Data Structures in Java

Lecture 18: Spanning Trees

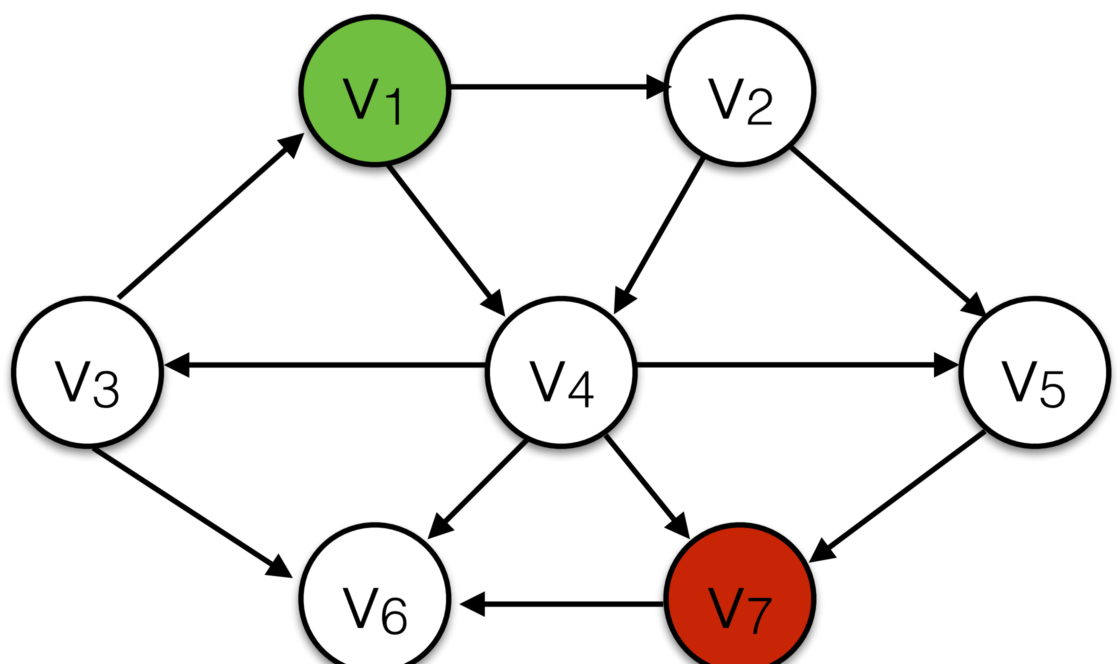
11/23/2015

Daniel Bauer

A General View of Graph Search

Goals:

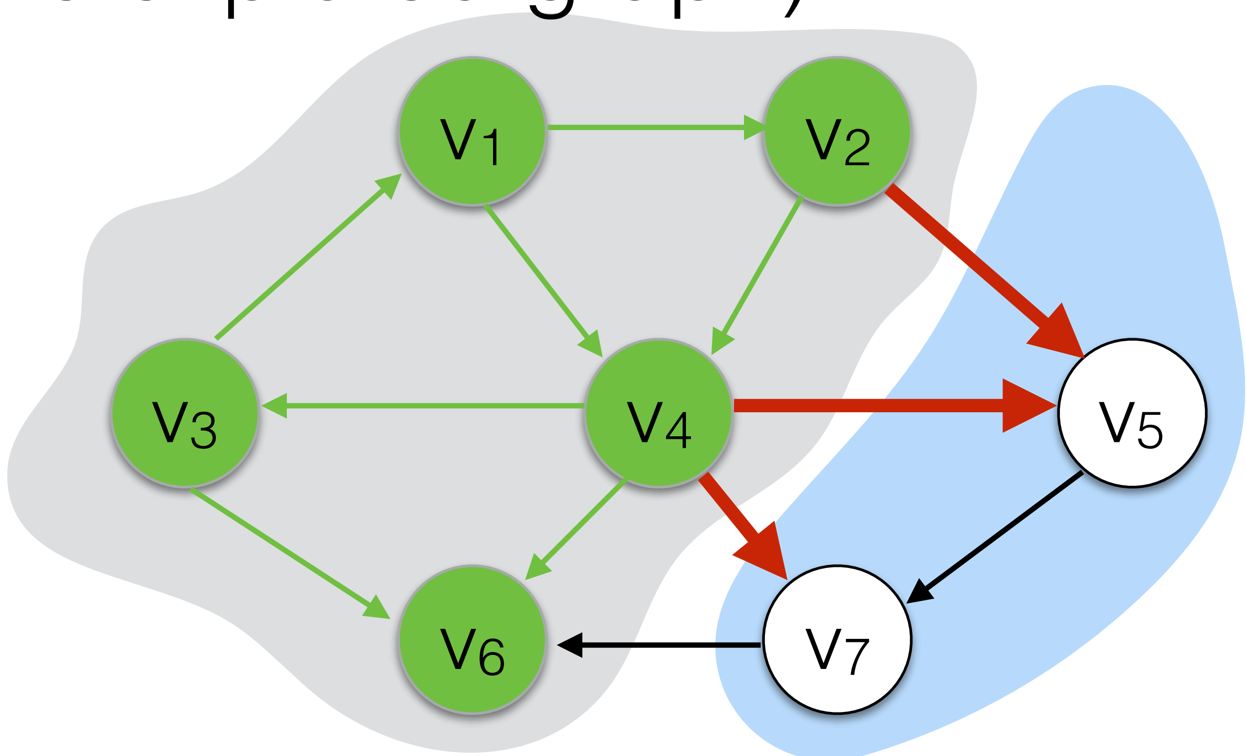
- Explore the graph systematically starting at *s* to
  - Find a vertex *t* / Find a path from *s* to *t*.
  - Find the shortest path from *s* to all vertices.
  - …



A General View of Graph Search

In every step of the search we maintain

- The part of the graph already explored.
- The part of the graph not yet explored.
- A data structure (an agenda) of *next* edges (adjacent to the explored graph).

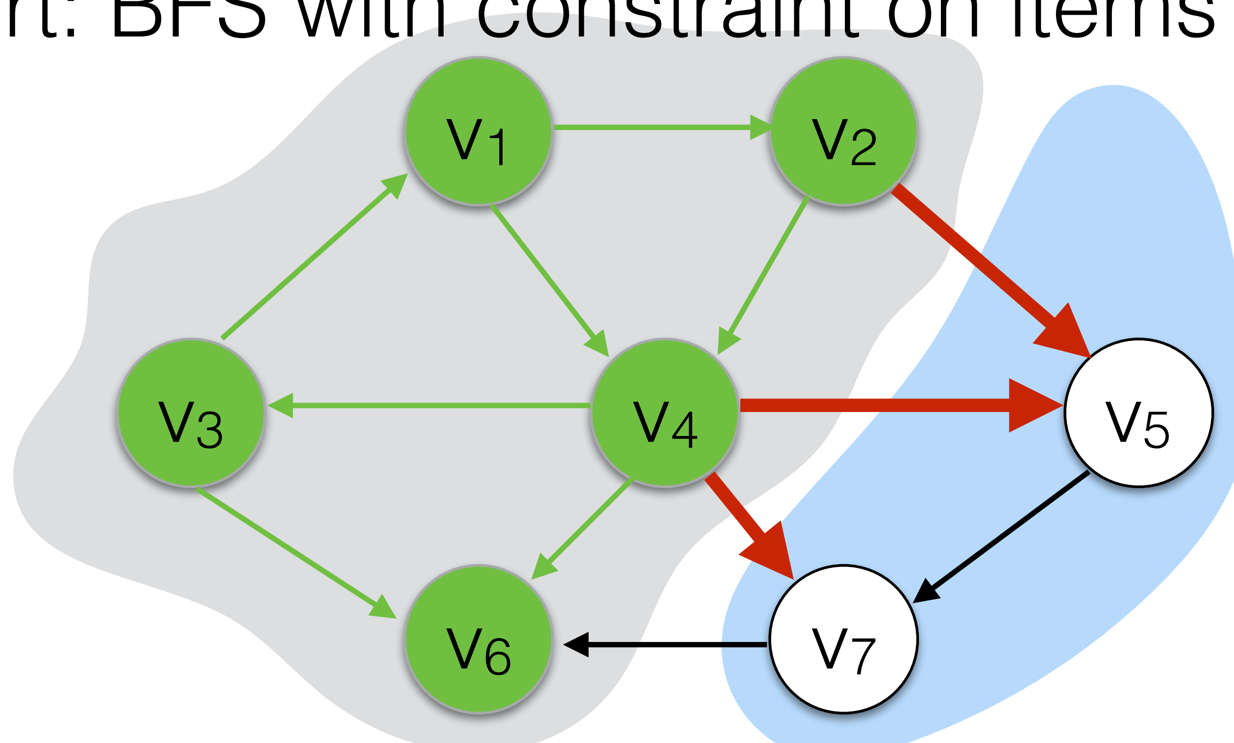


Agenda: (v2,v5), (v4,v5), (v4,v7)

A General View of Graph Search

The graph search algorithms discussed so far differ almost only in the type of agenda they use:

- DFS: uses a stack.
- BFS: uses a queue.
- Dijkstra's: uses a priority queue.
- Topological Sort: BFS with constraint on items in the queue.



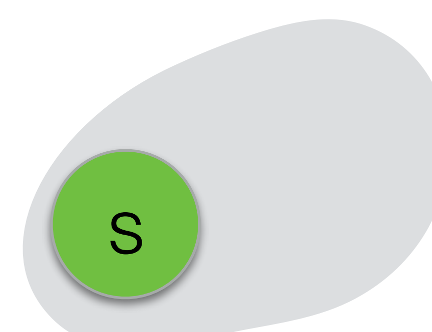
Agenda: (v2,v5), (v4,v5), (v4,v7)

Correctness of Dijkstra's Algorithm

- We want to show that Dijkstra's algorithm really finds the minimum path costs (we don't miss any shorter solutions by choosing the shortest edge greedily).

- Proof by induction on the set S of visited nodes.

- Base case:
  |S|=1. Trivial. Length shortest path is 0.



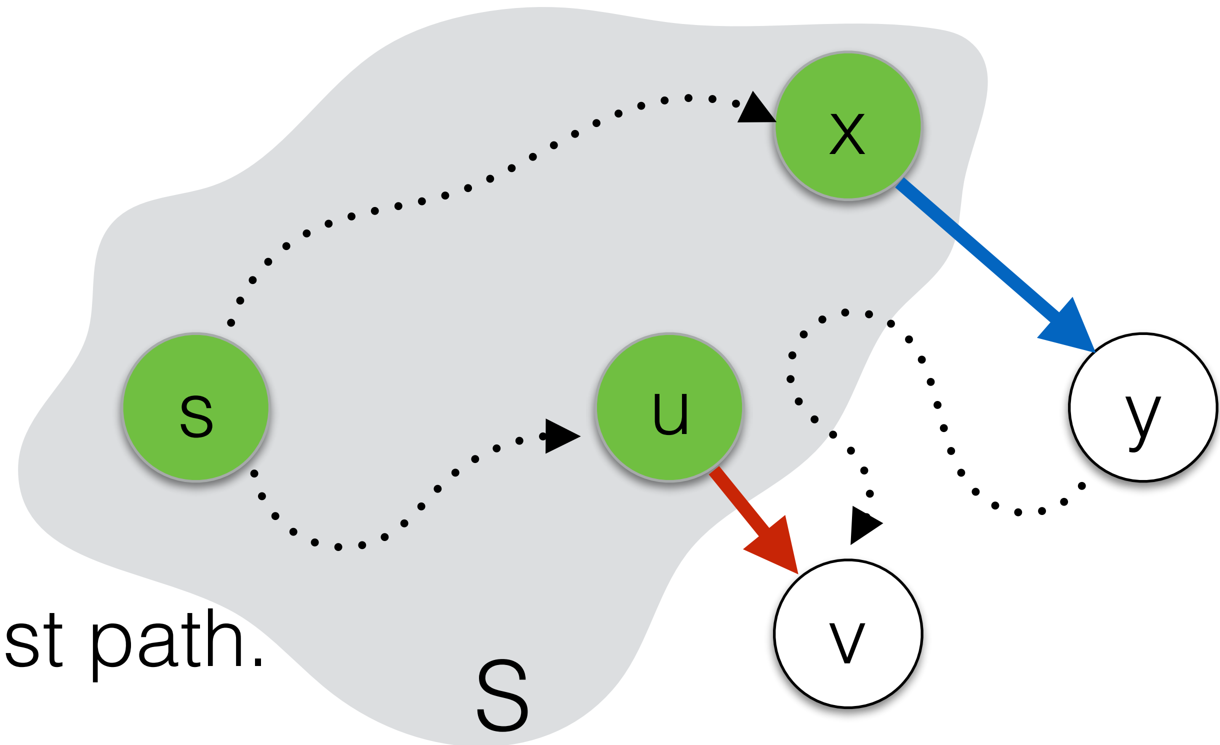
Correctness of Dijkstra's Inductive Step

- Assume the algorithm produces the minimal path cost from *s* for the subset S, |S| = k.

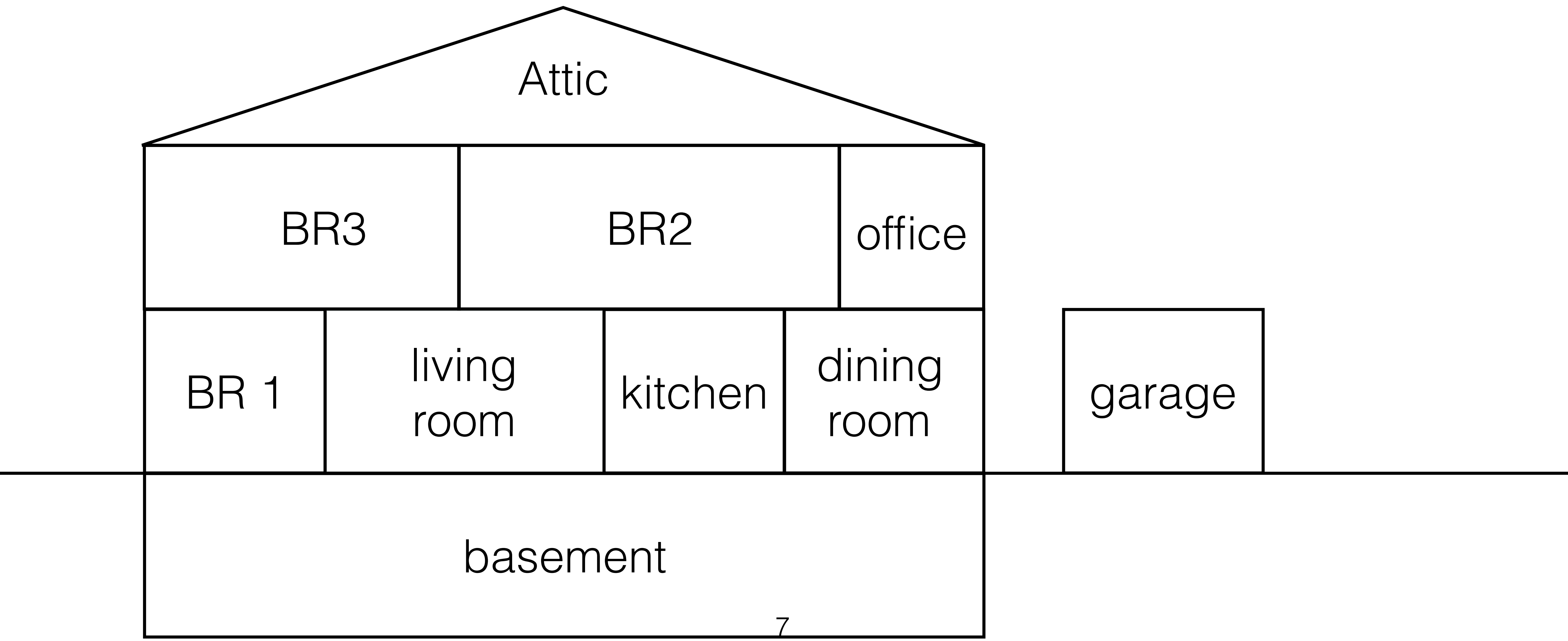
- Dijkstra's algorithm selects the next edge (u,v) leaving S.

- Assume there was a shorter path from s to v that does not contain (u,v).
    - Then that path must contain another edge (x,y) leaving S.
    - The cost of (x,y) is already higher than (u,v) because we didn't choose it before (u,v)

- Therefore (u,v) must be on the shortest path.



Designing a Home Network.



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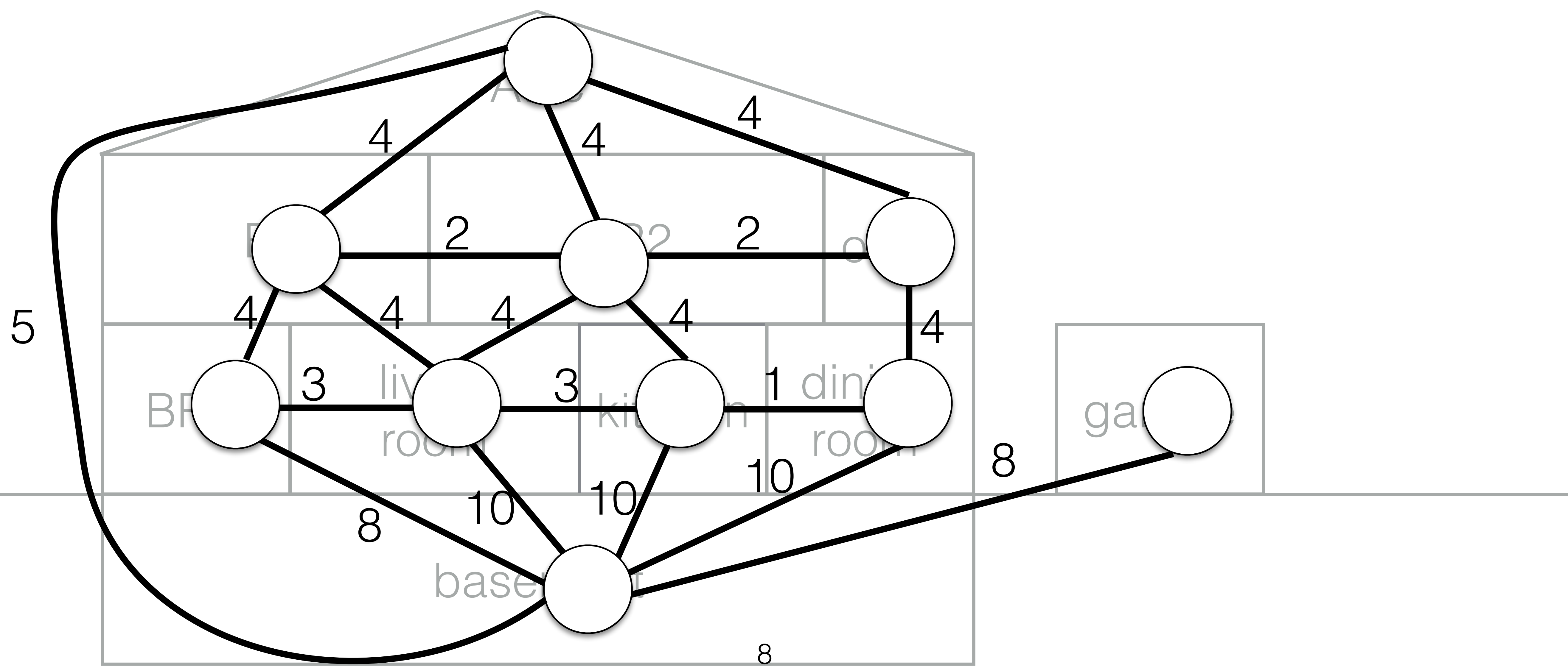
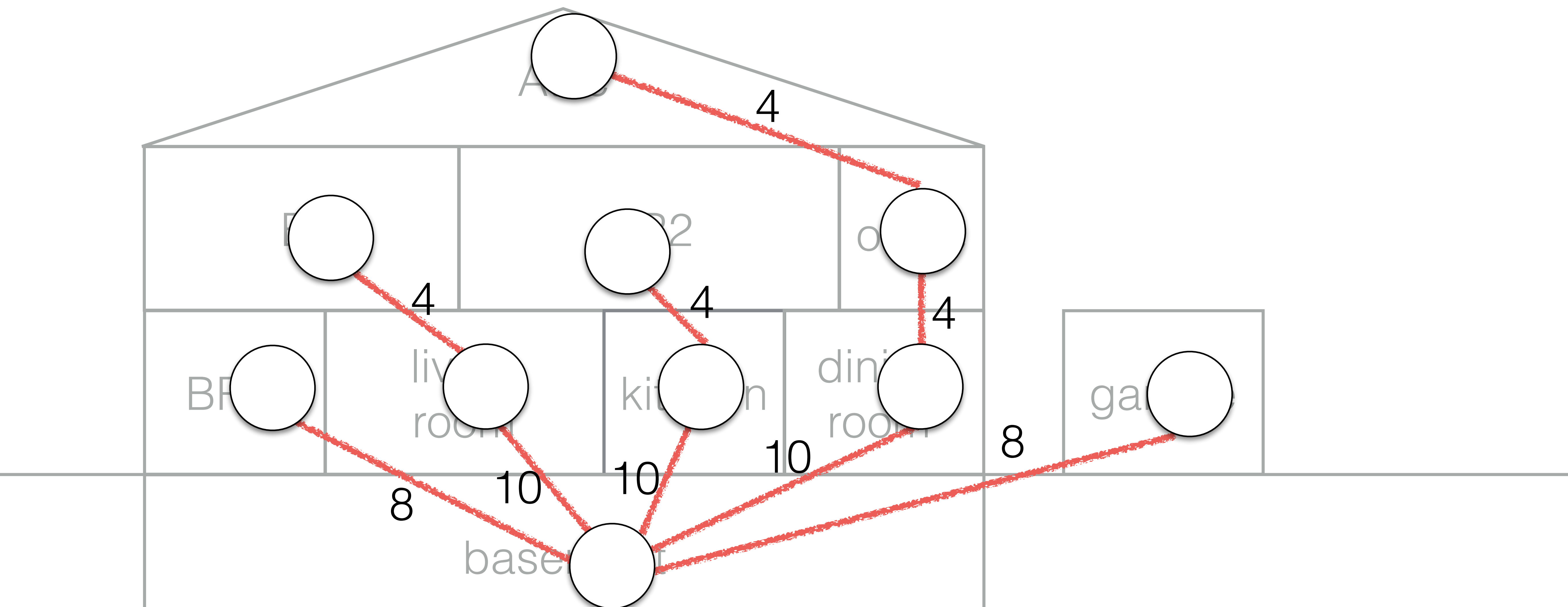


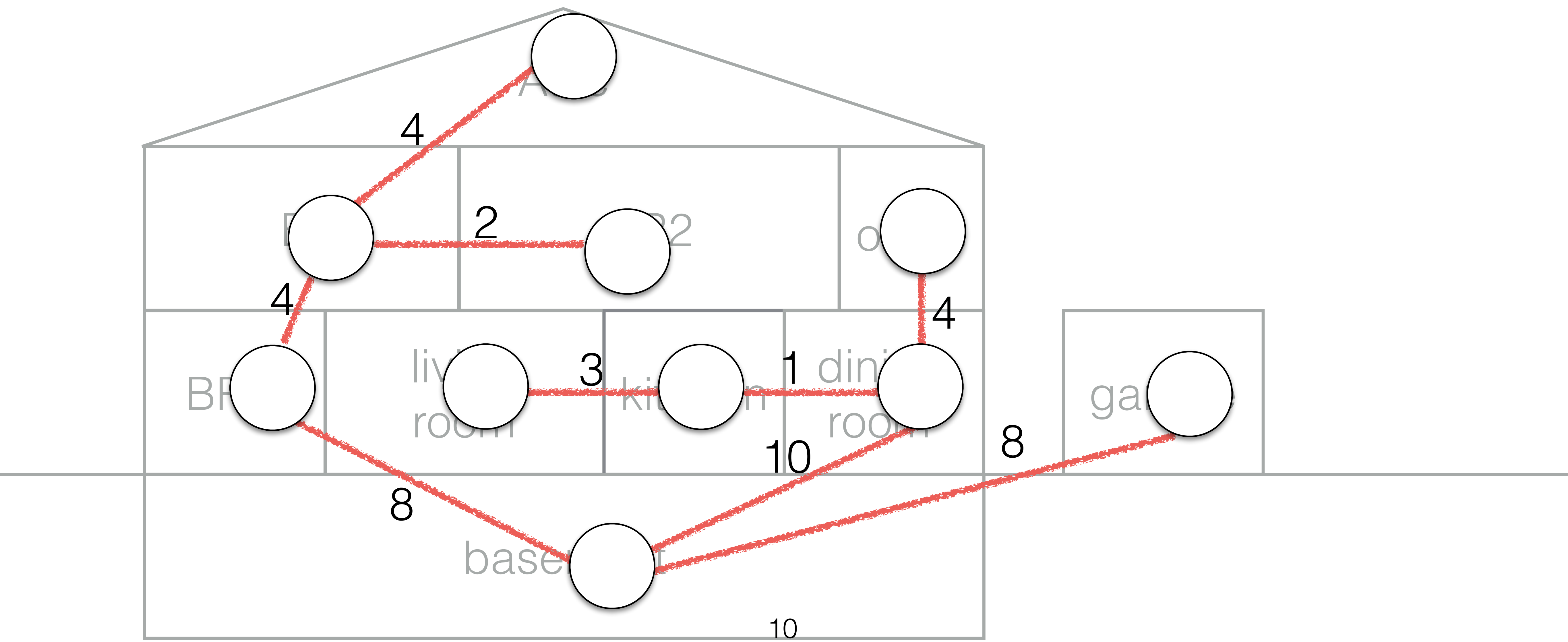
Designing a Home Network.

Total cost: 62



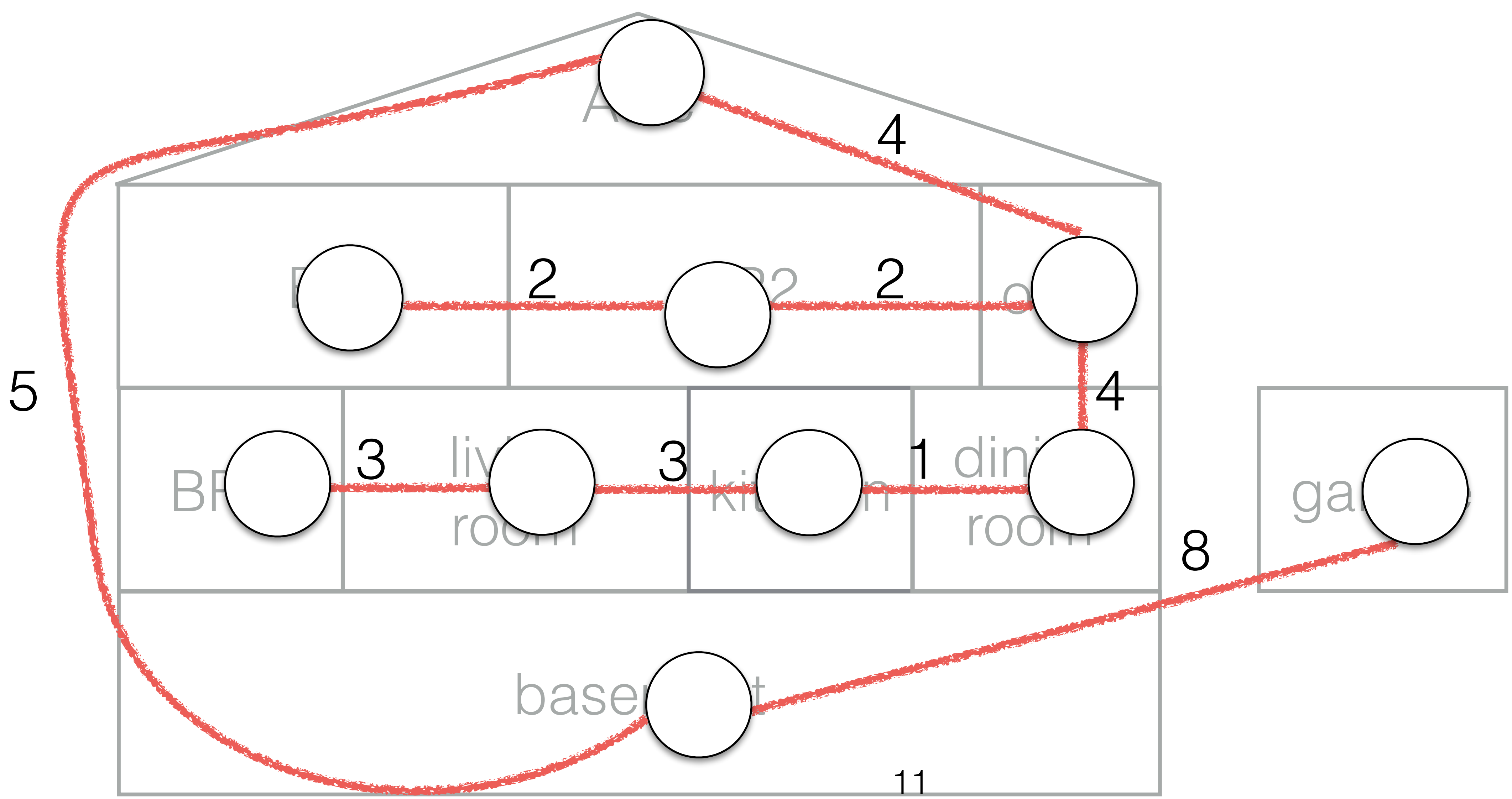
Designing a Home Network.

Total cost: 44



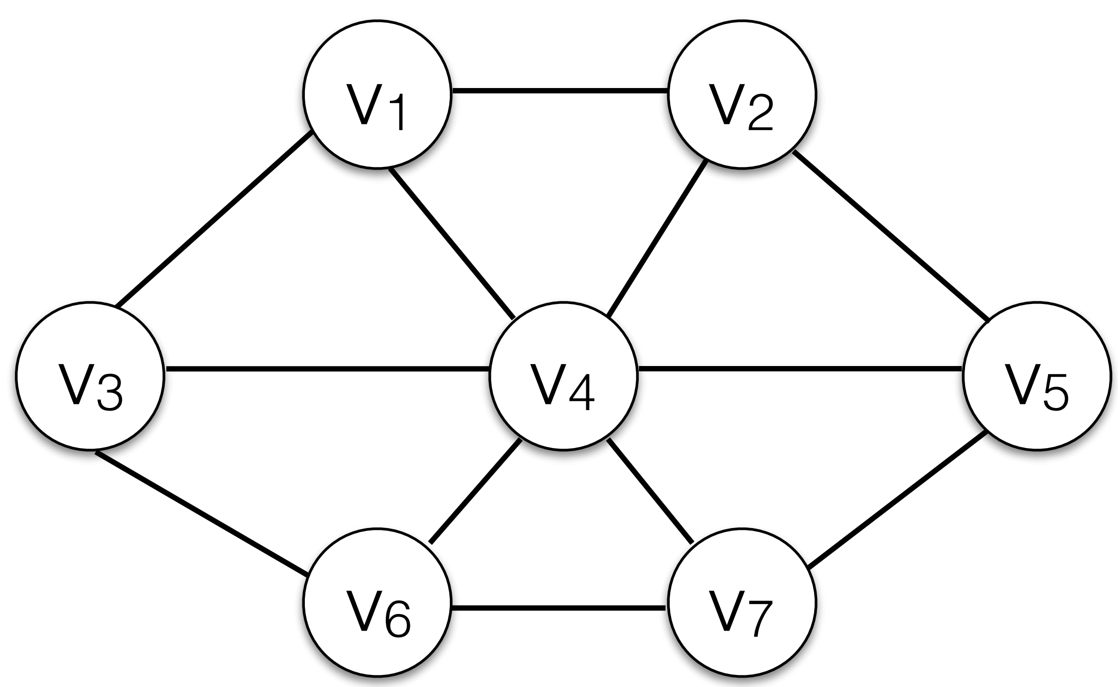
Designing a Home Network.

Total cost: 32



Spanning Trees

- Given an undirected, connected graph G=(V,E).
- A ***spanning tree*** is a tree that connects all vertices in the graph. $T=(V, E_T \subseteq E)$



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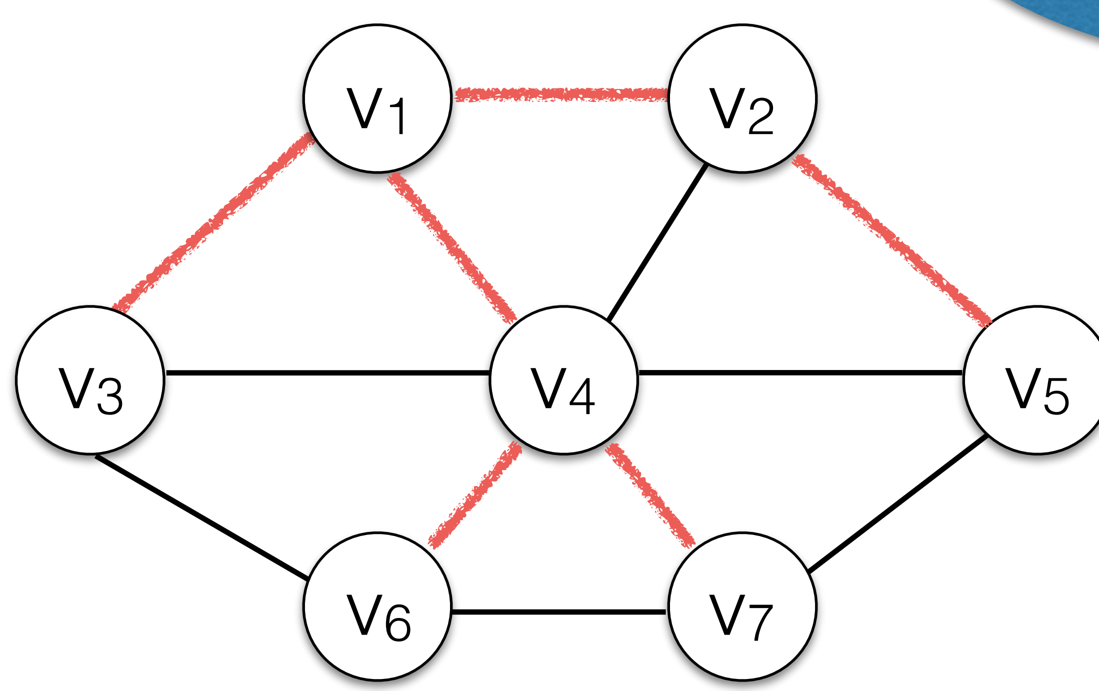
T is acyclic. There is a single path between any pair of vertices.



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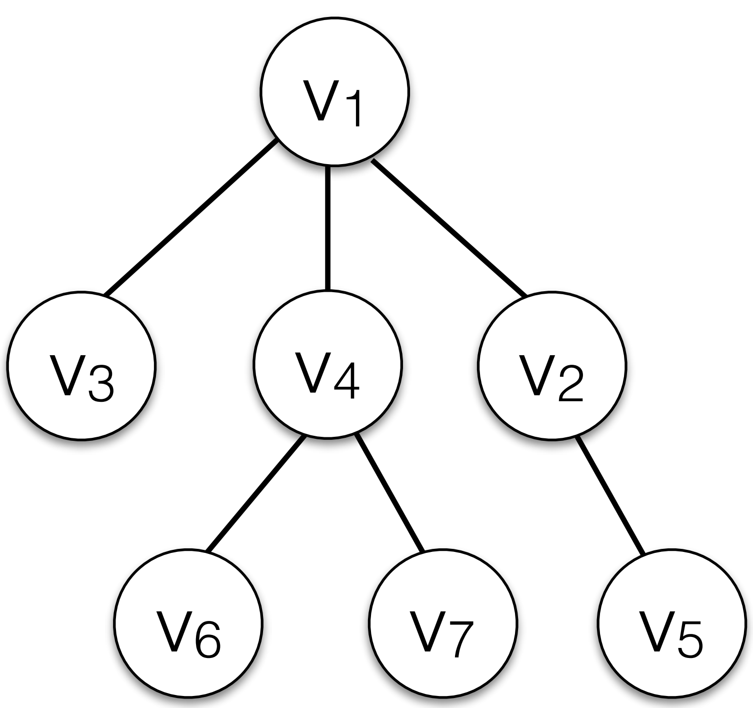

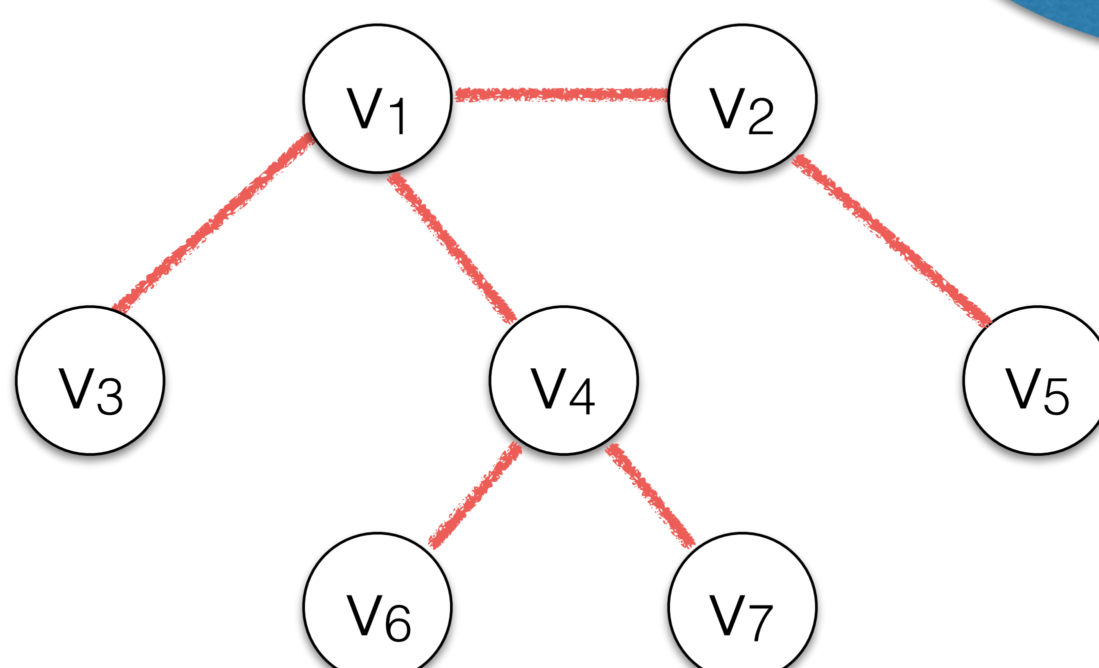

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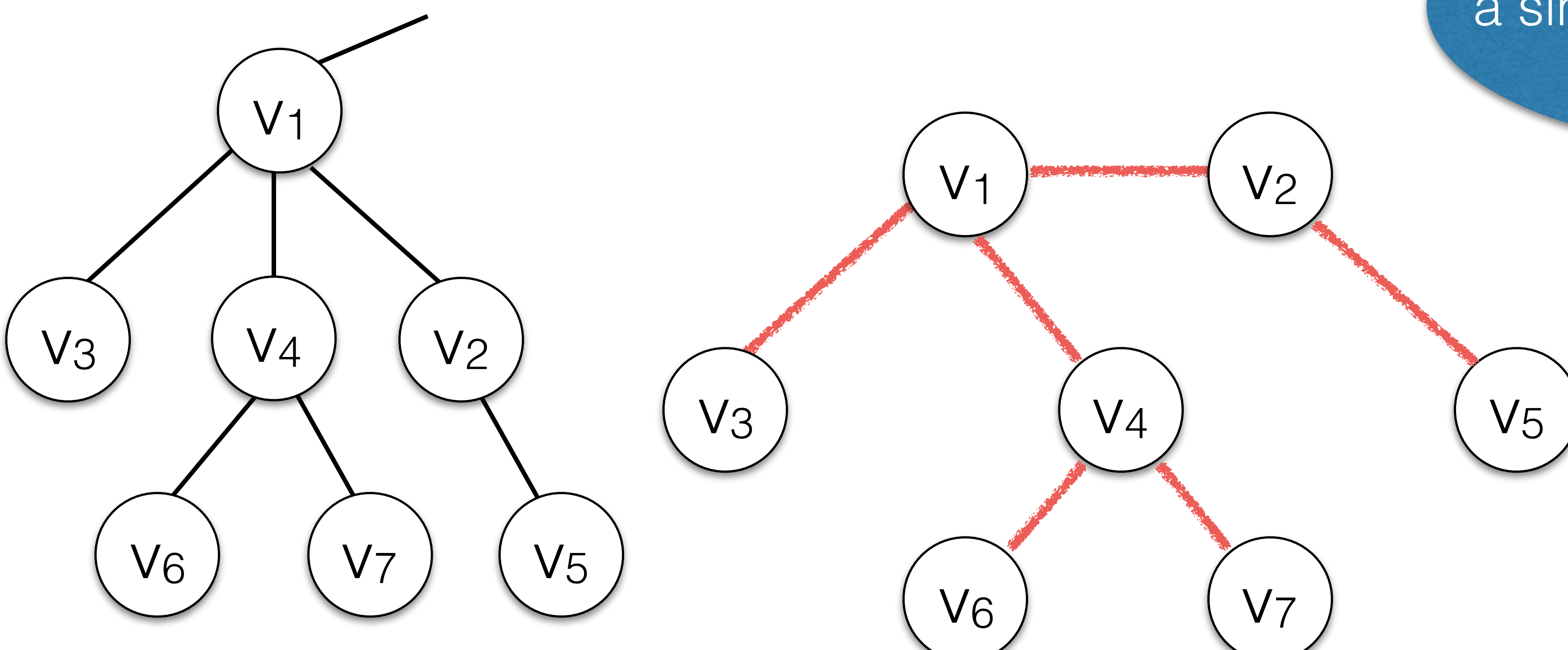

Any node can be the root of the spanning tree.

Spanning Trees

- Given an undirected, connected graph $G=(V,E)$.
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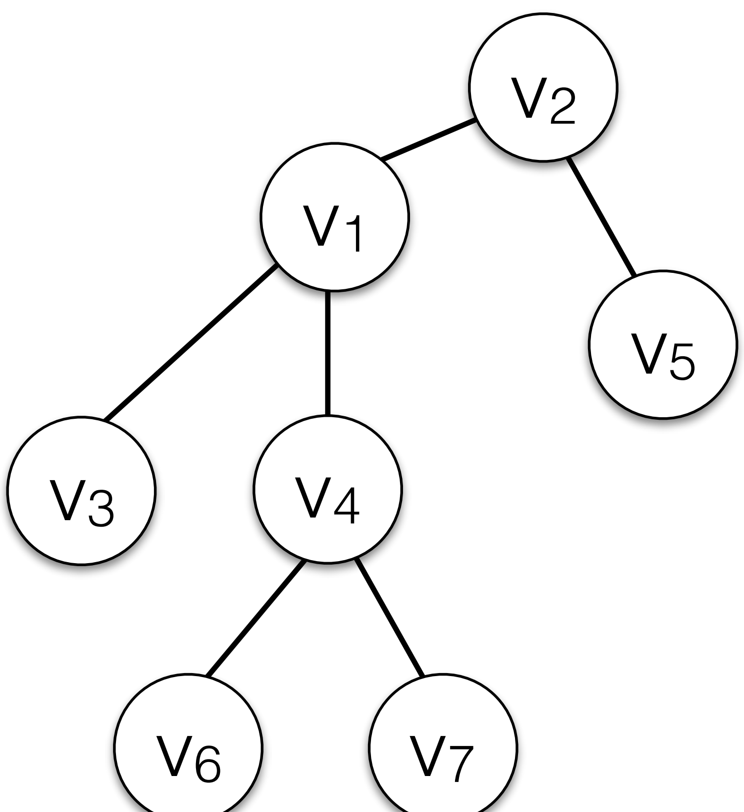
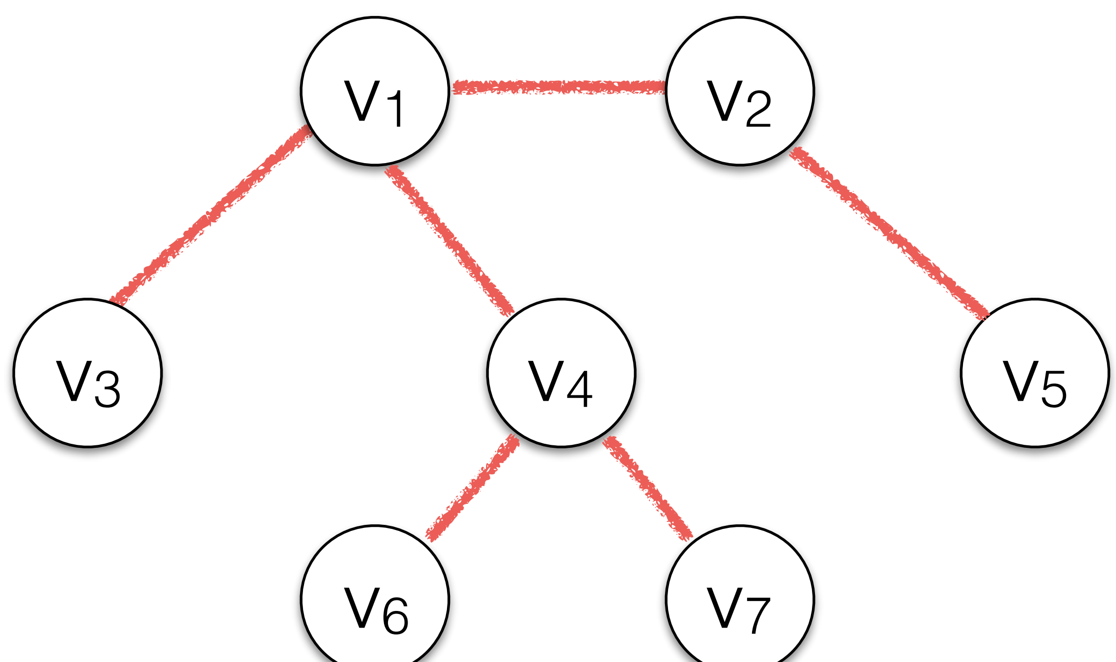




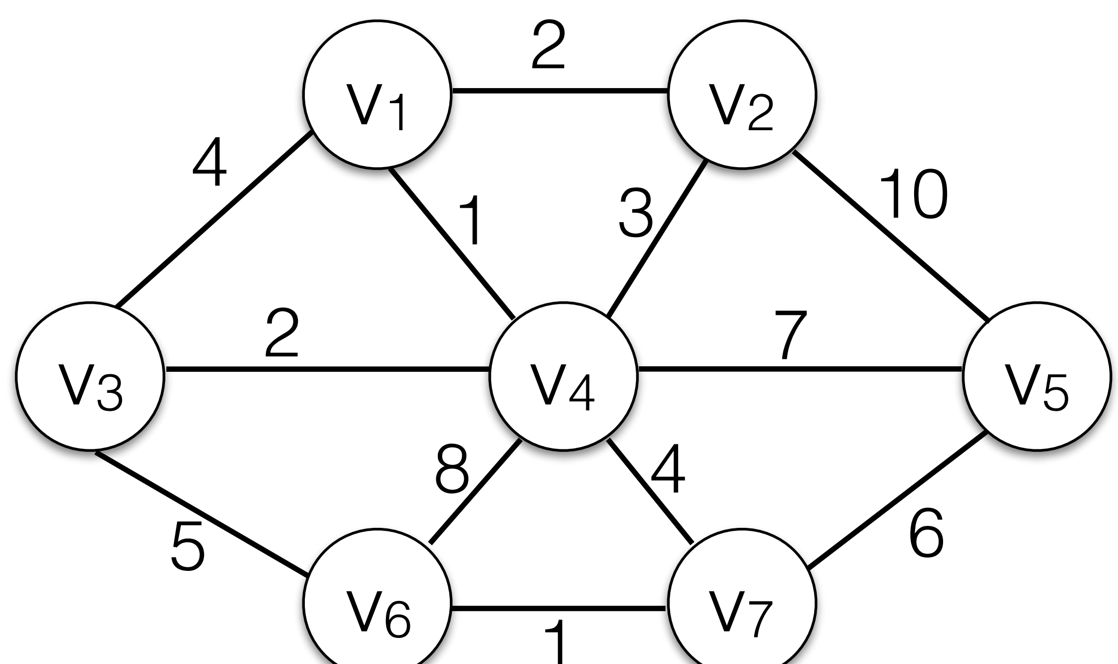
Number of edges in a spanning tree: $|V|-1$

Spanning Trees, Applications

- Constructing a computer/power networks (connect all vertices with the smallest amount of wire).
- Clustering Data.
- Dependency Parsing of Natural Language
  (directed graphs. This is harder).
- Constructing mazes.
- …
- Approximation algorithms for harder graph problems.
- …

Minimum Spanning Trees

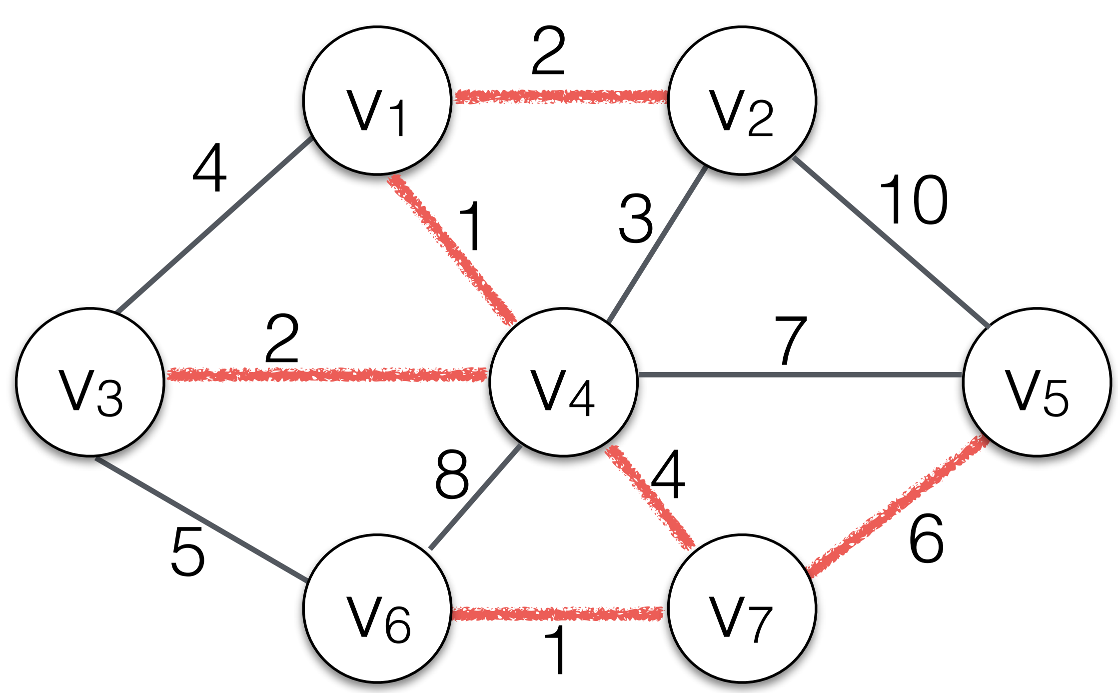
- Given a *weighted* undirected graph G=(E,V).
- A ***minimum spanning tree*** is a spanning tree with the minimum sum of edge weights.



Minimum Spanning Trees

- Given a *weighted* undirected graph G=(E,V).
- A ***minimum spanning tree*** is a spanning tree with the minimum sum of edge weights.

Total cost = 16



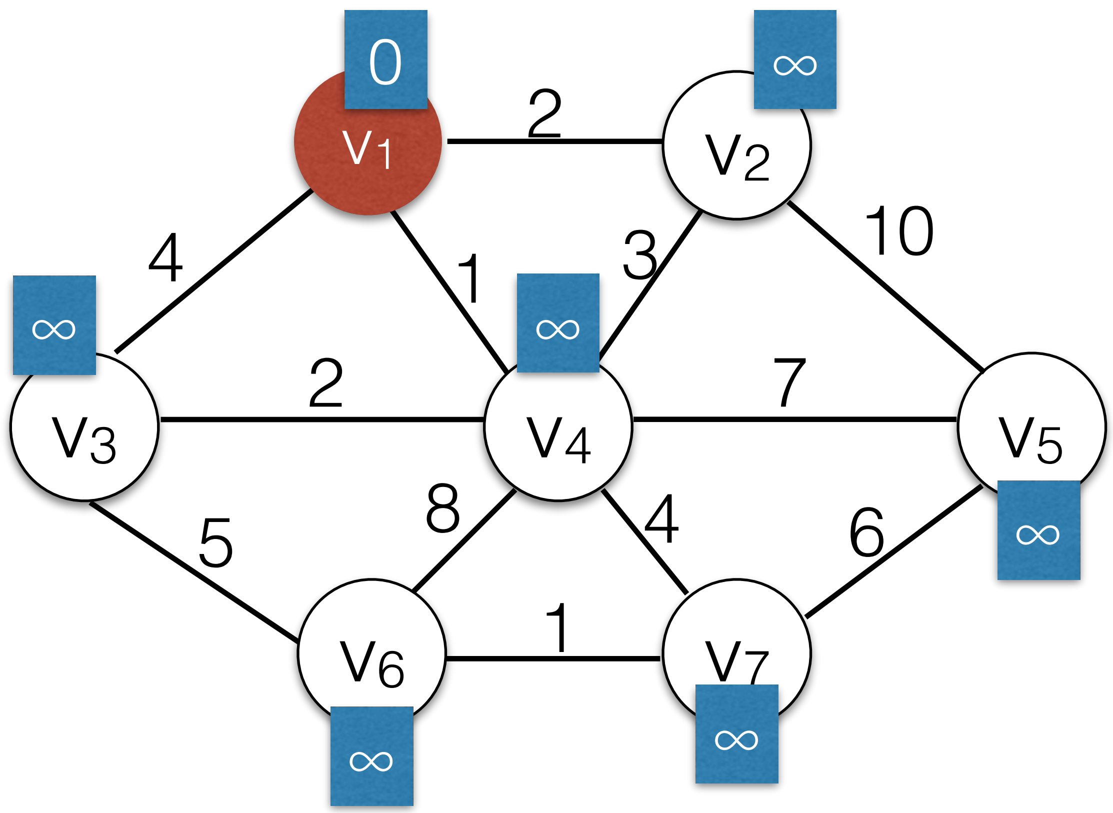
(often there are multiple minimum spanning trees)

Prim's Algorithm for finding MSTs

- Another greedy algorithm. A variant of Dijkstra's algorithm.

- Cost annotations for each vertex v reflect the lowest weight of an edge connecting v to other *vertices already visited.*
  - That means there might be a lower-weight edge from another vertices that have not been seen yet.
- Keep vertices on a priority queue and always expand the vertex with the lowest cost annotation first.

Prim’s Algorithm



Use a Priority Queue *q*

- for all v ∈ V
  set v.cost = ∞, set v.visited = false
- Choose any vertex s.
  set s.cost = 0, s.visited = true;
- q.insert(s)

- While *q* is not empty:
  - (costu, u) <- *q*.deleteMin()
  - if not u.visited:
    - u.visited = True
    - for each edge (u,v):
      - if not v.visited:
        - if (**cost(u,v)** < v.cost)
          - v.cost = cost(u,v)
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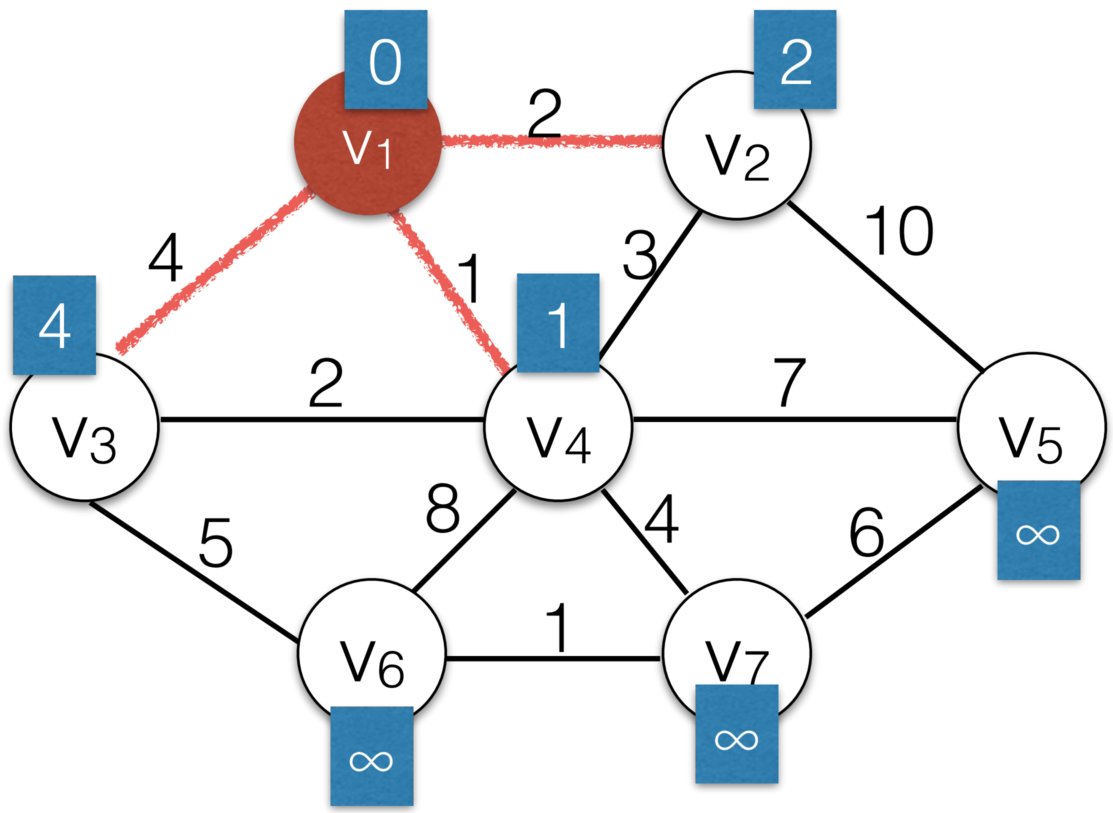
Prim's Algorithm



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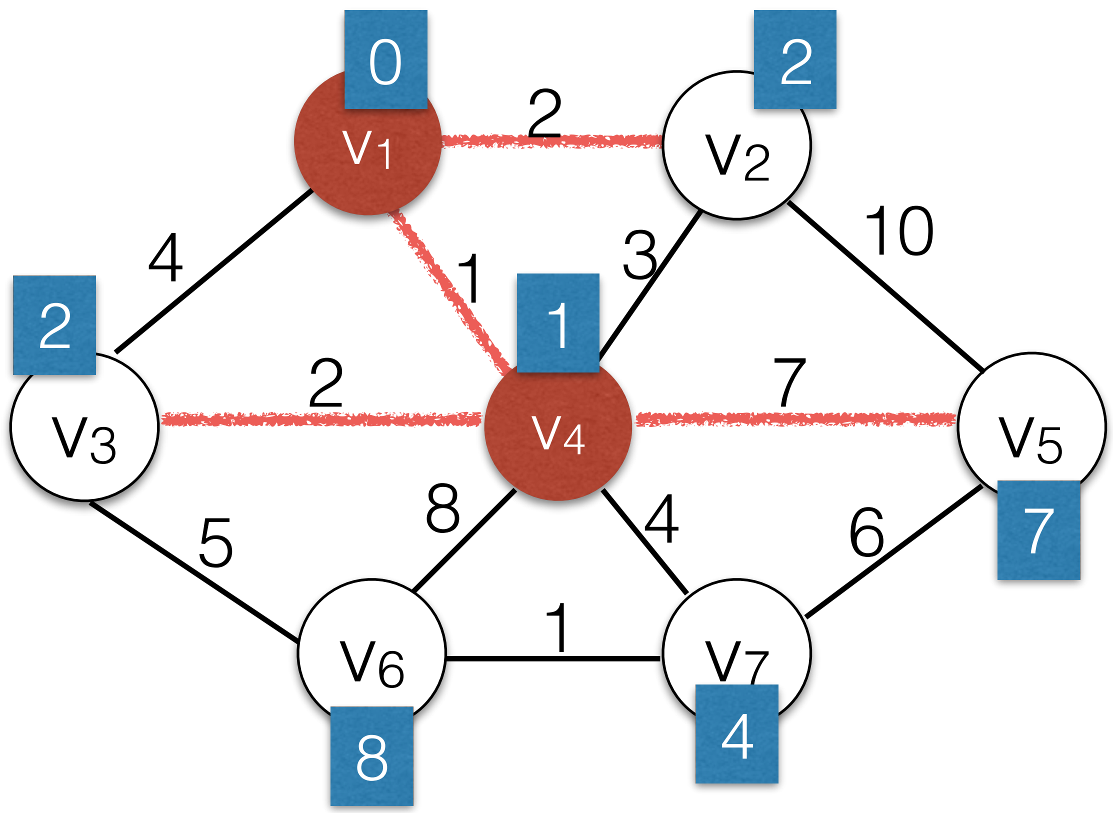

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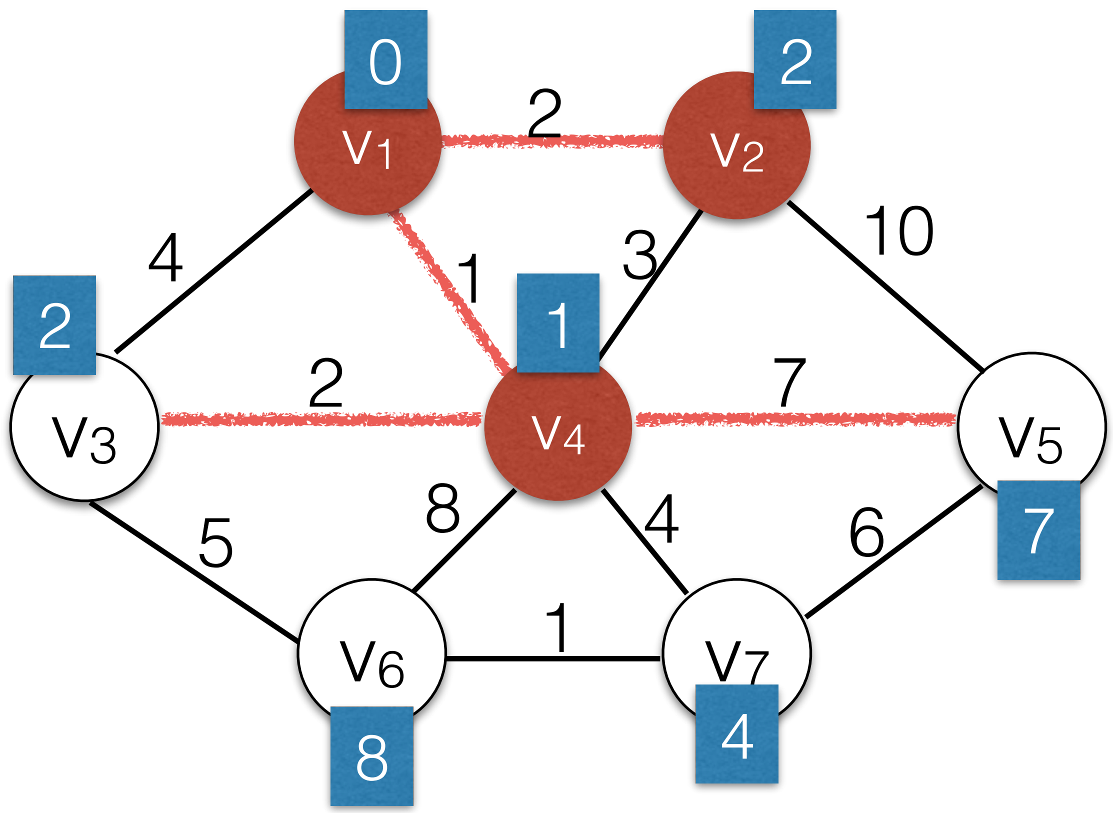

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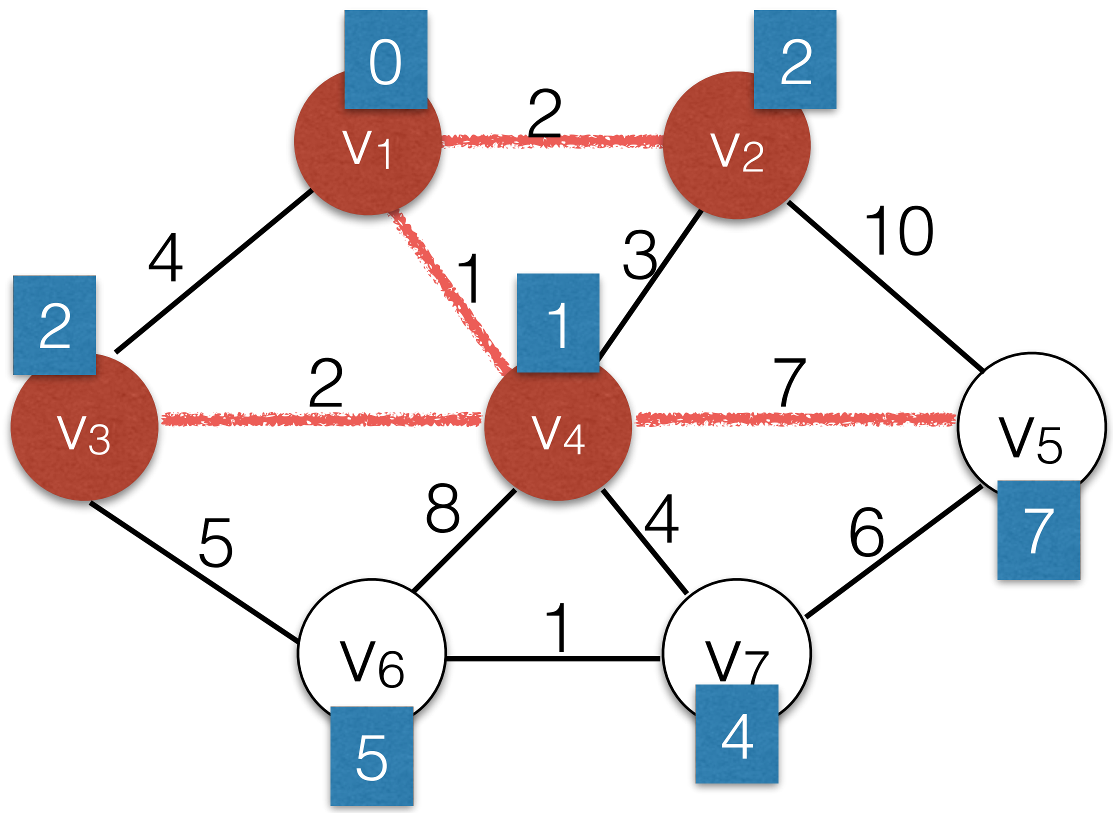

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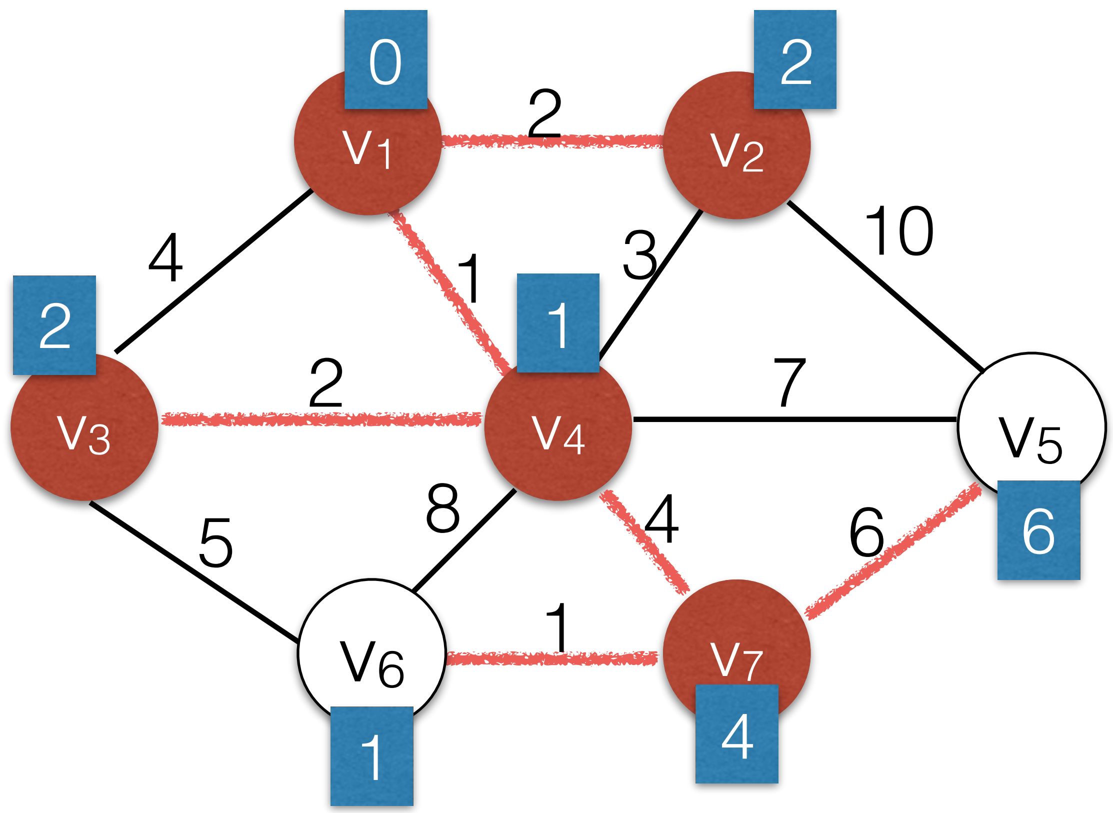

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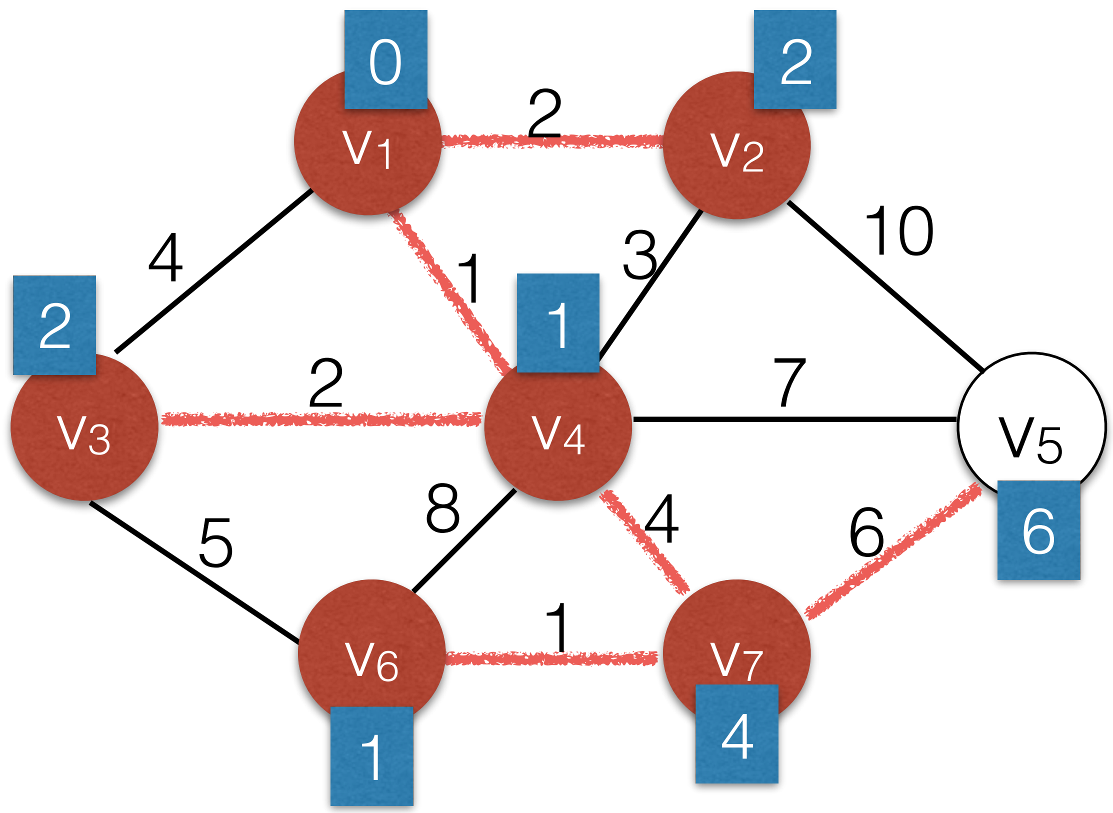

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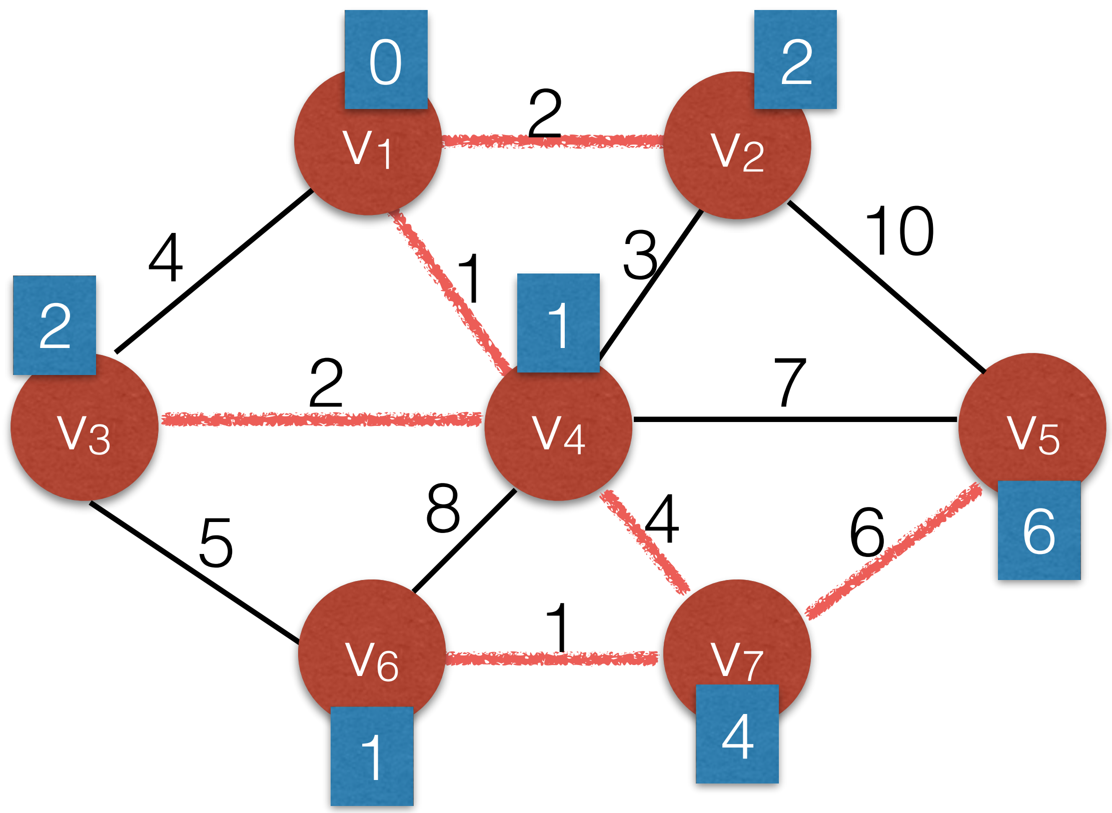

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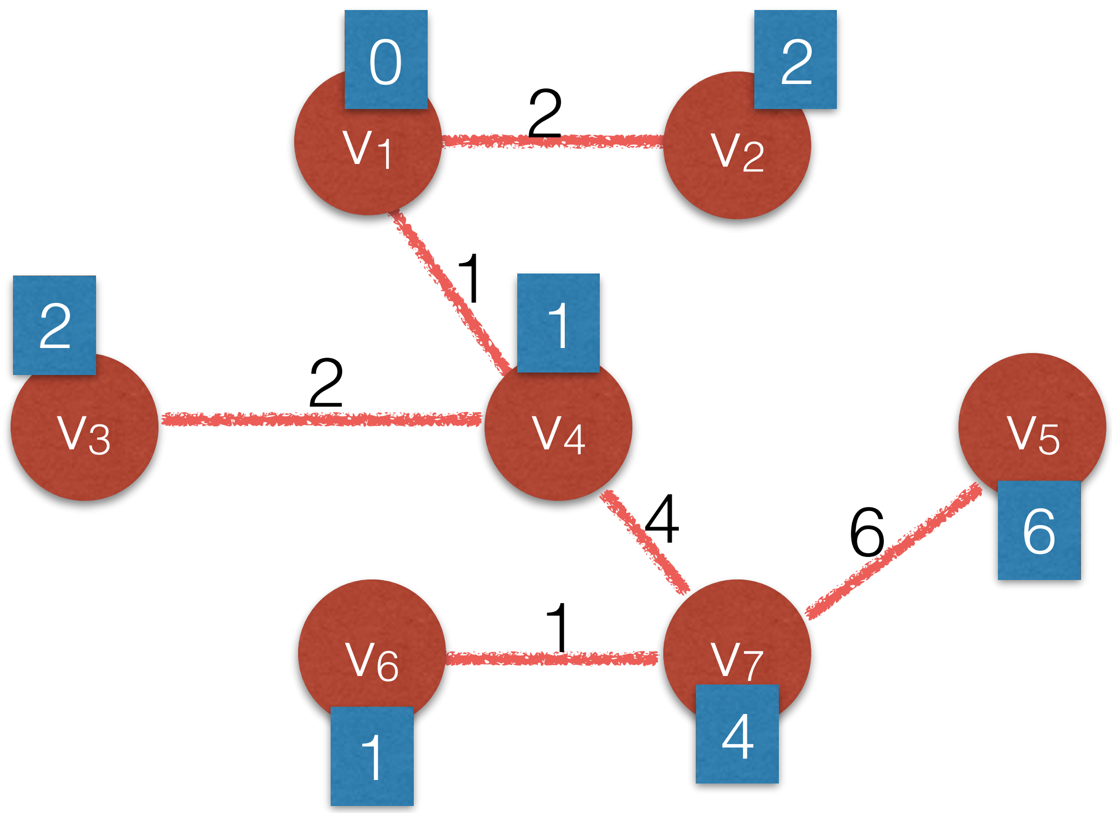

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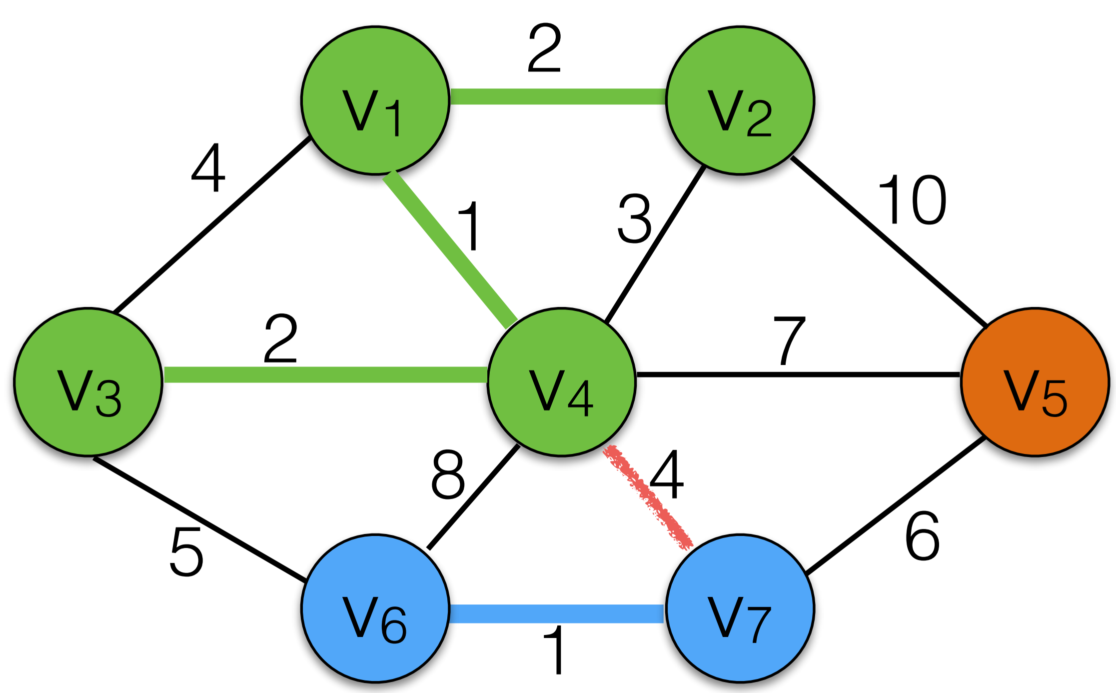
Running time: Same as Dijkstra's Algorithm
O(|E| log |V|)

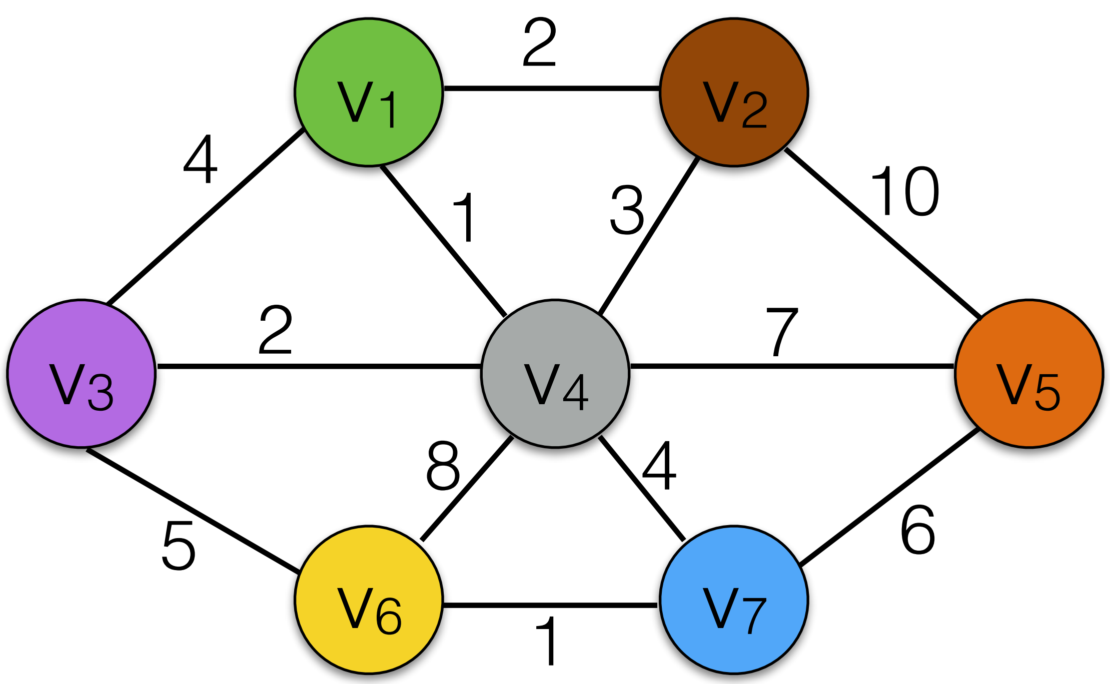
Kruskal's Algorithm for finding MSTs

- Kruskal's algorithm maintains a "forest" of trees.
- Initially each vertex is its own tree.
- Sort edges by weight. Then attempt to add them one-by one. Adding an edge merges two trees into a new tree.
- If an edge connects two nodes that are already in the same tree it would produce a cycle. Reject it.



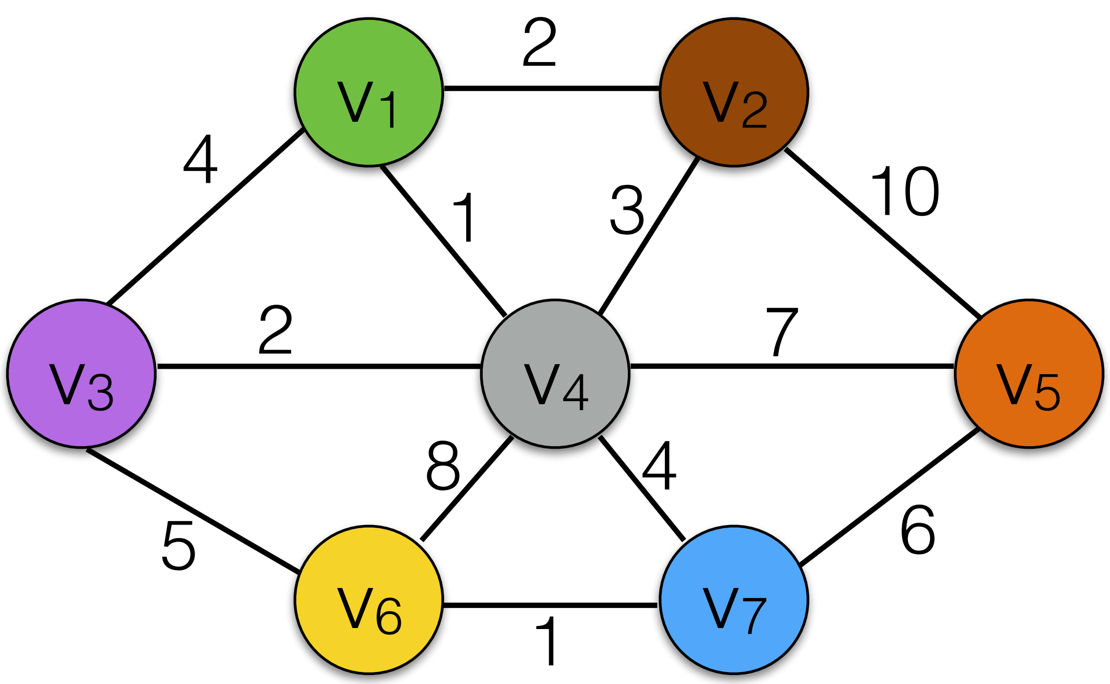
Kruskal's Algorithm

Sort edges (or keep them on a heap)



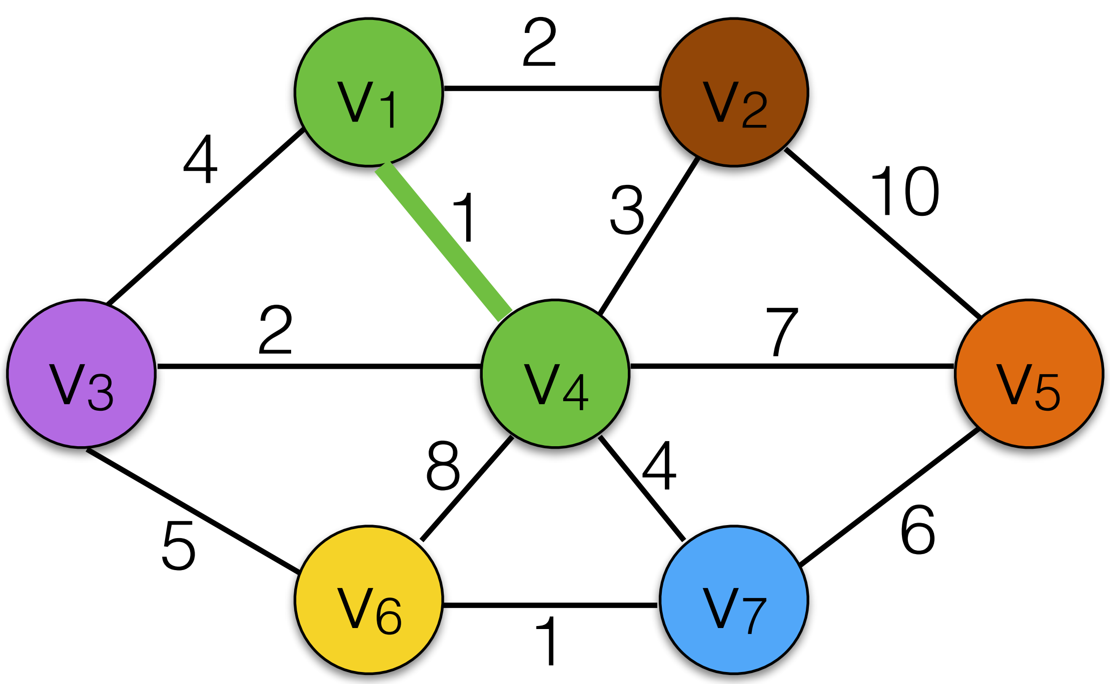
(v1,v2)	2
(v1,v3)	4
(v1,v4)	1
(v2,v4)	3
(v2,v5)	10
(v3,v4)	2
(v3,v6)	4
(v4,v5)	7
(v4,v6)	8
(v4,v7)	4
(v5,v7)	6
(v6,v7)	1

Kruskal's Algorithm



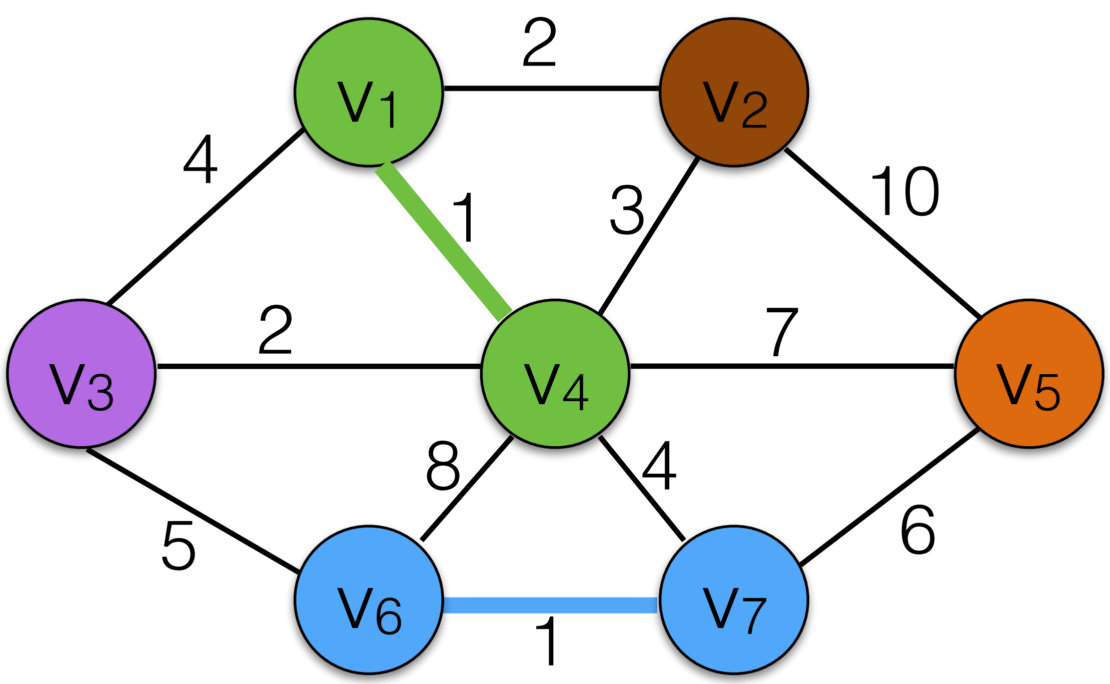
(v1,v4)	1
(v6,v7)	1
(v1,v2)	2
(v3,v4)	2
(v2,v4)	3
(v1,v3)	4
(v3,v6)	4
(v4,v7)	4
(v5,v7)	6
(v4,v5)	7
(v4,v6)	8
(v2,v5)	10

Kruskal's Algorithm



(v1,v4)	1	OK
(v6,v7)	1	
(v1,v2)	2	
(v3,v4)	2	
(v2,v4)	3	
(v1,v3)	4	
(v3,v6)	4	
(v4,v7)	4	
(v5,v7)	6	
(v4,v5)	7	
(v4,v6)	8	
(v2,v5)	10	

Kruskal's Algorithm



(v1,v4)	1	OK
(v6,v7)	1	OK
(v1,v2)	2	
(v3,v4)	2	
(v2,v4)	3	
(v1,v3)	4	
(v3,v6)	4	
(v4,v7)	4	
(v5,v7)	6	
(v4,v5)	7	
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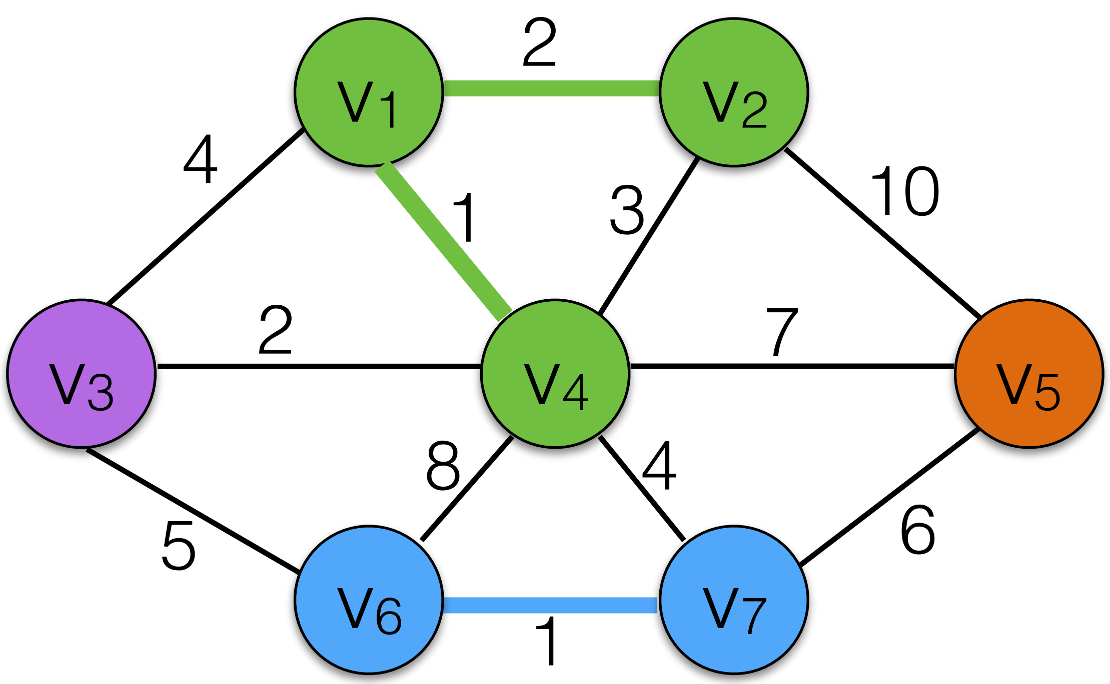
(v2,v5)	10	

Kruskal's Algorithm



(v1,v4)	1	OK
(v6,v7)	1	OK
(v1,v2)	2	OK
(v3,v4)	2	
(v2,v4)	3	
(v1,v3)	4	
(v3,v6)	4	
(v4,v7)	4	
(v5,v7)	6	
(v4,v5)	7	
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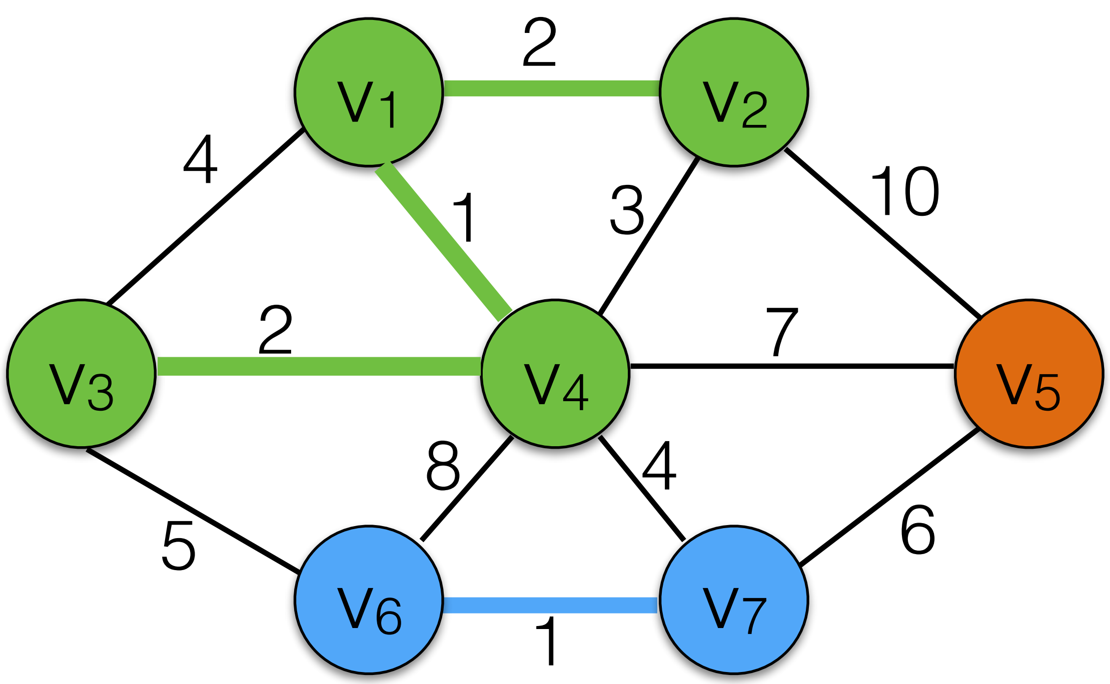
Kruskal's Algorithm



(v1,v4)	1	OK
(v6,v7)	1	OK
(v1,v2)	2	OK
(v3,v4)	2	OK
(v2,v4)	3	
(v1,v3)	4	
(v3,v6)	4	
(v4,v7)	4	
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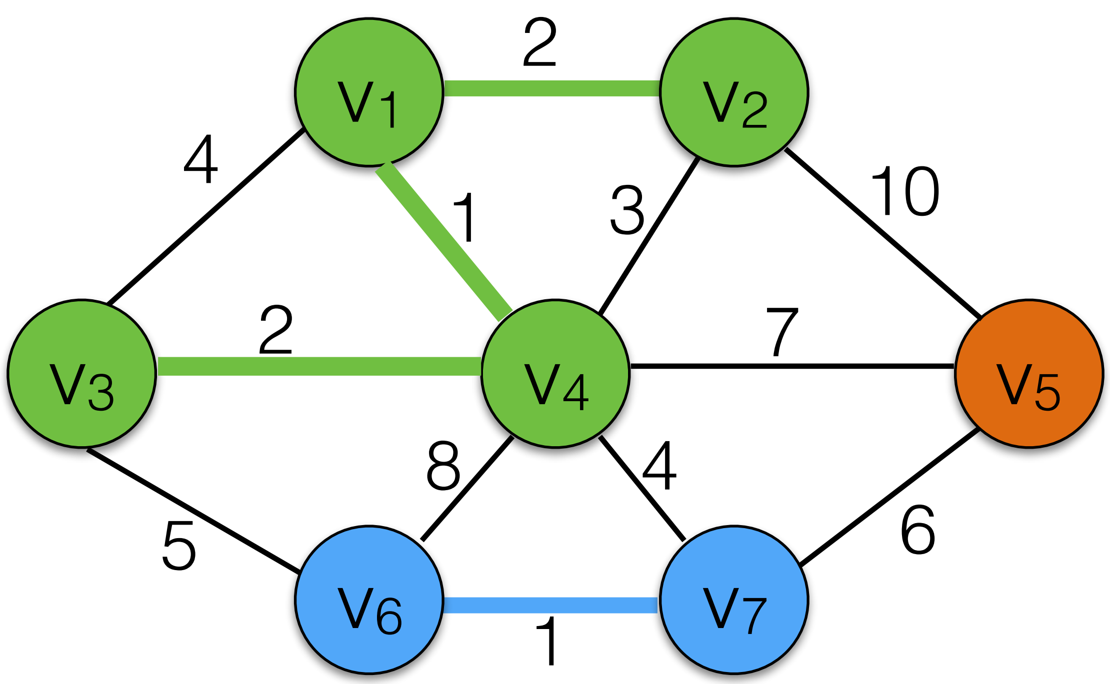
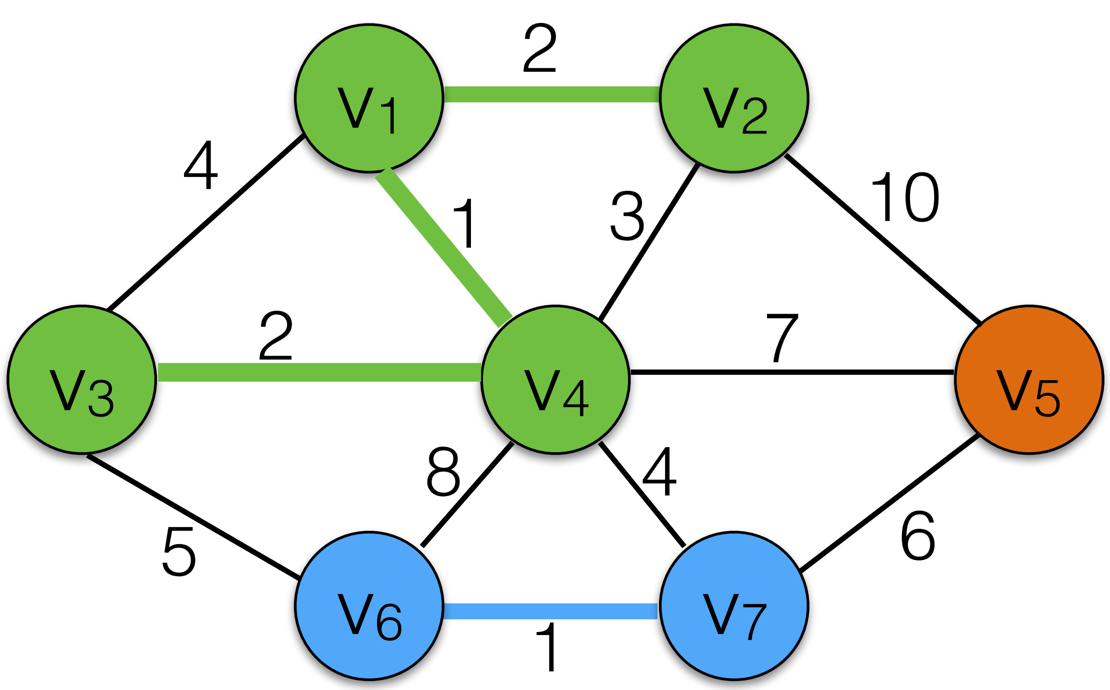
Kruskal's Algorithm



(v1,v4)	1	OK
(v6,v7)	1	OK
(v1,v2)	2	OK
(v3,v4)	2	OK
(v2,v4)	3	reject
(v1,v3)	4	
(v3,v6)	4	
(v4,v7)	4	
(v5,v7)	6	
(v4,v5)	7	
(v4,v6)	8	
(v2,v5)	10	

Kruskal's Algorithm



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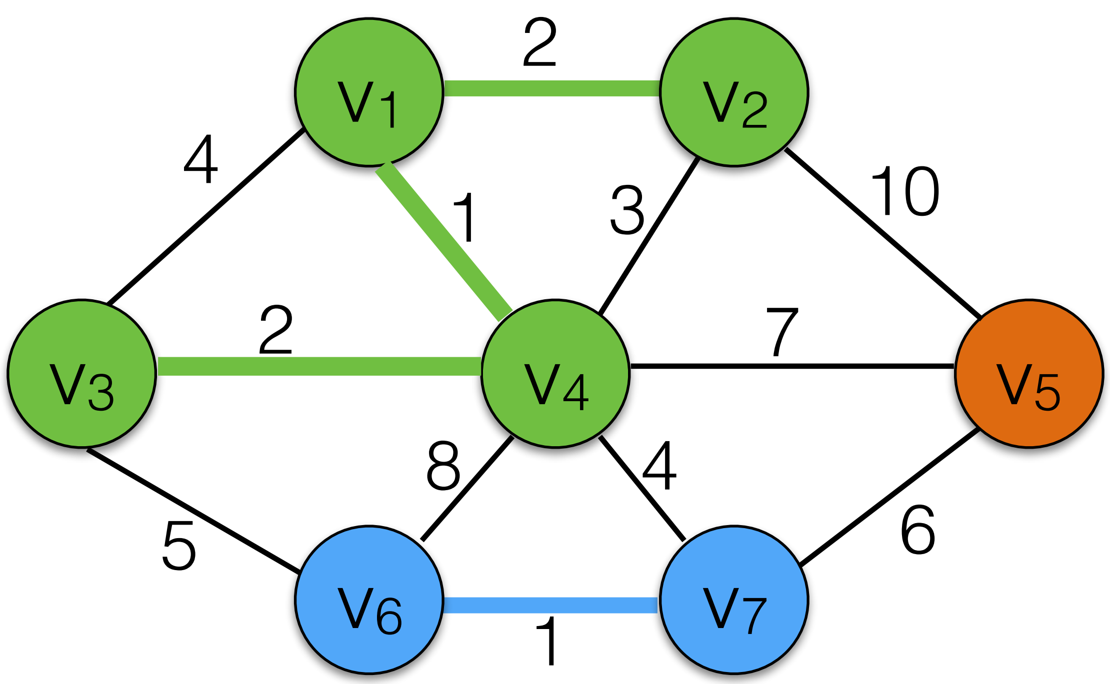
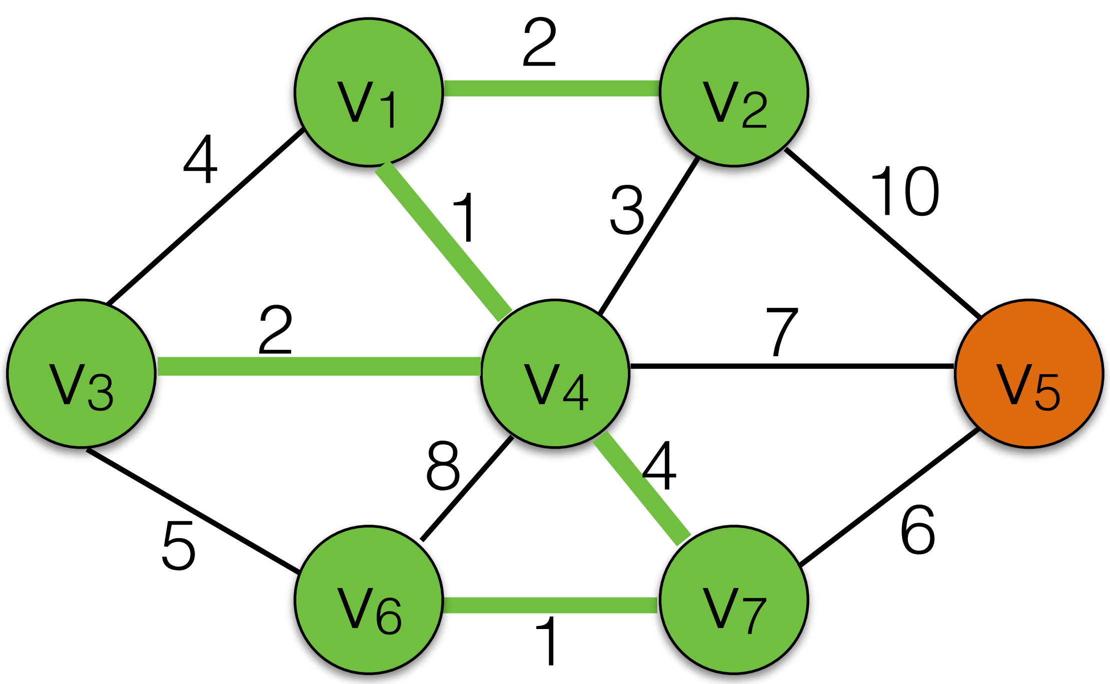
(v2,v4)	3	reject
(v1,v3)	4	reject
(v3,v6)	4	
(v4,v7)	4	
(v5,v7)	6	
(v4,v5)	7	
(v4,v6)	8	
(v2,v5)	10	

Kruskal's Algorithm



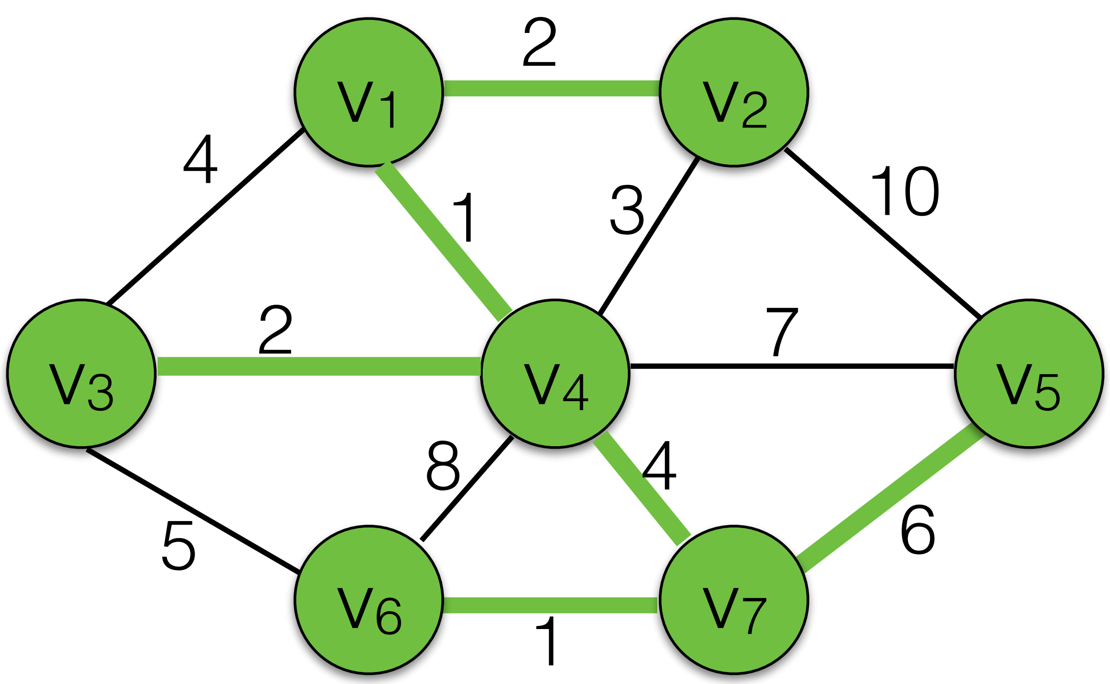
(v1,v4)	1	OK
(v6,v7)	1	OK
(v1,v2)	2	OK
(v3,v4)	2	OK
(v2,v4)	3	reject
(v1,v3)	4	reject
(v3,v6)	4	reject
(v4,v7)	4	
(v5,v7)	6	
(v4,v5)	7	
(v4,v6)	8	
(v2,v5)	10	

Kruskal's Algorithm



(v1,v4)	1	OK
(v6,v7)	1	OK
(v1,v2)	2	OK
(v3,v4)	2	OK
(v2,v4)	3	reject
(v1,v3)	4	reject
(v3,v6)	4	reject
(v4,v7)	4	OK
(v5,v7)	6	
(v4,v5)	7	
(v4,v6)	8	
(v2,v5)	10	

Kruskal's Algorithm



(v1,v4)	1	OK
(v6,v7)	1	OK
(v1,v2)	2	OK
(v3,v4)	2	OK
(v2,v4)	3	reject
(v1,v3)	4	reject
(v3,v6)	4	reject
(v4,v7)	4	OK
(v5,v7)	6	OK
(v4,v5)	7	
(v4,v6)	8	
(v2,v5)	10	

Implementing Kruskal's Algorithm

- Try to add edges one-by-one in increasing order. Build a heap in $O(|E|)$. Each deleteMin takes $O(\log |E|)$
- How to maintain the forest?
  - Represent each tree in the forest as a set.
  - When adding an edge, check if both vertices are in the same set. If not, take the union of the two sets.
  - This can be done efficiently using a *disjoint set* data structure *(Weiss Chapter 8).*

Total turns out to be: $O(|E| \log |V|)$

Application: Hierarchical Clustering

- This is a very common data analysis problem.
- Group together data items based on similarity (defined over some feature set).
- Discover classes and class relationships.

Zoo Data Set

101 animals

represent each data item as a vector of integers (15 attributes).

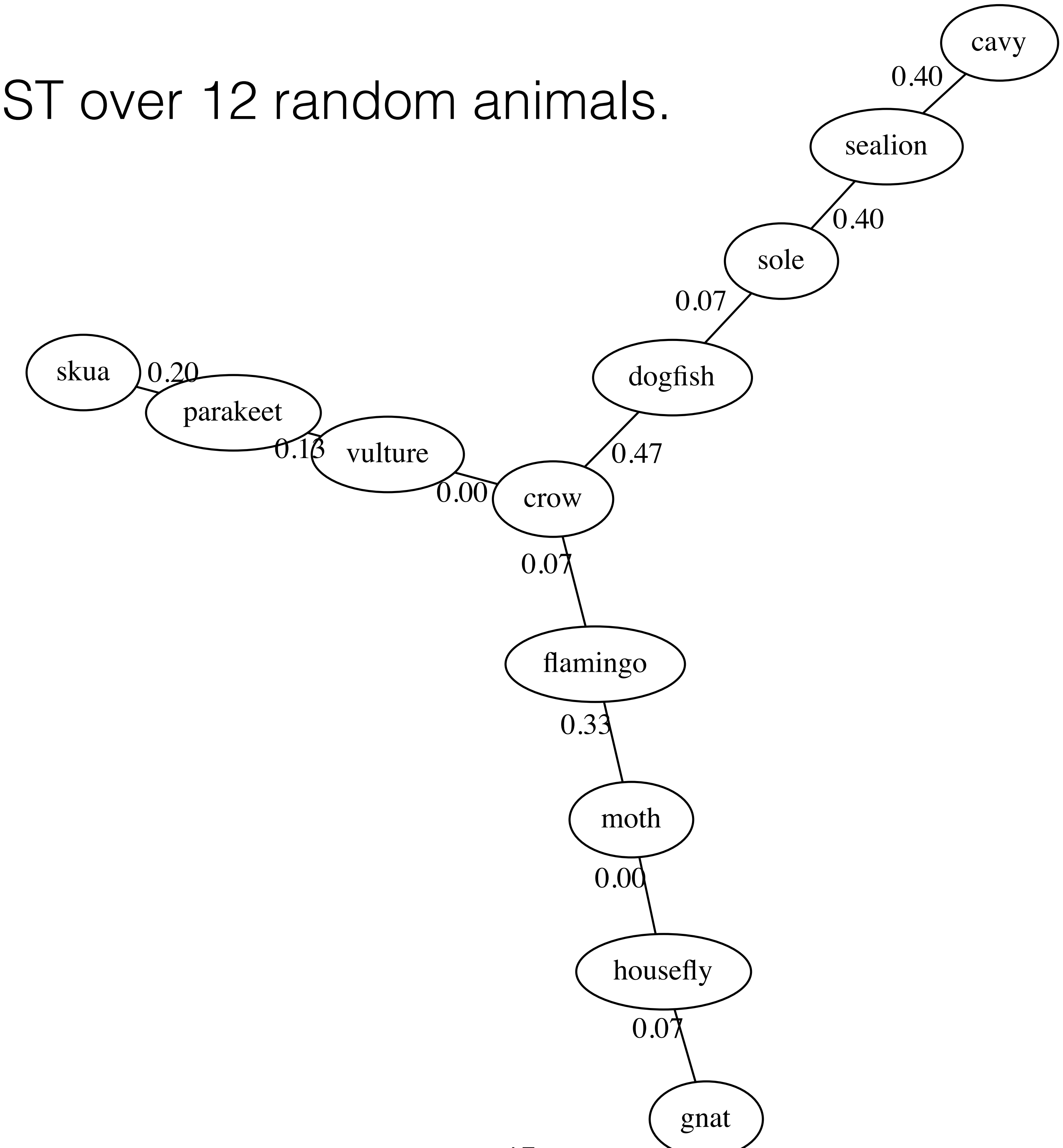
	bear	chicke	tortoise	flea
hair	1	0	0	0
feathers	0	1	0	0
eggs	0	1	1	1
milk	1	0	0	0
airborne	0	1	0	0
aquatic	0	0	0	0
predator	1	0	0	0
toothed	1	0	0	0
backbone	1	1	1	0
breathes	1	1	1	1
venomou	0	0	0	0
fins	0	0	0	0
legs	4	2	4	6
tail	0	1	1	0
domestic	0	1	0	0

...

https://archive.ics.uci.edu/ml/datasets/Zoo

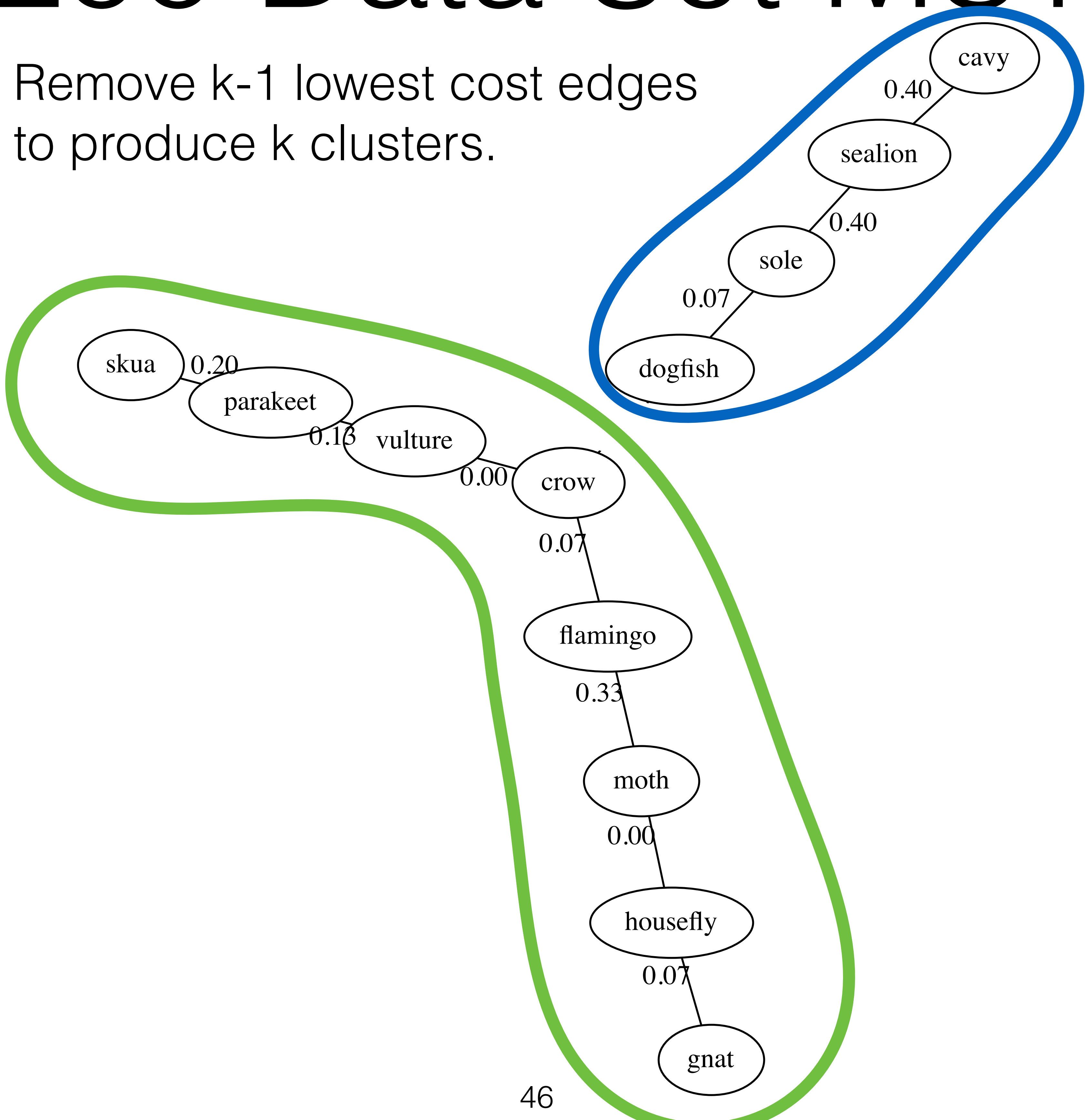
Zoo Data Set MST

- MST over 12 random animals.



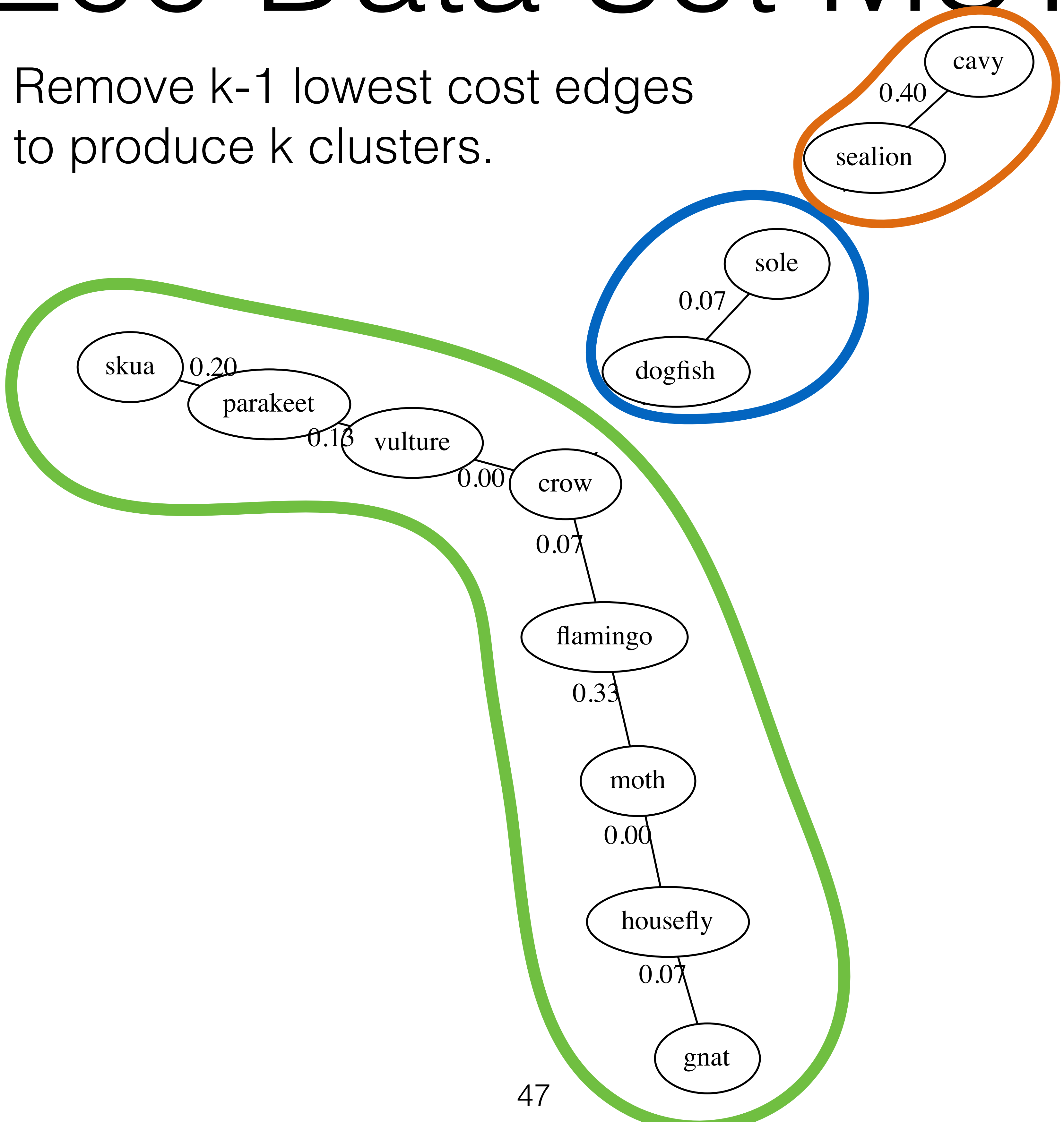
Zoo Data Set MST

- Remove k-1 lowest cost edges to produce k clusters.



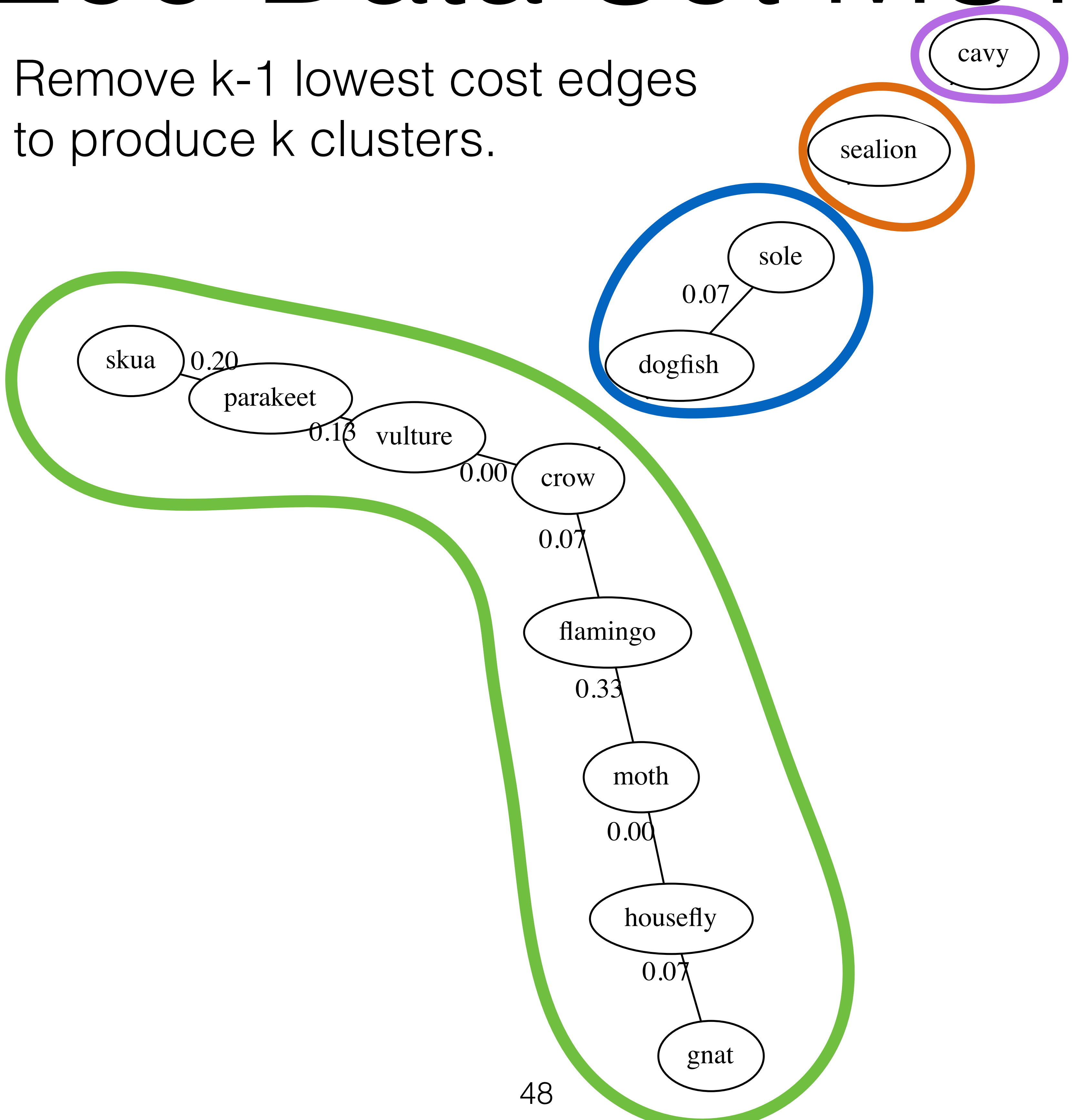
Zoo Data Set MST

- Remove k-1 lowest cost edges to produce k clusters.



Zoo Data Set MST

- Remove k-1 lowest cost edges to produce k clusters.



Zoo Data Set MST

- Remove k-1 lowest cost edges to produce k clusters.