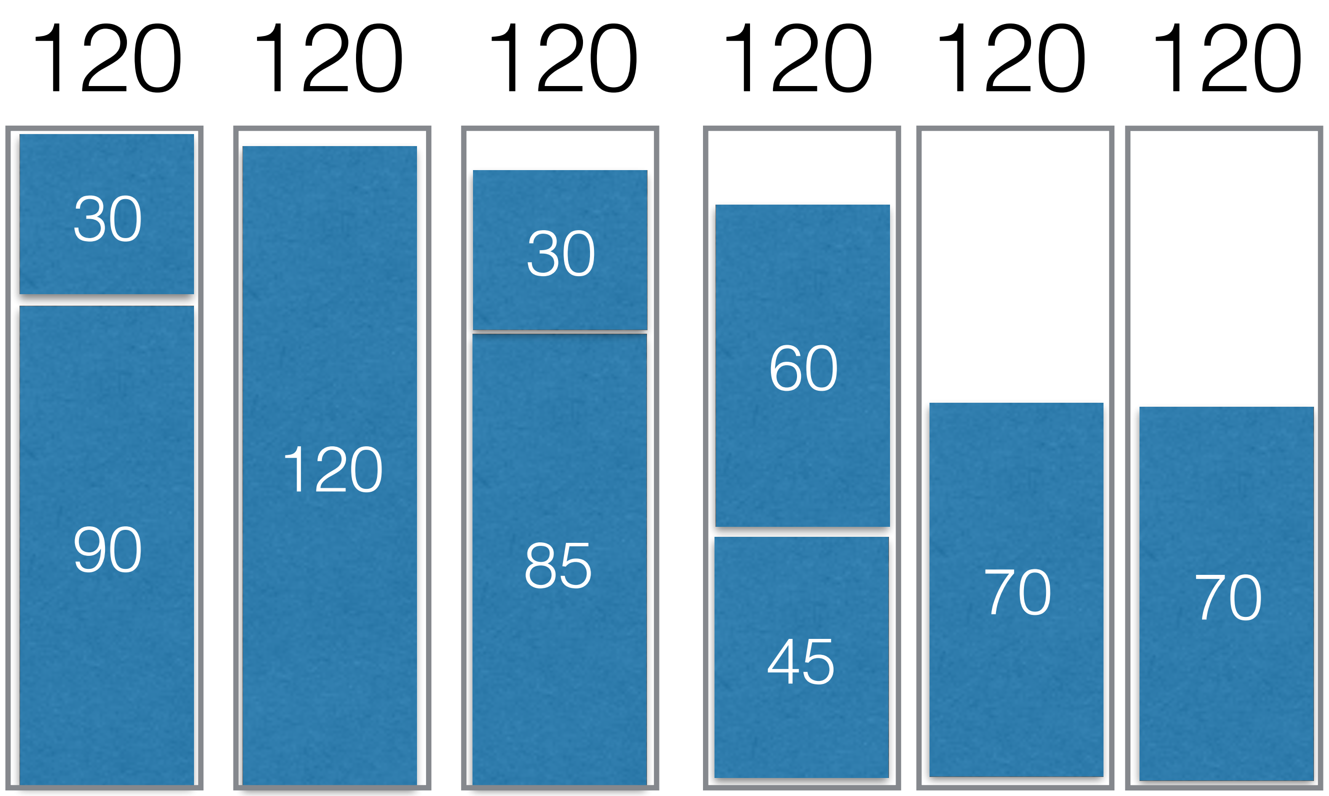
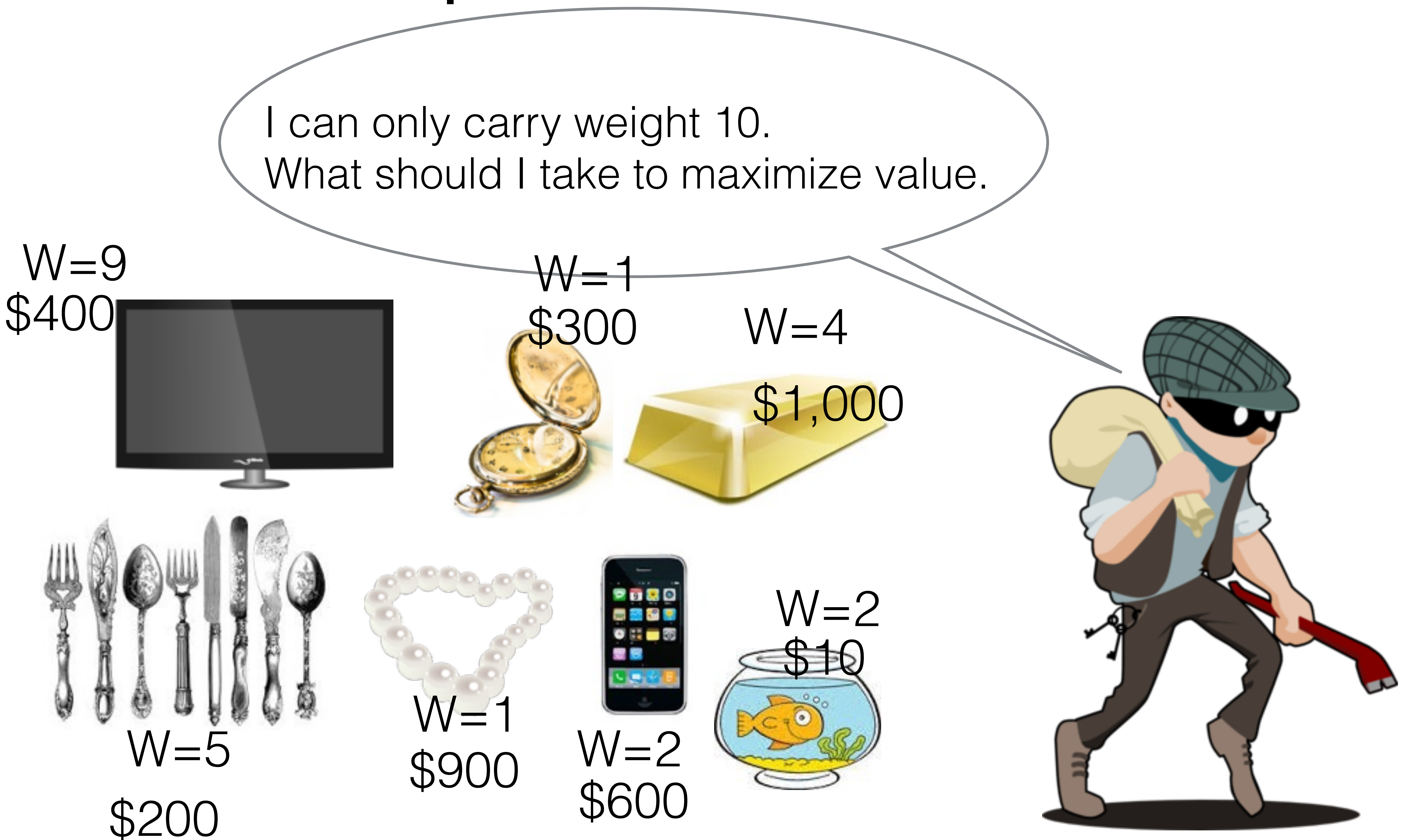


Knapsack Problem



Knapsack Problem

- Given N items, each with a value v_i and a weight w_i.
- Select a subset of the items to pack in a knapsack, such that
  - the total weight does not exceed some limit W
  - the sum of values is maximized.

Decision Problems

- A decision problem has, for each input, exactly two possible outcomes, YES or NO.

- “Does this Graph contain an Euler Circuit”
  “Does this Graph contain a Hamiltonian Cycle”

From Combinatorial Optimization to Decision Problems

- Any combinatorial optimization problem can be re-phrased as a decision problem by asking if a decision that is better than a certain threshold exists.

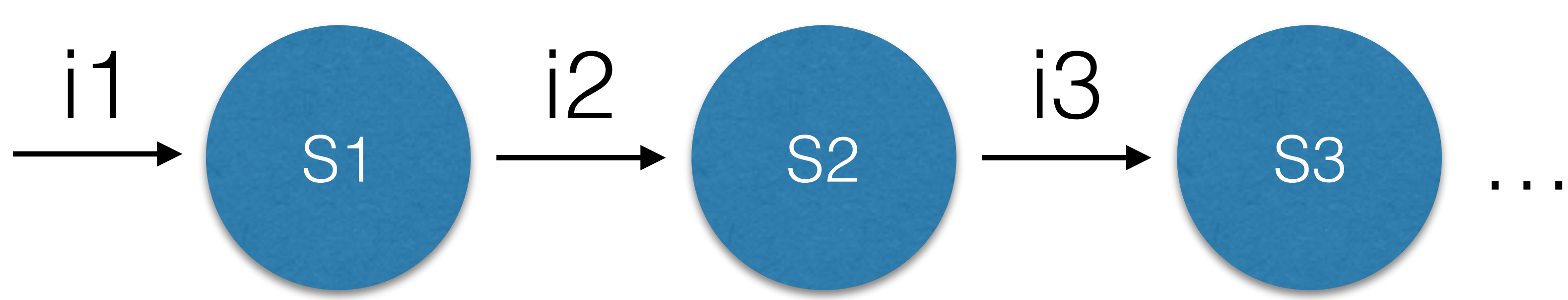
  - For instance, for TSP:
    "Is there a simple cycle that visits all vertices and has total cost $\leq K$"

- Observation:
  Solving the optimization problem is at least as hard as solving the decision problem.

Deterministic and Non-Deterministic Machines

- The “state” of a computation consists of all current data (input, memory, CPU registers,…) and the last program instruction.

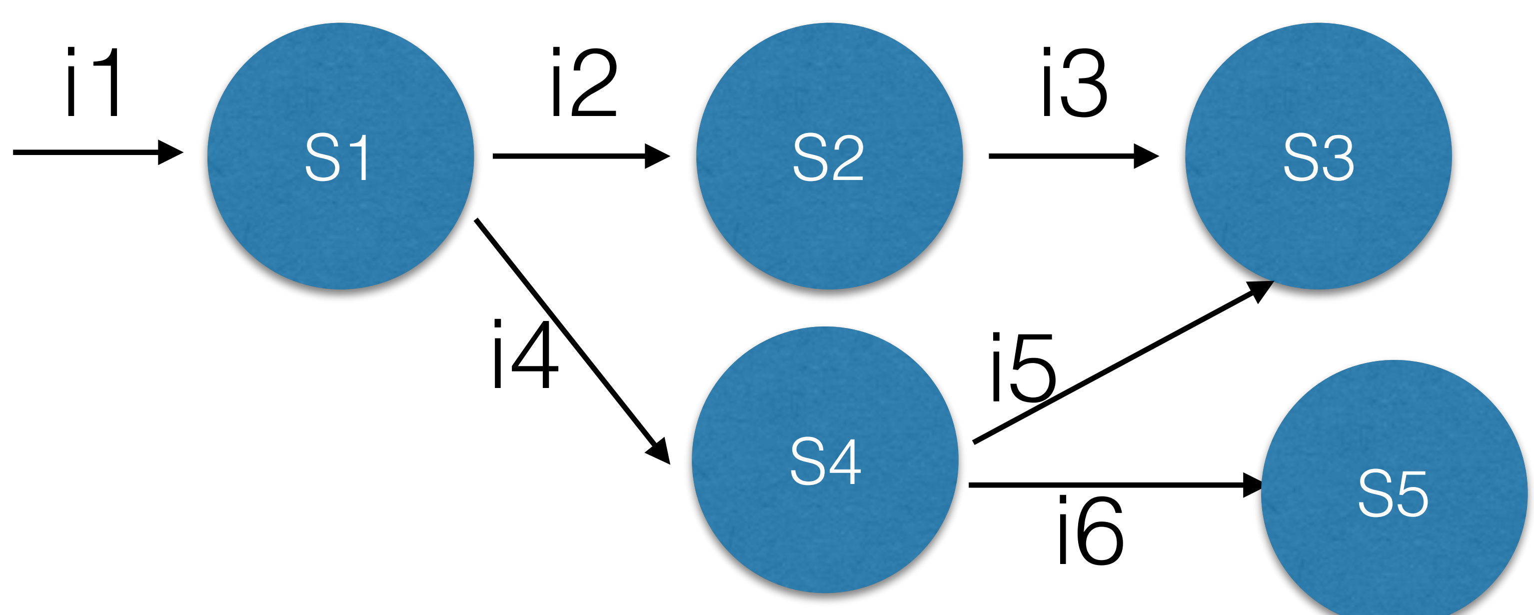
- Given any state, a deterministic machine goes to a unique next instruction.



Deterministic and Non-Deterministic Machines

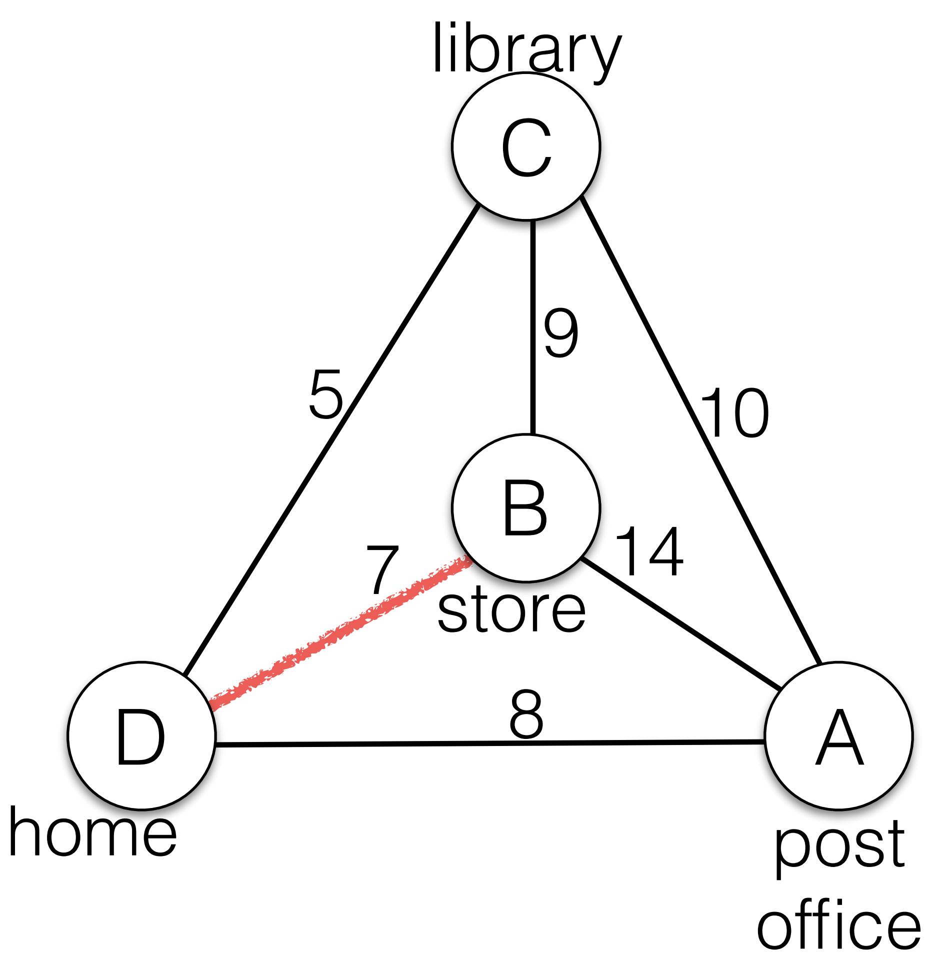
- A non-deterministic machine could be in ANY number of states at the same time.

- Equivalently, a non-deterministic machine contains an "oracle" that tells it the optimal instruction (of several multiple instructions) to execute in each state.



TSP with an Oracle

- State of the computation: Visited nodes, previous path.
- Same algorithm as greedy algorithm,
  but now the oracle tells us which edge to follow next.



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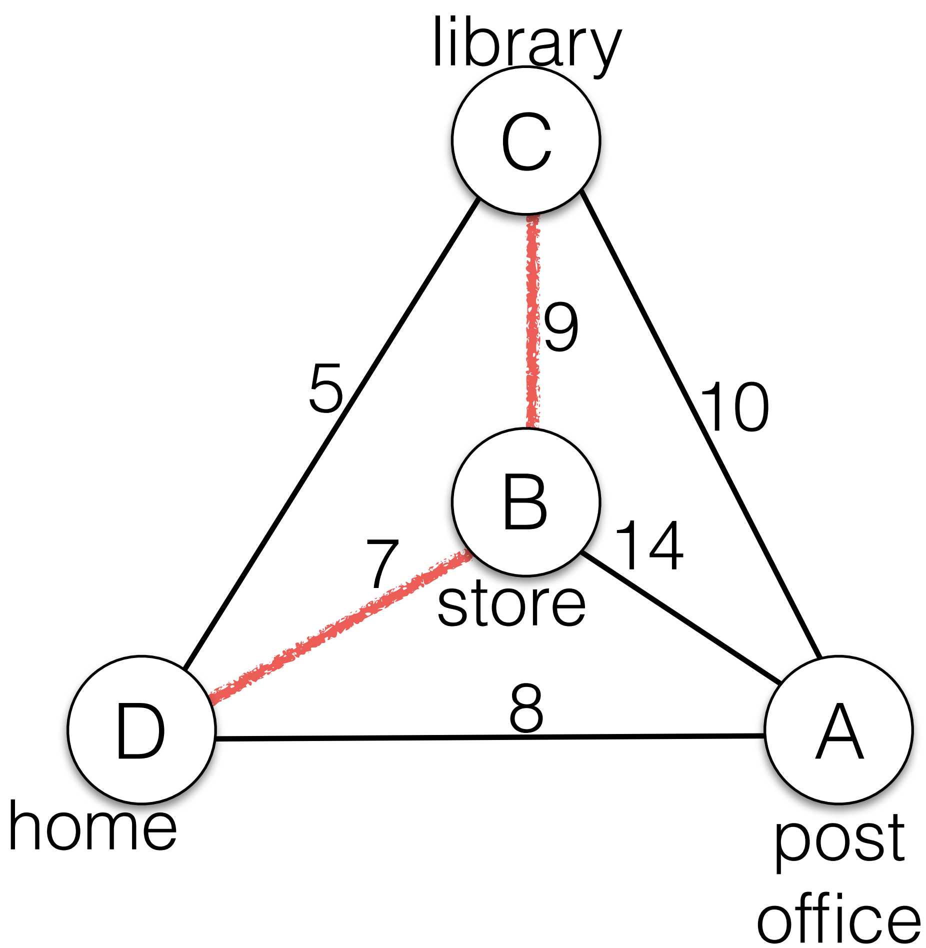

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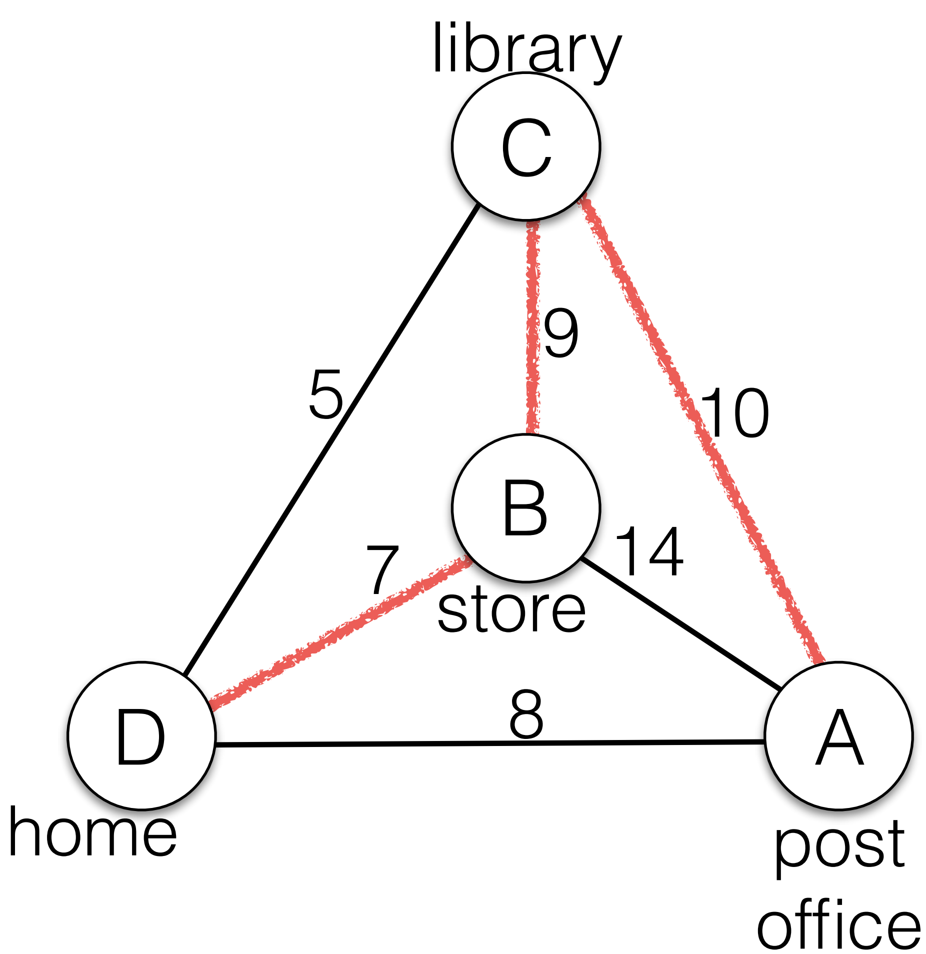

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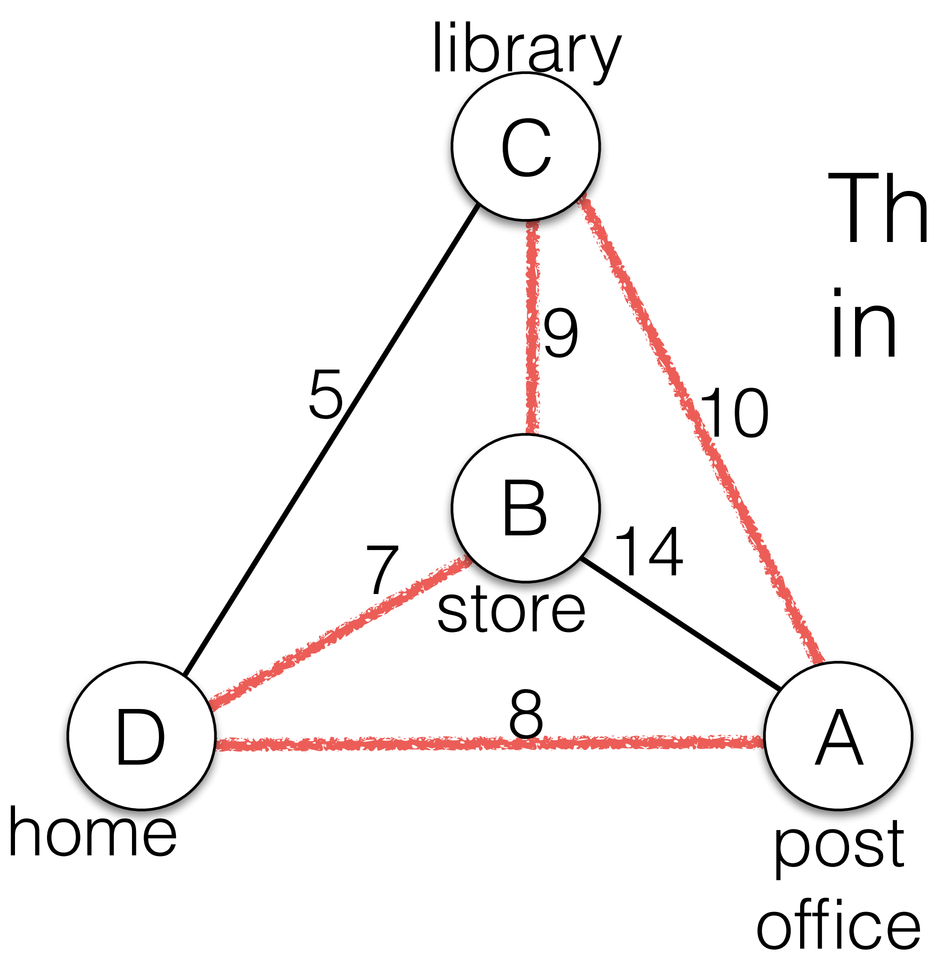


This algorithm produces an optimal tour in linear time!

Unfortunately, a real oracle is not realistic.
(But we can have a limited amount of parallelism).

The Class of NP Problems

- NP (*Nondeterministic Polynomial Time*) is the the class of problems for which a polynomial running time algorithm is known to exist on a non-deterministic machine.

- How do we know that a problem is in NP.

- Are there problems that are not in NP?

How Do We Know If a Problem Is in NP?

- Assume a decision problem produces YES on some input and some proof/“certificate” for this result.

- A decision problem is in NP if we can verify, in deterministic polynomial time, that the proof for a YES instance is correct.

- Examples:
  - An algorithm determines that a graph contains a Hamiltonian cycle and provides such a cycle as proof.
  - A spanning tree of cost $< K$ is proof that such a spanning tree exists in a graph.

Undecidable Problems

- Are there problems that are impossible to solve?

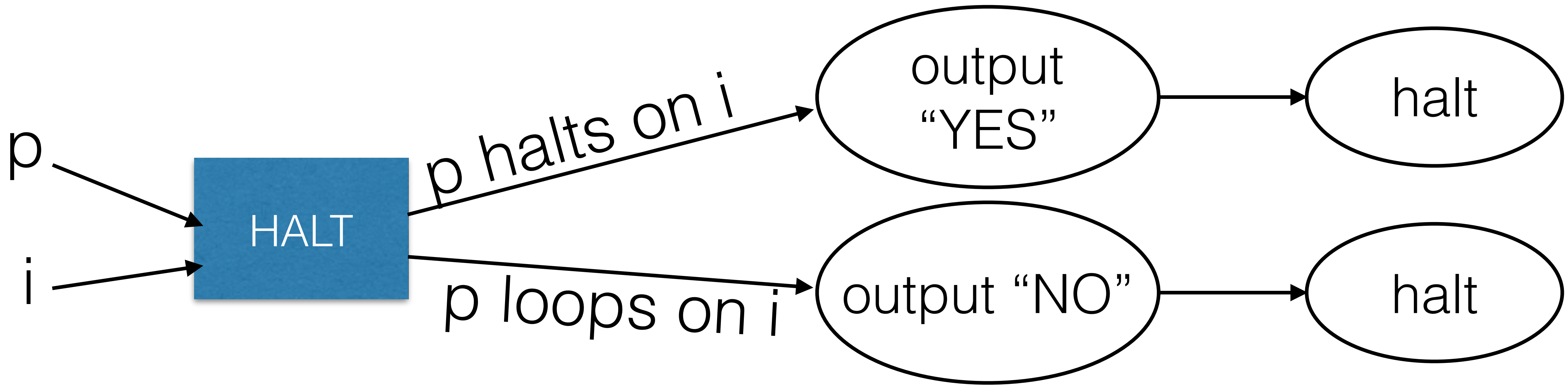
- Halting Problem:
  - Given a program description and some input, determine if the program will terminate (halt) or run forever (loop).

This problem is **recursively undecidable.**

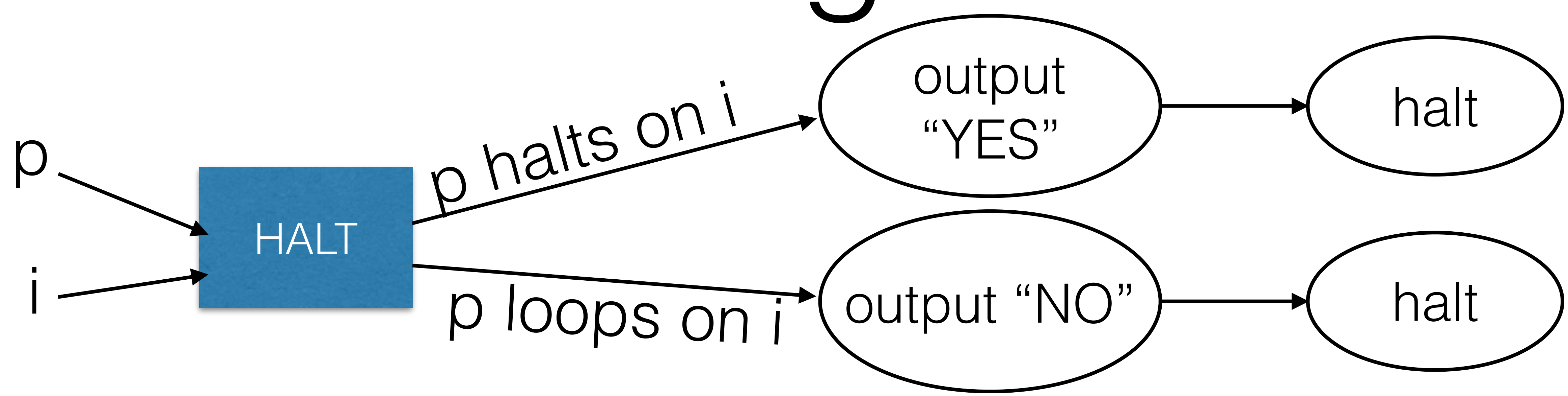
Turing 1936

The Halting Problem

- Assume we wrote a program called HALT(p,i)
  - HALT outputs "YES" and halts if *p* halts on *i.*
  - HALT outputs "NO" and halts if *p* loops on *i*.

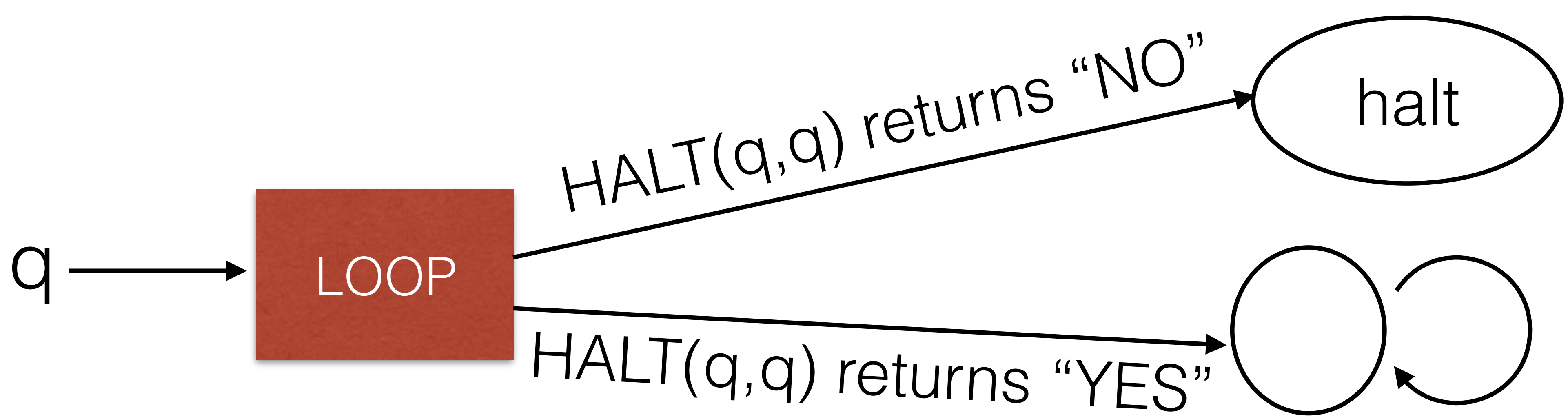


The Halting Problem

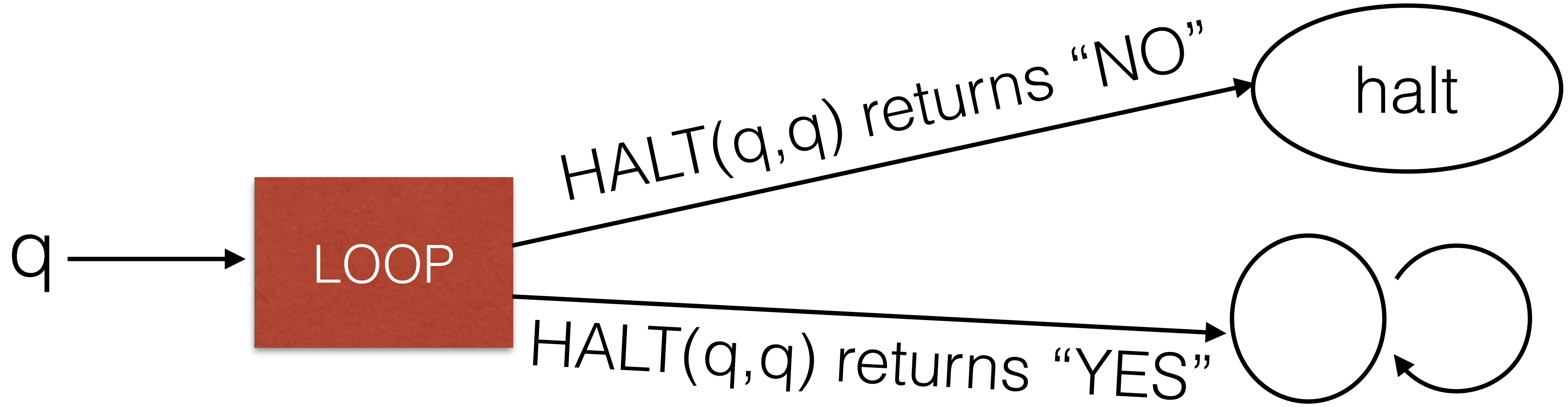


Write another program called LOOP(*q*)

- LOOP halts if HALT(q,q) returns "YES"
- LOOP loops if HALT(q,q) returns "NO"



The Halting Problem



What happens if we run LOOP(LOOP)?

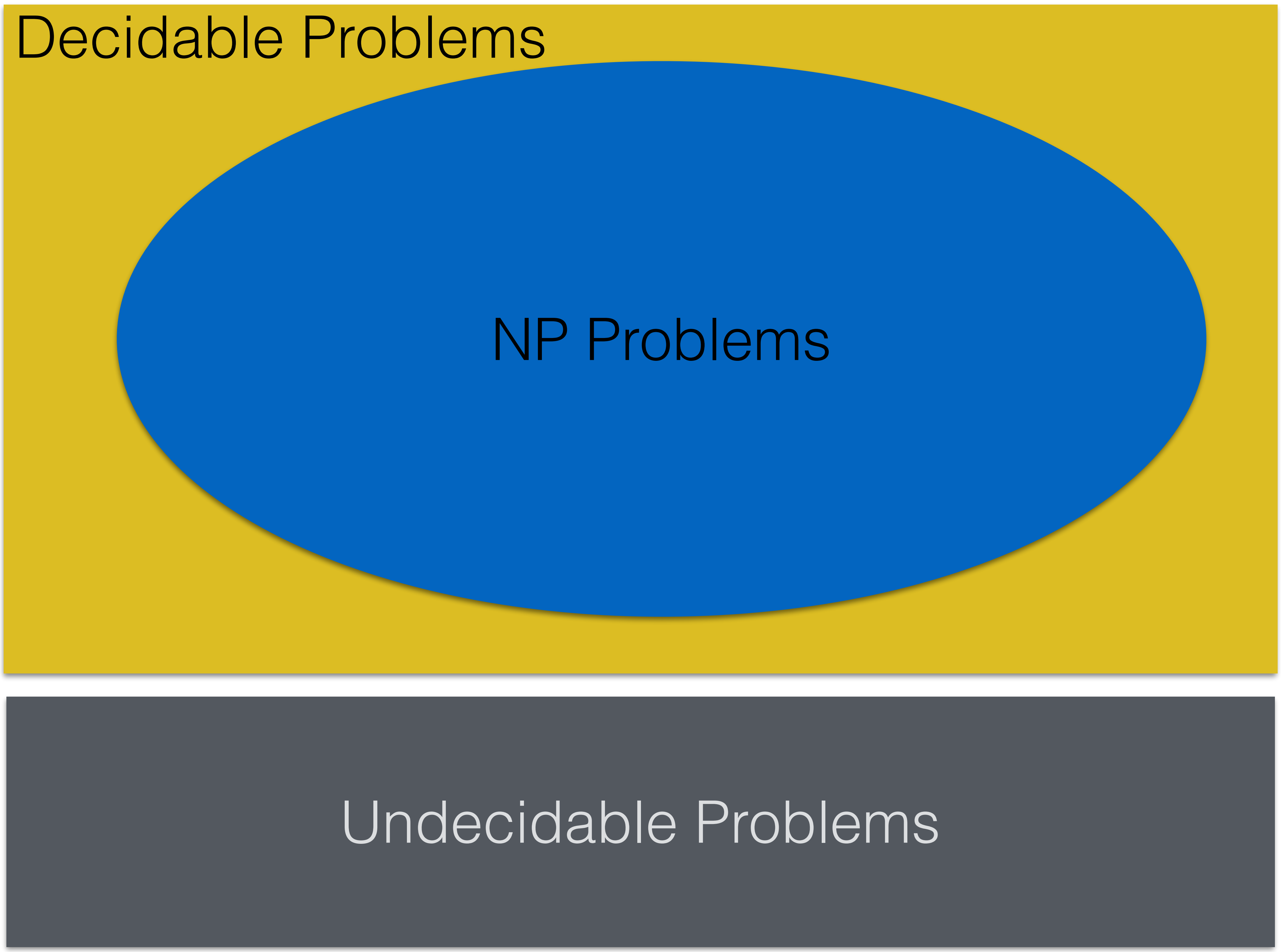
- Assume LOOP(LOOP) halts
    - Then HALT(LOOP,LOOP) must have returned "NO".

- Assume LOOP(LOOP) loops.
    - Then HALT(LOOP,LOOP) must have returned "YES".

A decidable problem that is (probably) not in NP

- Consider the problem of deciding if a graph does NOT have a hamiltonian cycle.
- No NP algorithm is known for this problem.
- Intuitively, a proof would require to list all possible cycles. Verifying the proof means to show that none of them is Hamiltonian, one by one.

NP Problems

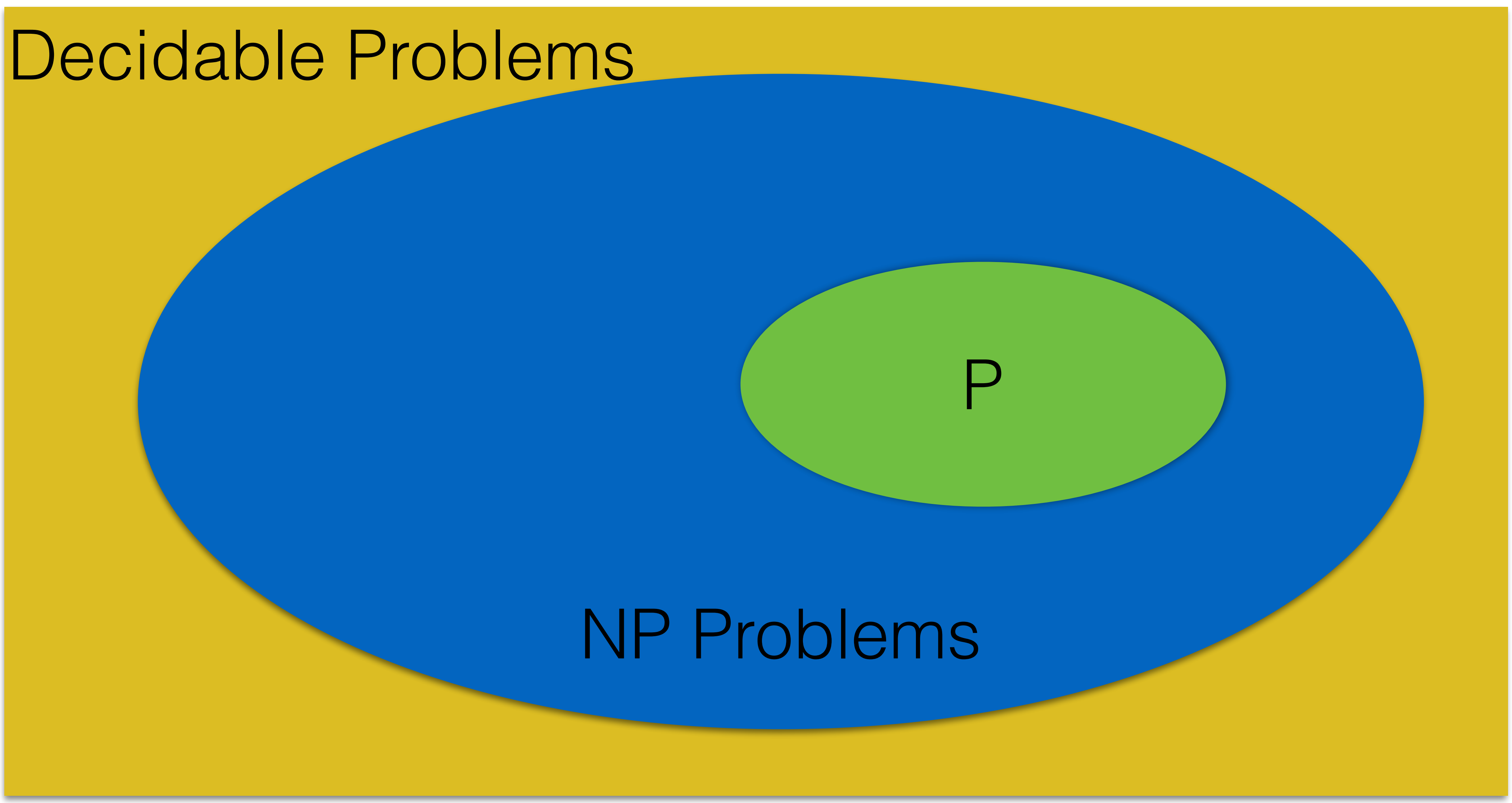


P and NP Problems

- The class P contains all problems that are solvable in polynomial time on a deterministic machine (most of the problems discussed in this course are in P).
- Clearly, all problems in P are also in NP.
- Surprisingly, it is **unknown** if there are problems in NP (i.e. with proofs that can be verified in polynomial time), that cannot be SOLVED in polynomial time.

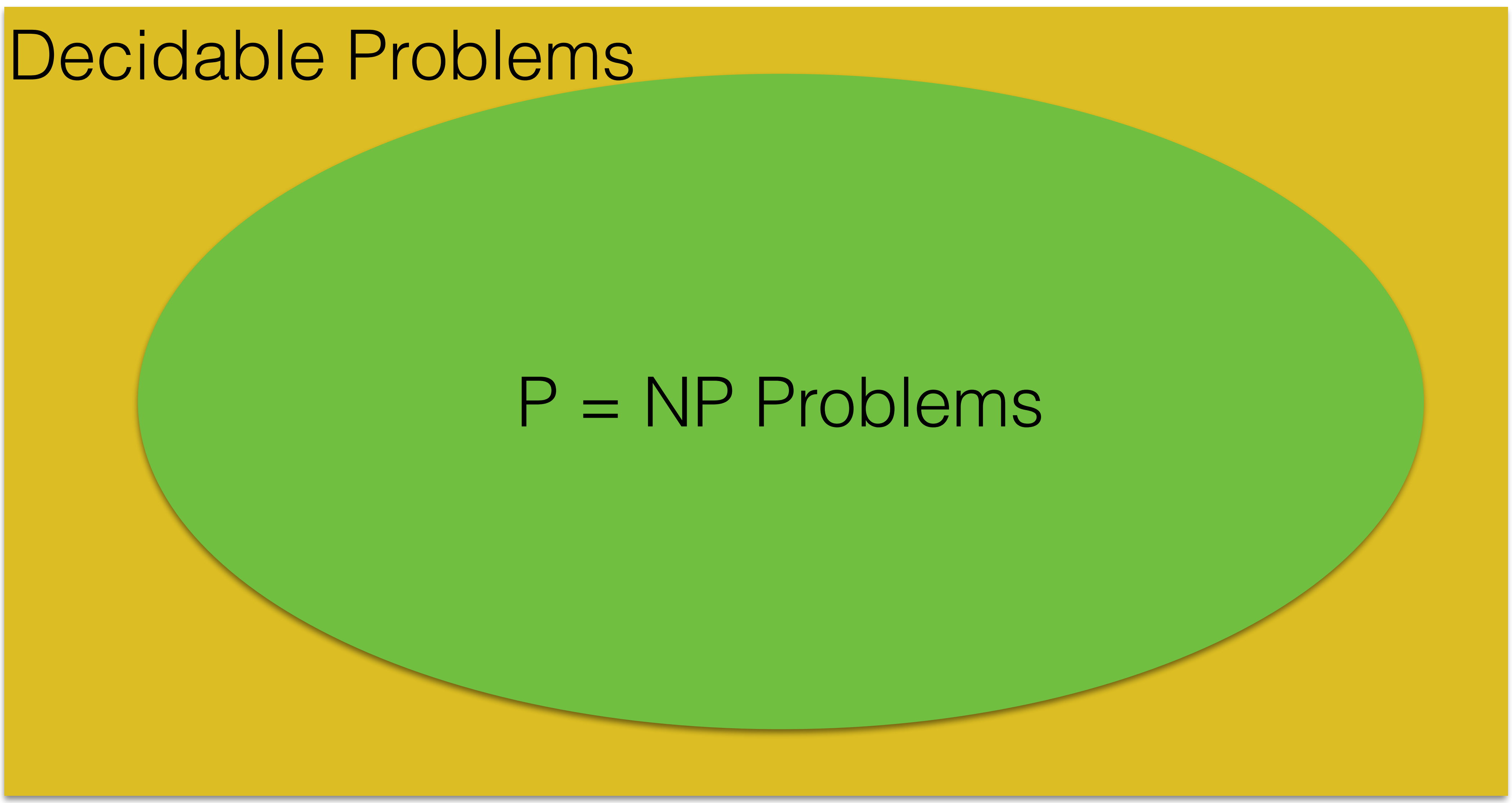
$P = NP$ vs. $P \neq NP$

P and NP Problems



if $P \neq NP$

P and NP Problems



if P = NP

NP-Complete Problems

- An NP problem is NP-Complete if it is at least as hard as any problem in NP.

- How de we know that a given problem p is NP complete?

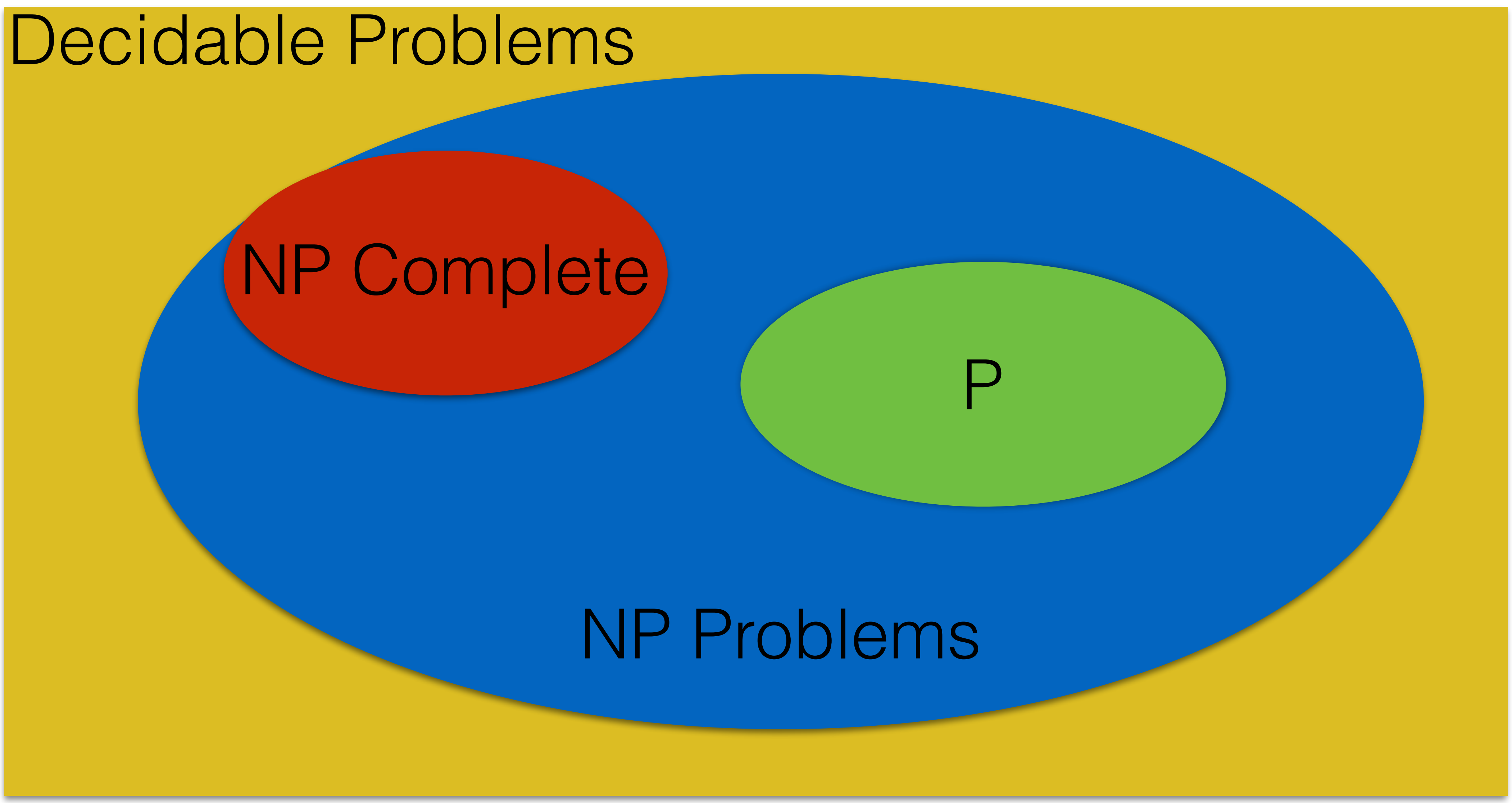
  - Any instance of any problem q in NP can be *transformed* into an instance of p in polynomial time.

  - This is also called a **reduction of q to p.**

Reductions

- Provide a mapping so that any instance of q can be transformed into an instance of p.
- Solve p and then map the result back to q.
- These mappings must be computable in polynomial time.

Problem Classes

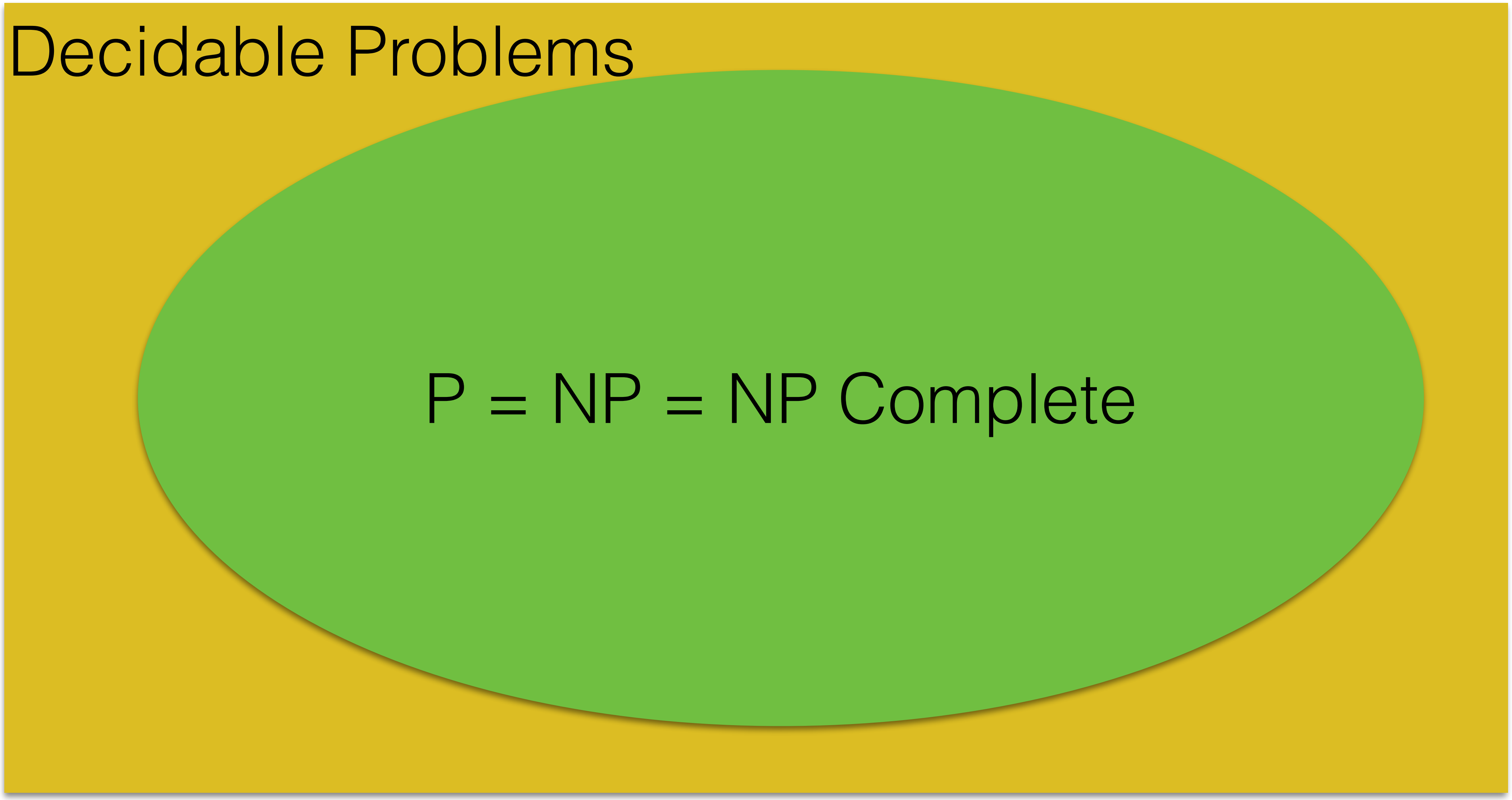


if $P \neq NP$

Importance of the NP-Complete Class

- Any other problem in NP can be transformed into an NP-Complete problem.
- If a polynomial time solution exists for any of these problems, there is a polynomial time solution for all problems in NP!
- To show that a new problem is NP-Complete, we show that another NP-complete problem can be reduced to it.

P and NP Problems



if P = NP

Example Reduction

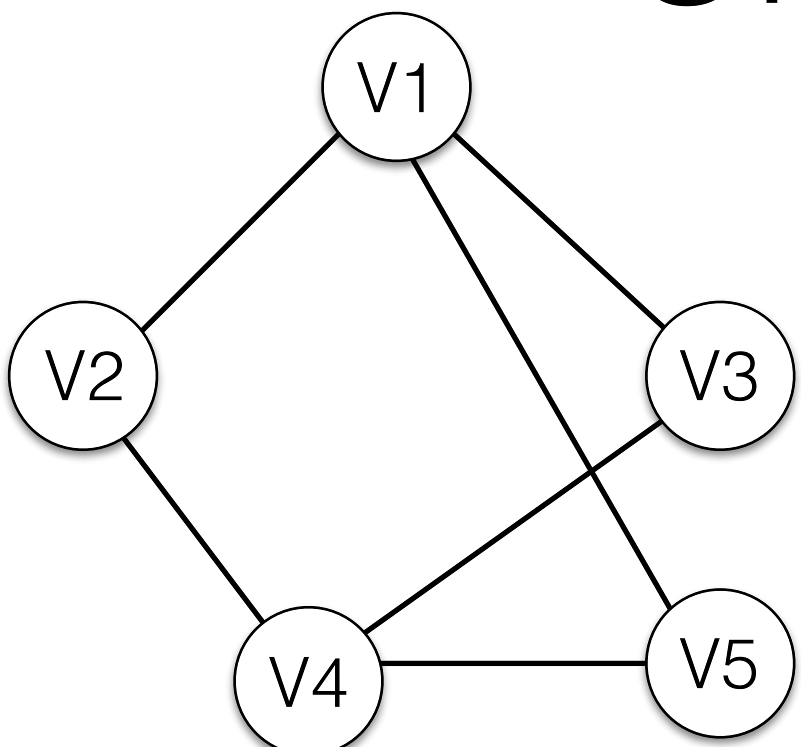
- Assume we know the Hamiltonian Circuit problem is NP-Complete.
- To show that TSP is NP-Complete, we can reduce Hamiltonian Circuit to it.

$$\begin{array}{c}\text{Hamiltonian Circuit (known to be NP-Complete)} \\ \downarrow \\ \text{Traveling Salesman Problem}\end{array}$$

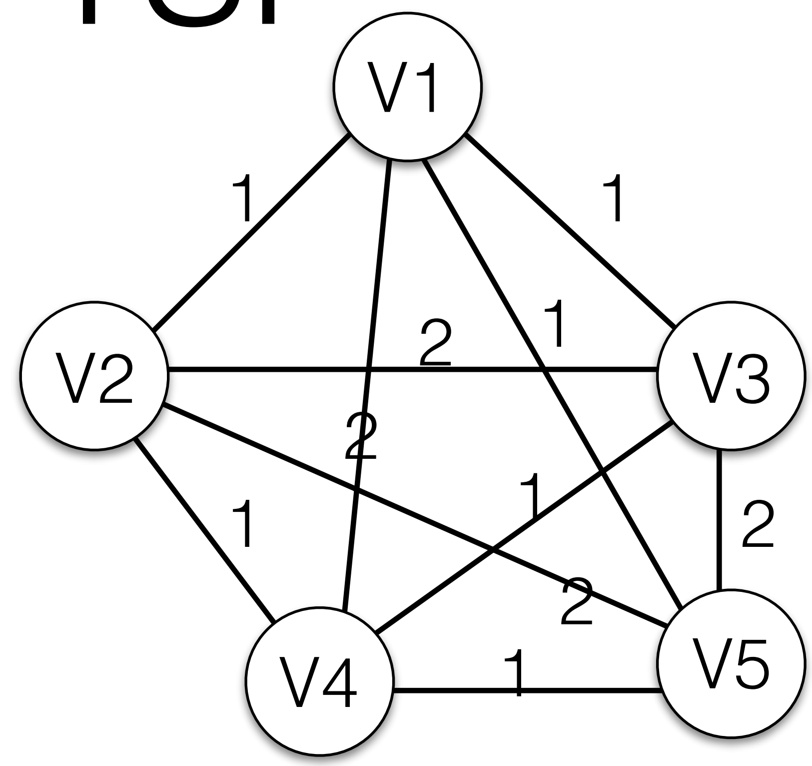
Reducing Hamiltonian Circuit to TSP

- We want to know if the input graph G = (V,E) contains a Hamiltonian Circuit.
- Construct a complete graph G’ over V.
- Set the cost of all edges in G’ that are also in E to 1.0. Set the cost of all other edges to 2.0.

Reducing Hamiltonian Circuit to TSP



Input graph G for Hamiltonian Circuit



Input graph G' for TSP

- Resulting TSP decision problem:
  - Does G' contain a TSP tour with cost ≤ |V|
- G contains a Hamiltonian Circuit if and only if G' contains a TSP tour with cost = |V|