Data Structures in Java

Lecture 17: Traversing Graphs. Shortest Paths.

10/18/2015

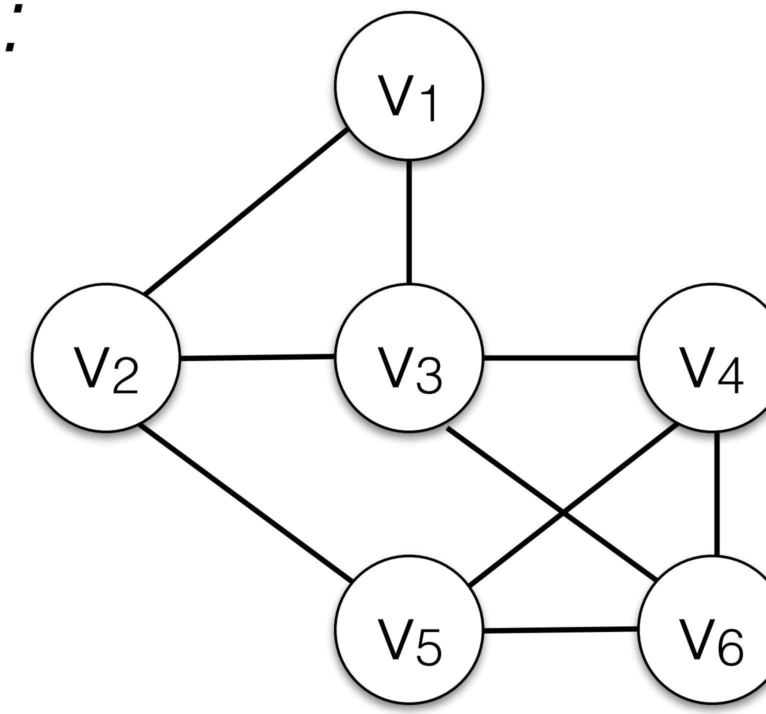
Daniel Bauer

Today: Graph Traversals

- Depth First Search (a generalization of pre-order traversal on trees to graphs, users a Stack)
- Breadth First Search (uses a Queue)
- Dijkstra's algorithm to find weighted shortest paths (uses a Priority Queue)
- Topological sort for Directed Acyclic Graphs.
  - Application: Shortest Project Completion Time.

Graphs

- A **Graph** is a pair of two sets *G=(V,E):*
  - V: the set of **vertices** (or **nodes**)
  - E: the set of **edges**.
    - each edge is a pair (v,w) where v,w $\in$ V

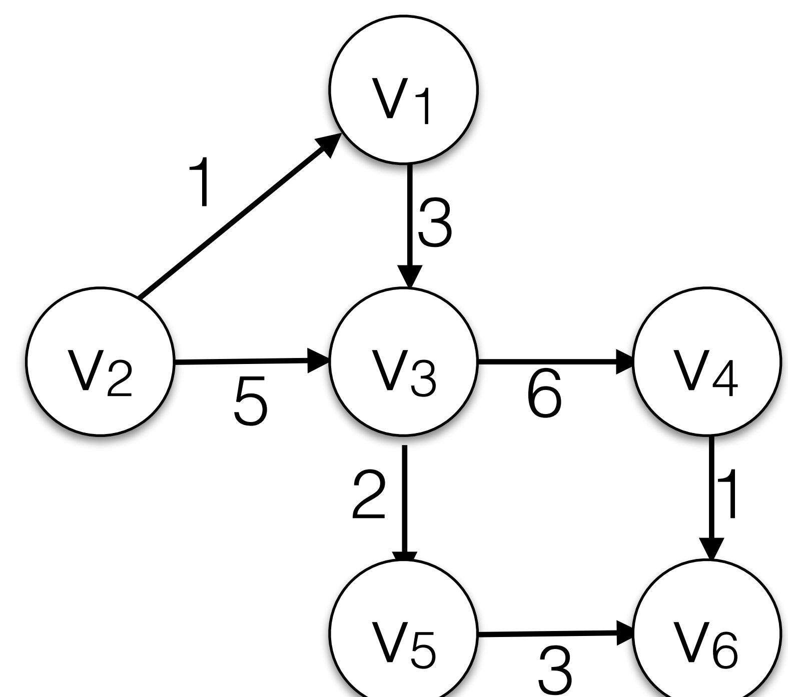


$$V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$$
$$E = \{(v_1, v_2), (v_1, v_3), (v_2, v_3),(v_2, v_5),(v_3, v_4), (v_3, v_6),(v_4, v_5), (v_4, v_6), (v_5, v_6)\}$$

Edges

- Graphs may be **directed** or **undirected**.
  - In directed graphs, the edge pairs are ordered.

- Edges often have some weight or cost associated with them (**weighted** graphs).



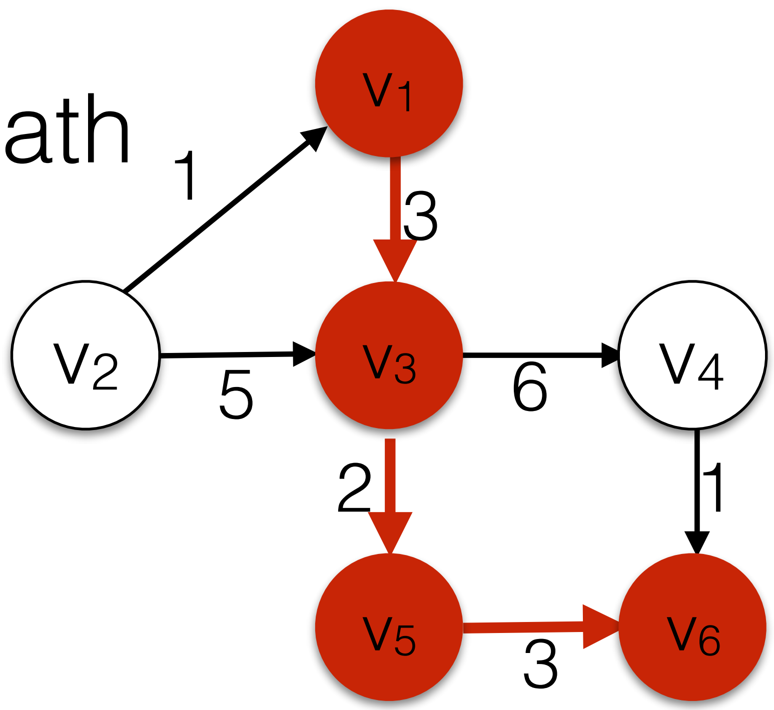
$V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$

$E = \{(v_1, v_3), (v_2, v_1), (v_2, v_3), (v_3, v_4), (v_3, v_5), (v_4, v_6), (v_5, v_6)\}$

directed and weighted graph

Paths

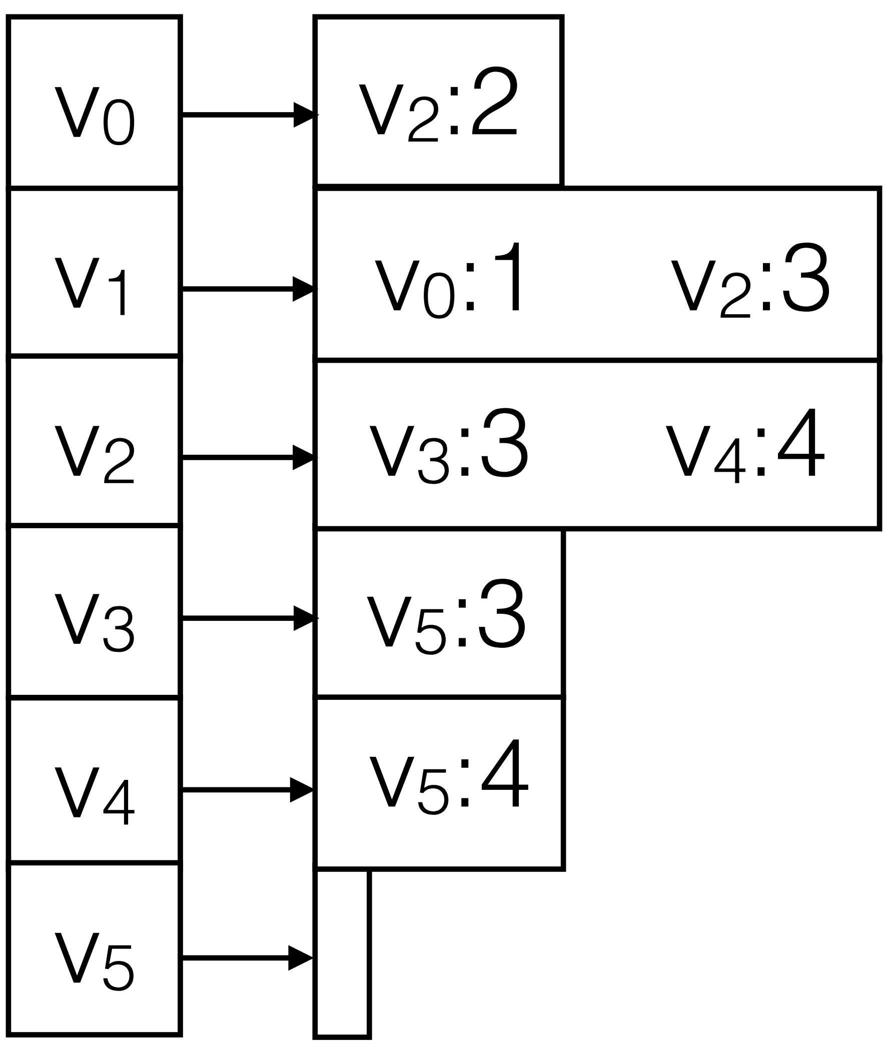
- Vertex w is **adjacent** to vertex v iff $(w,v) \in E$.
- A **path** is a sequence of vertices $w_1, w_2, \ldots, w_k$ such that $(w_i, w_{i+1}) \in E$.
- **length** of a path:
  k-1 = number of edges on path
- **cost** of a path:
  Sum of all edge costs.

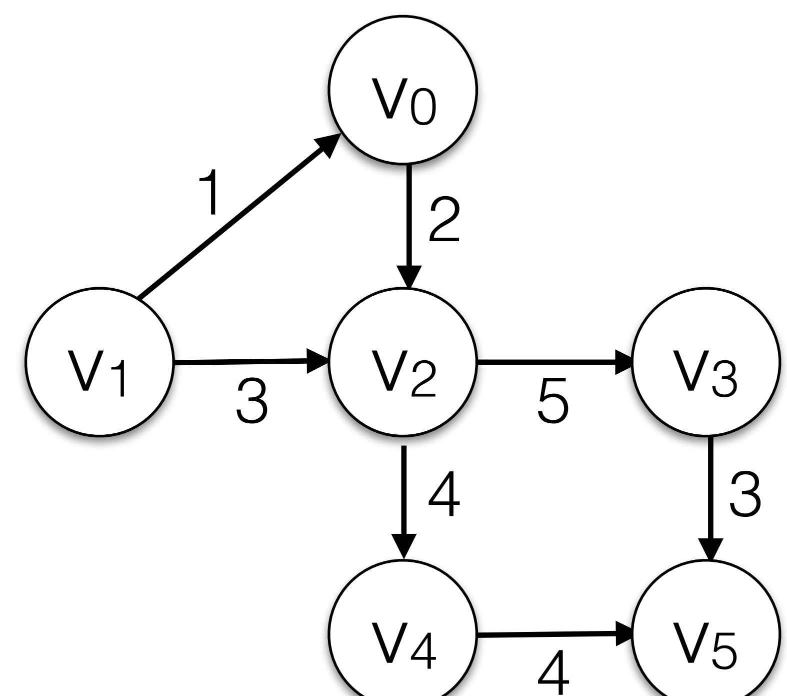


Path from v_1 to v_6, length 3, cost 8
$(v_1, v_3), (v_3, v_5), (v_5, v_6)$

Representing Graphs

- Represent graph G = (E,V), option 2: **Adjacency Lists**
  - For each vertex, keep a list of all adjacent vertices.





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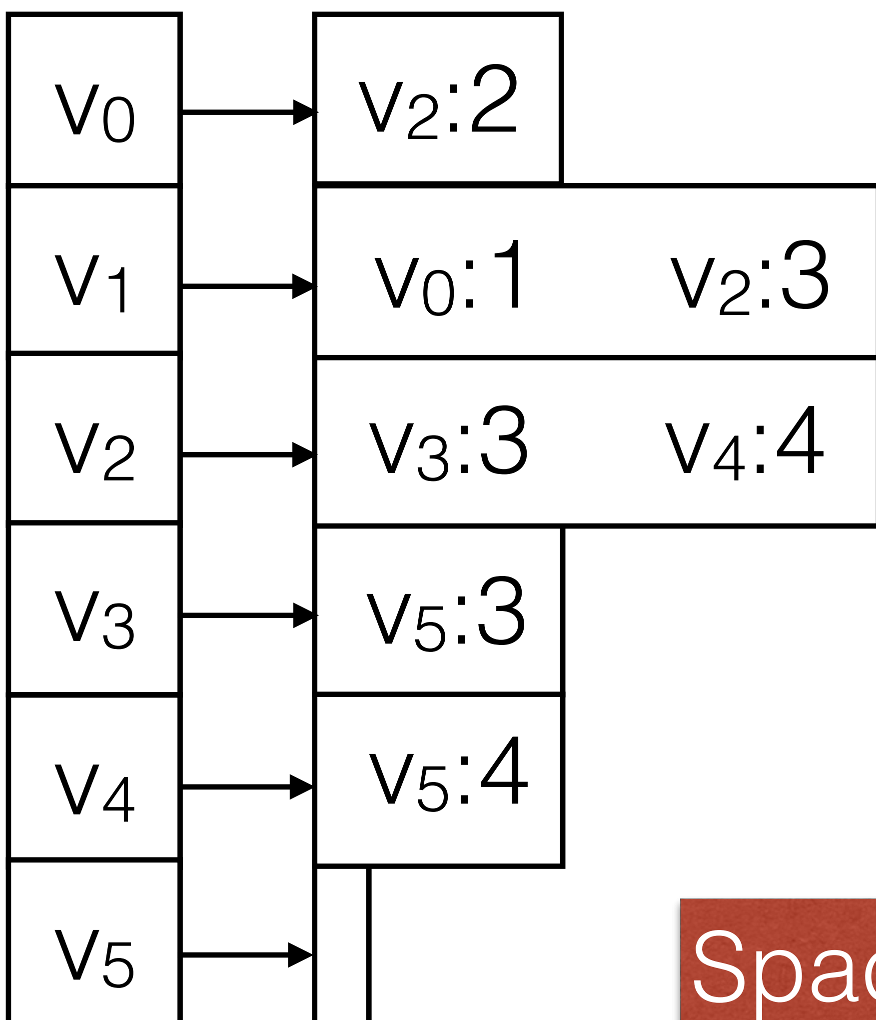
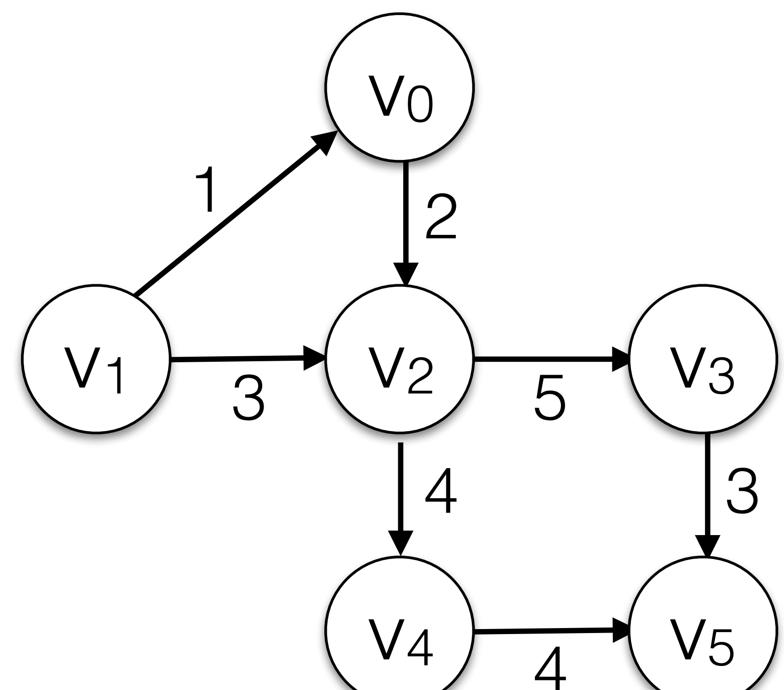
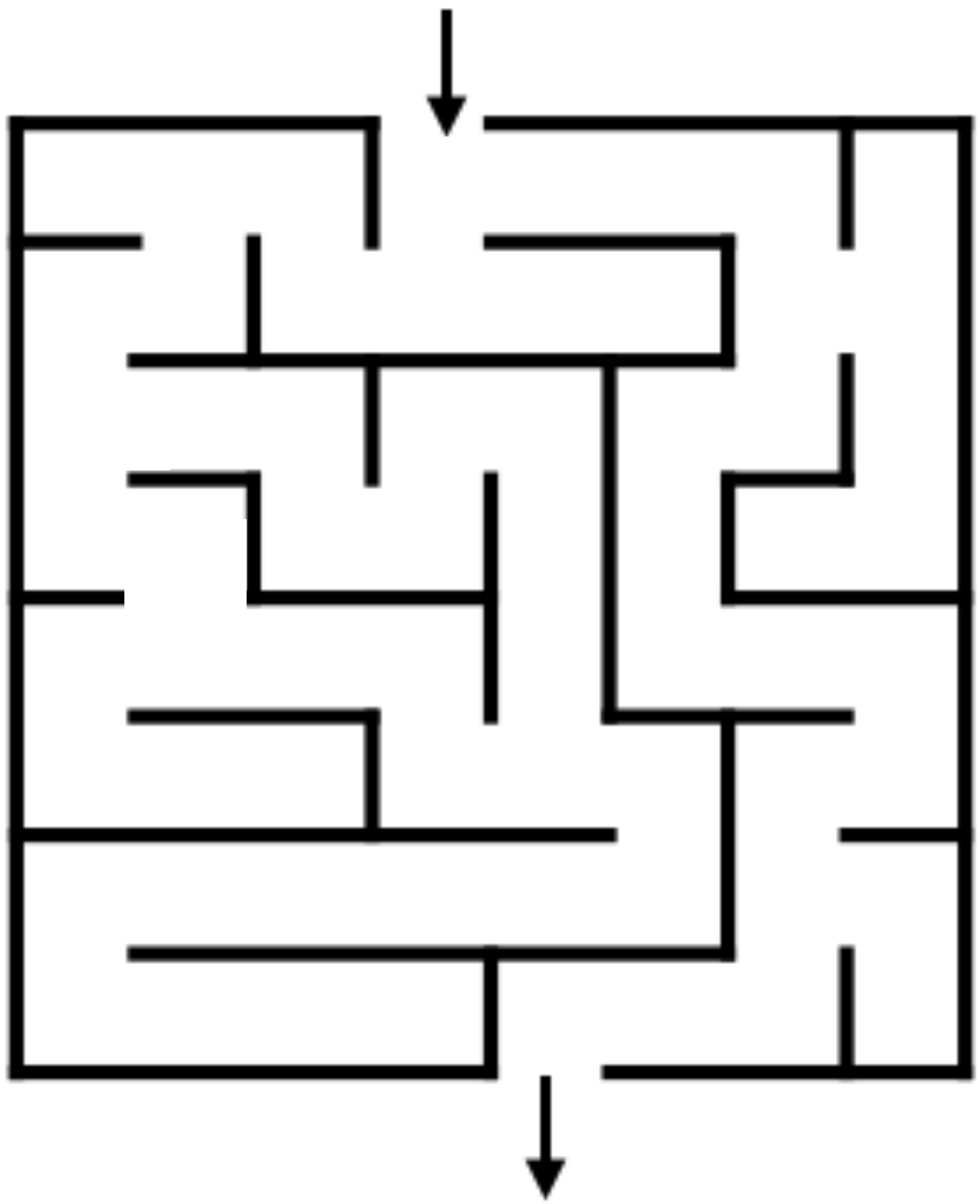




Space requirement: $\Theta(|V| + |E|)$

Graph Search



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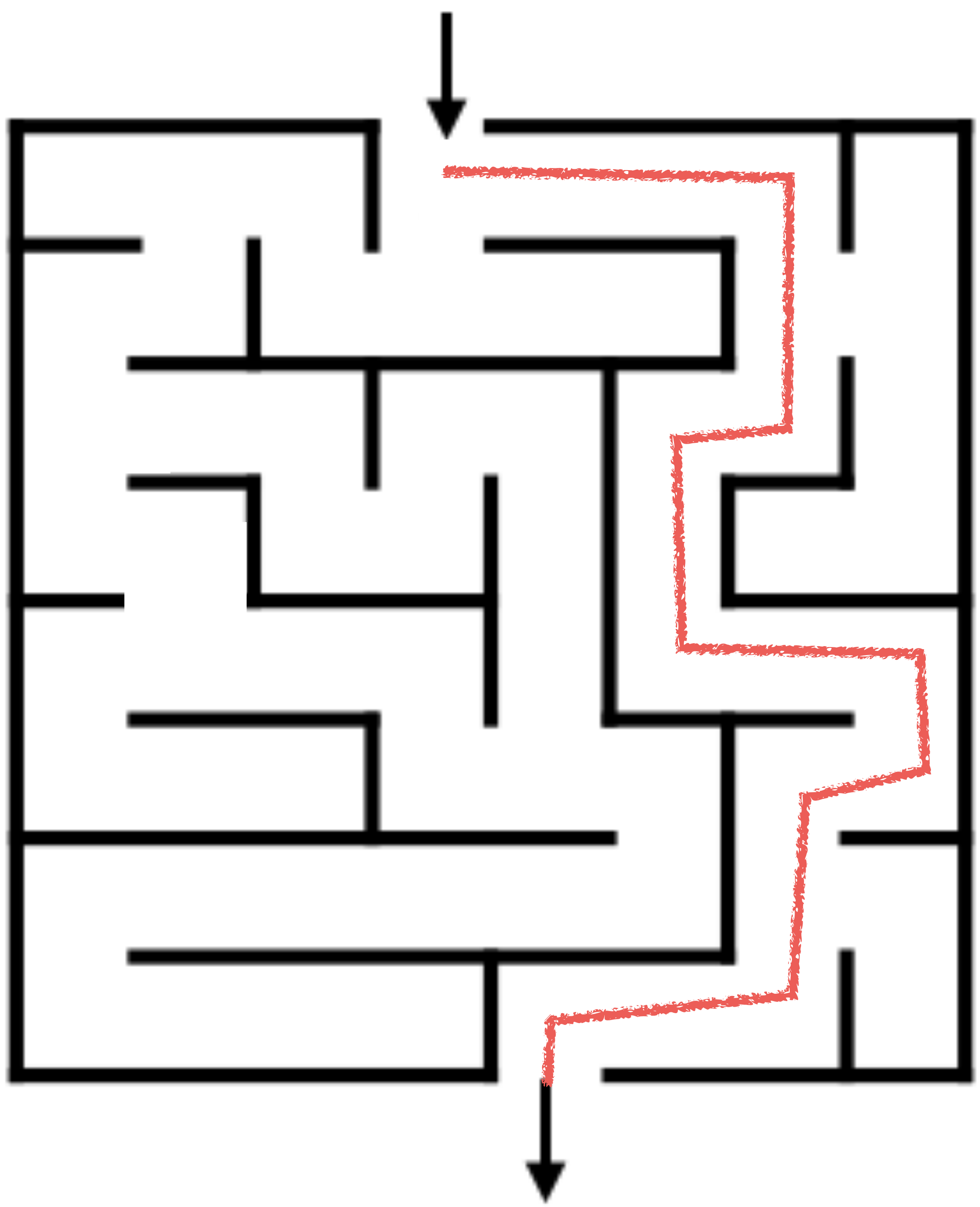

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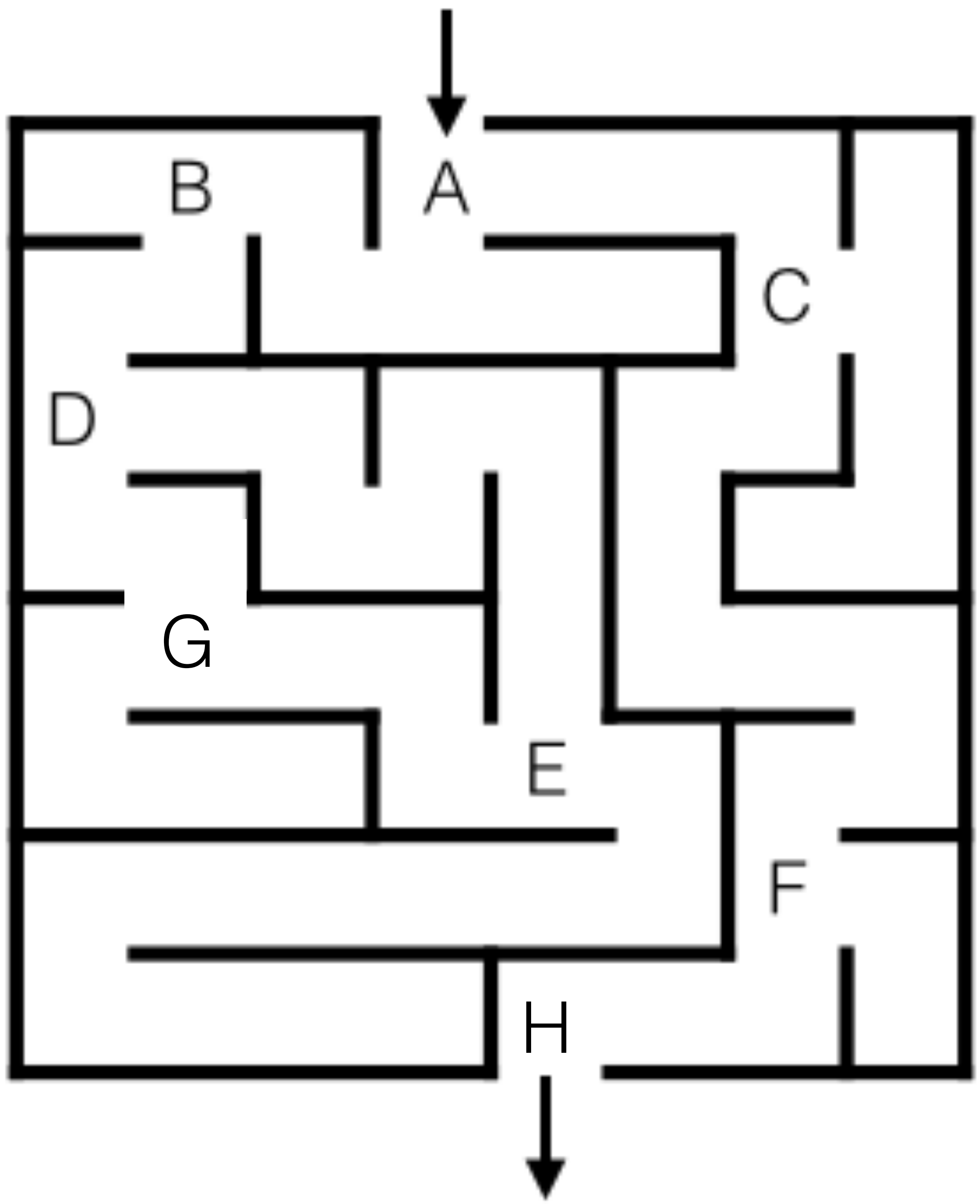
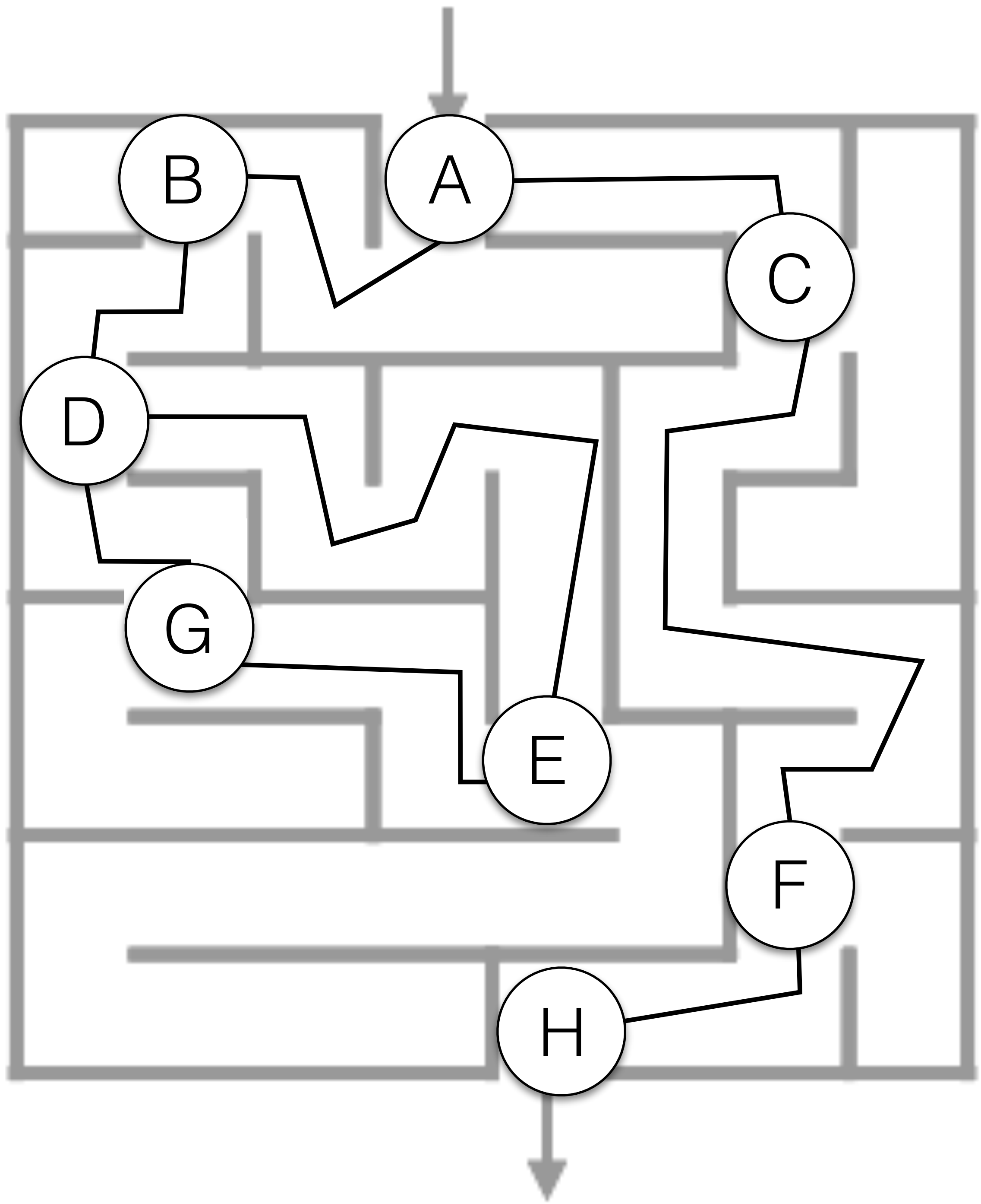


Letters indicate junctions where a decision must be made.

Graph Search



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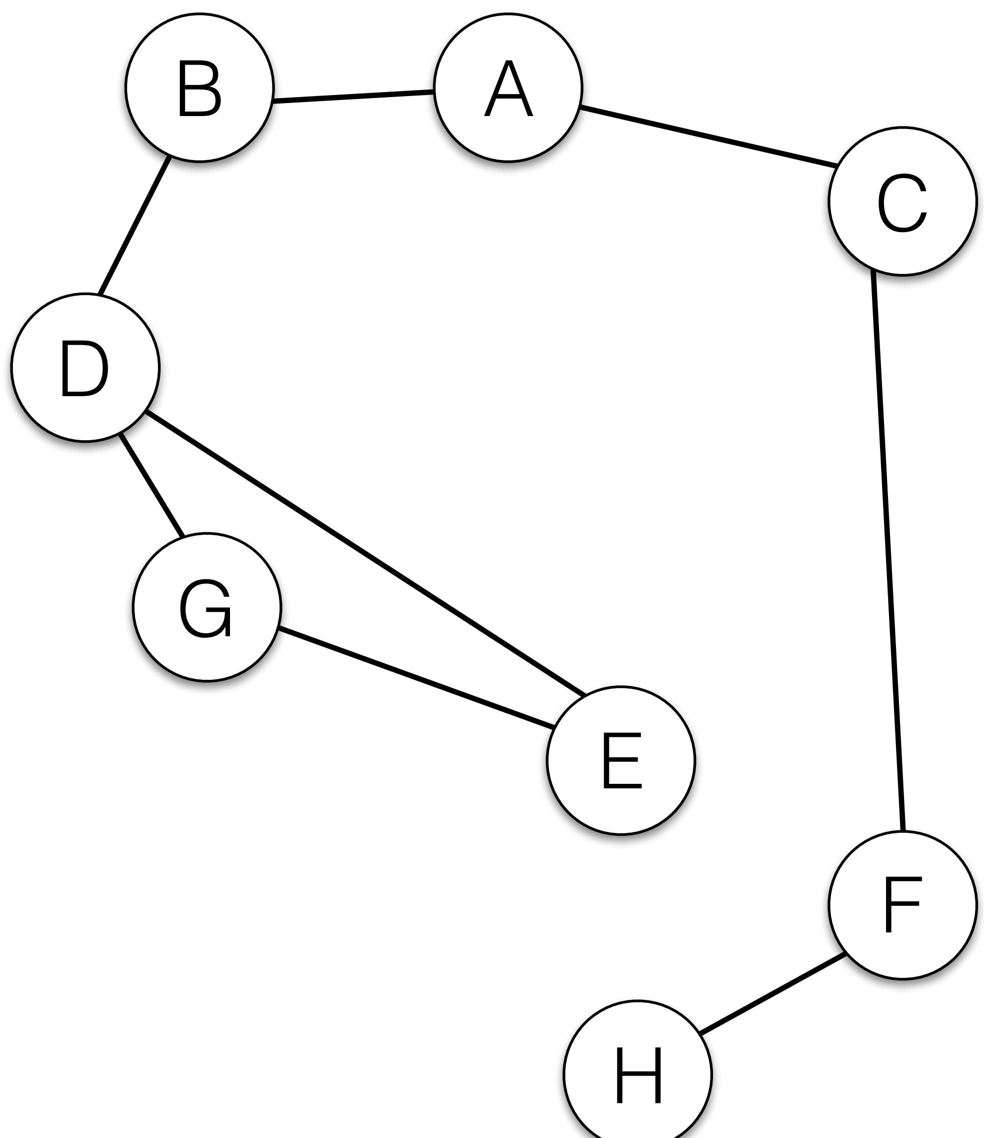
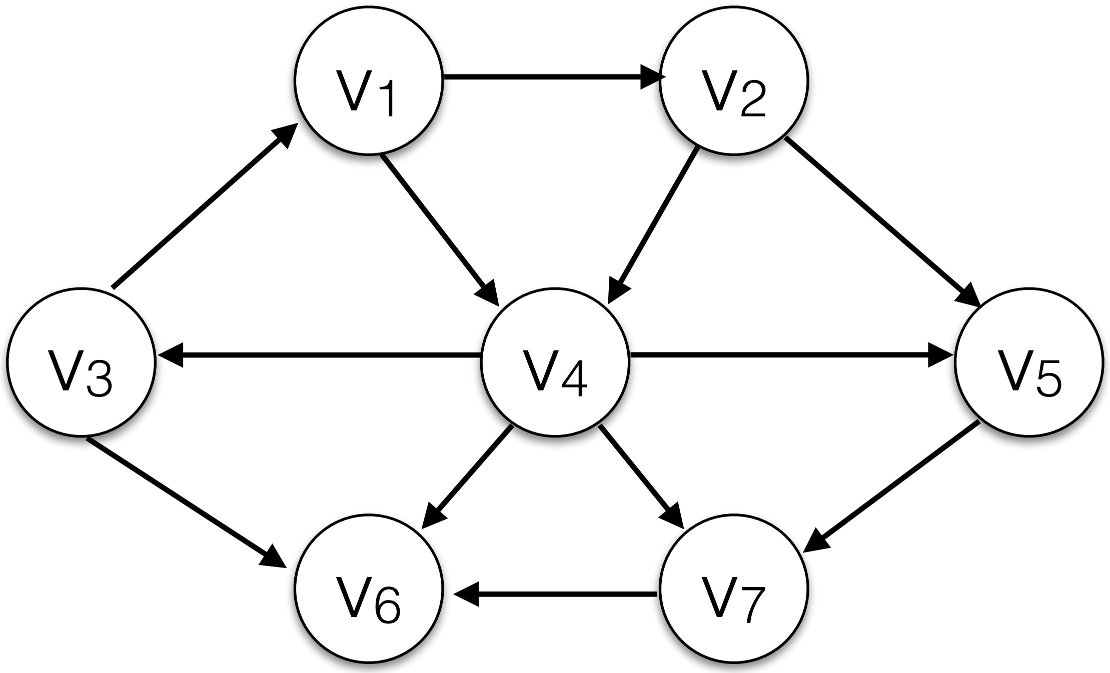


Graph Search: Depth First Search (DFS)

- Goal: Systematically explore the graph, starting at vertex *s* (source) touching all edges.
- Graph Traversals are the core ingredient of most graph algorithms.



Use a stack.

- Push *s* to the stack.
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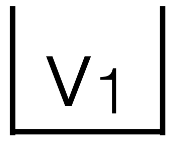

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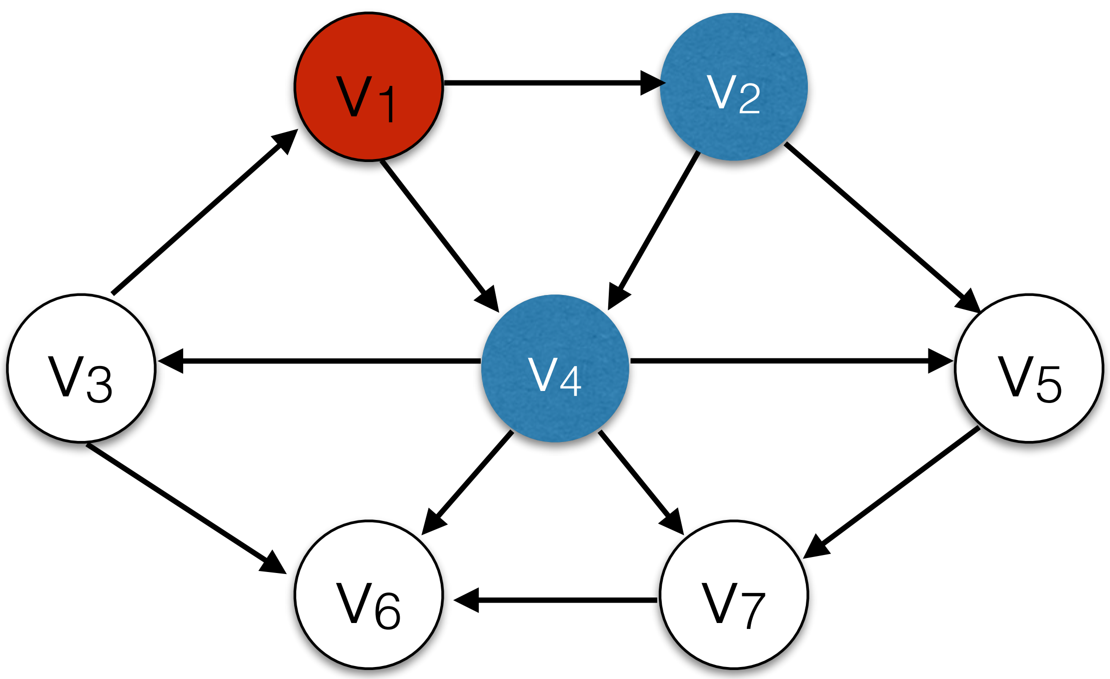

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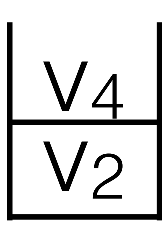

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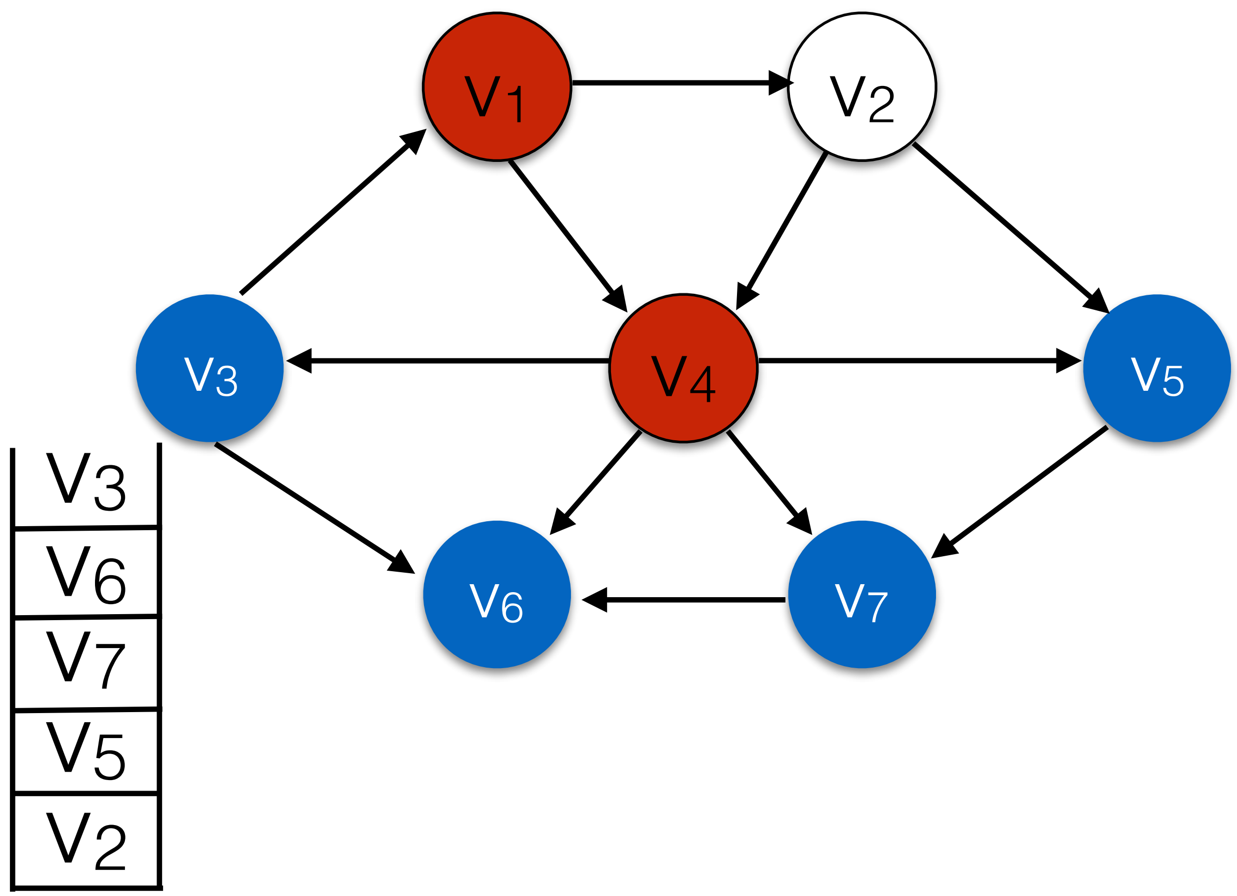

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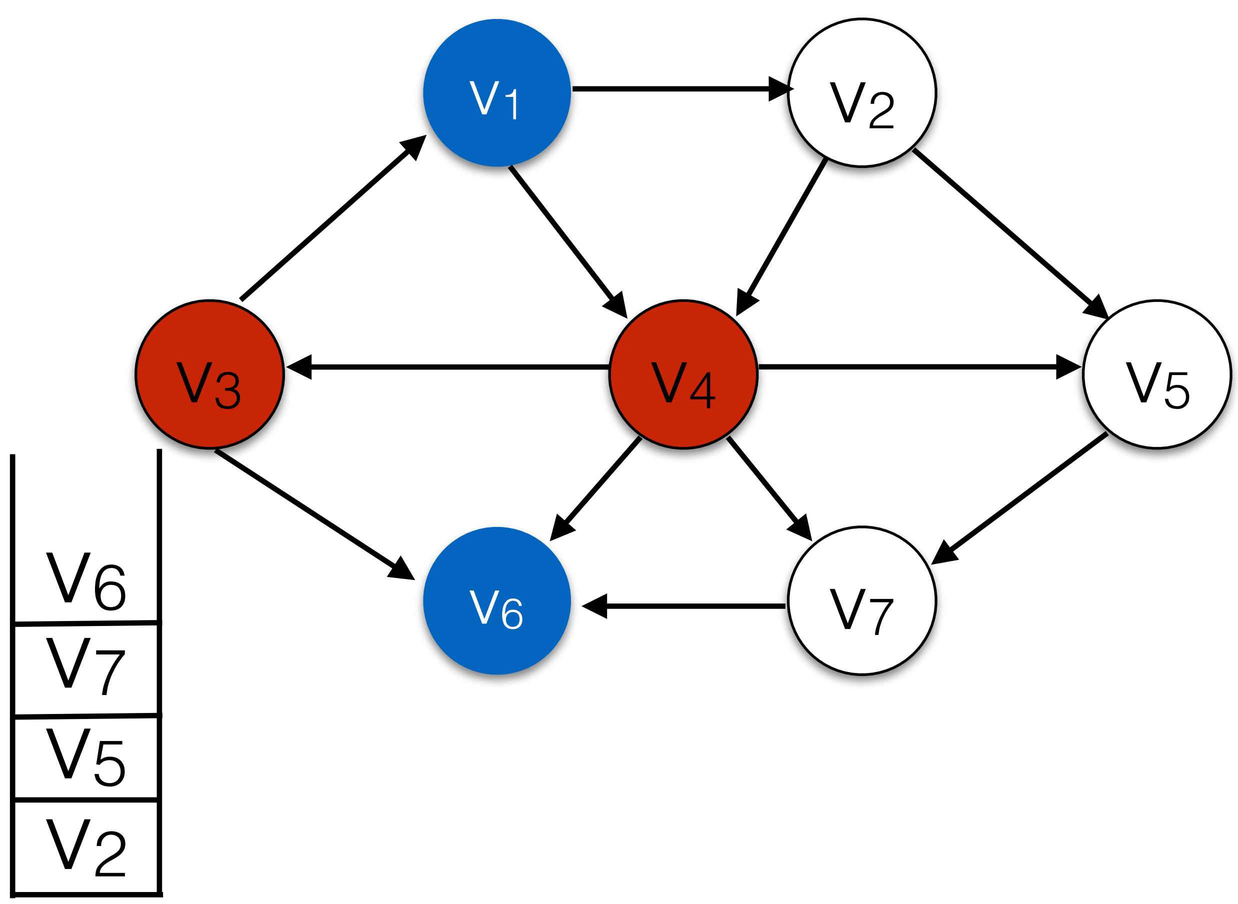

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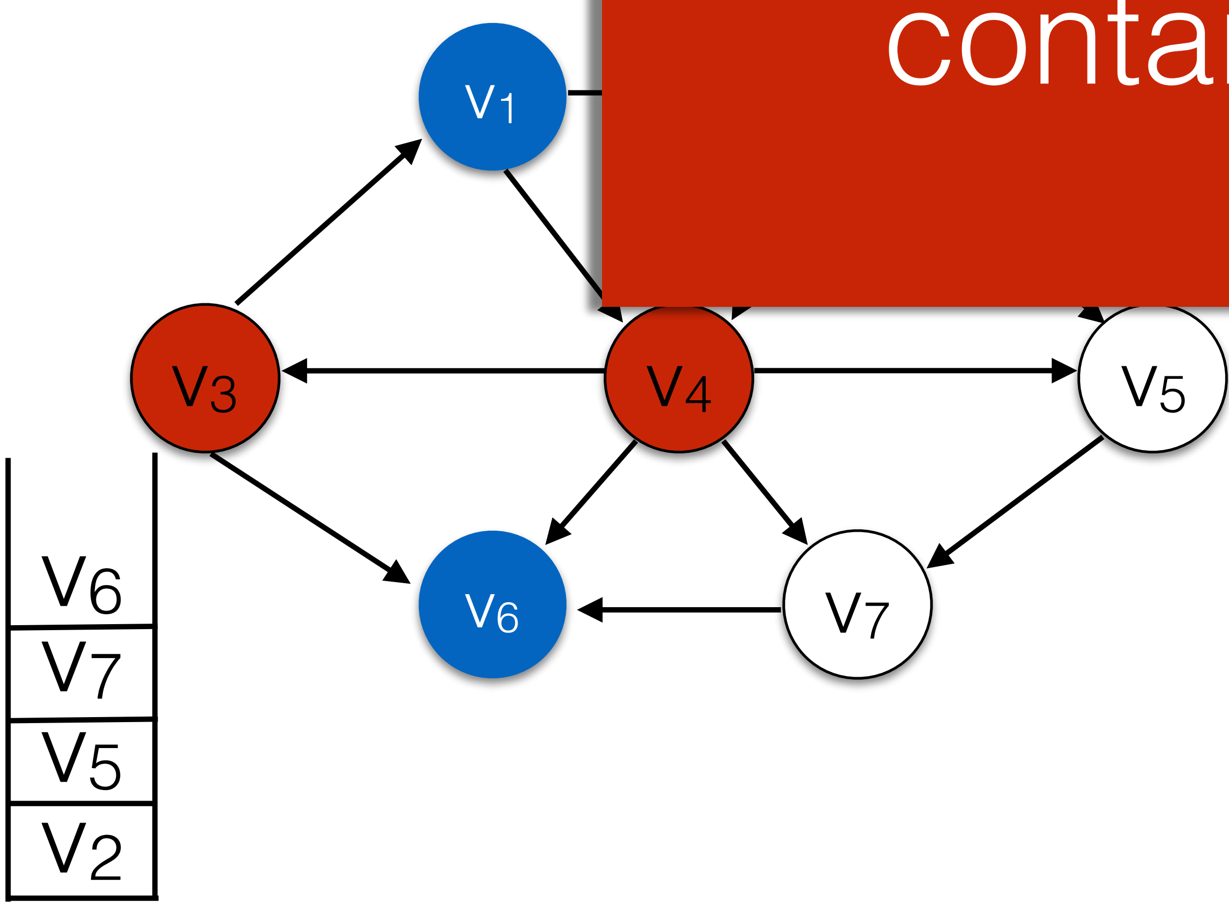
Problem: This Graph contains cycles!



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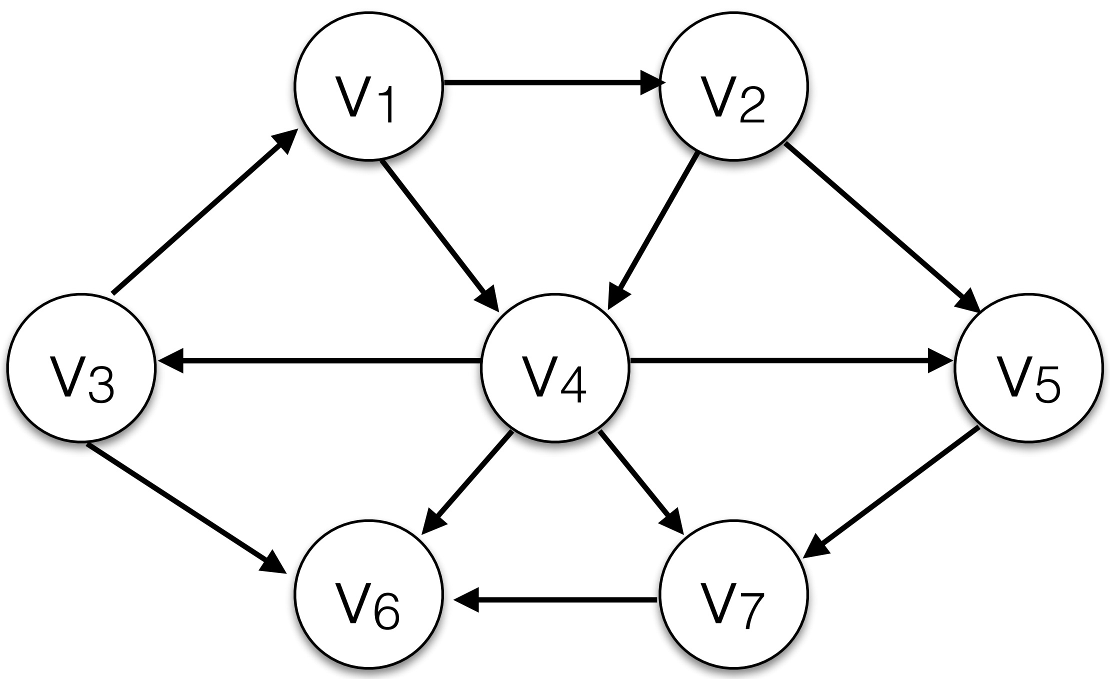
Depth First Search (DFS) with *Visited* Set



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v_1

visited {}

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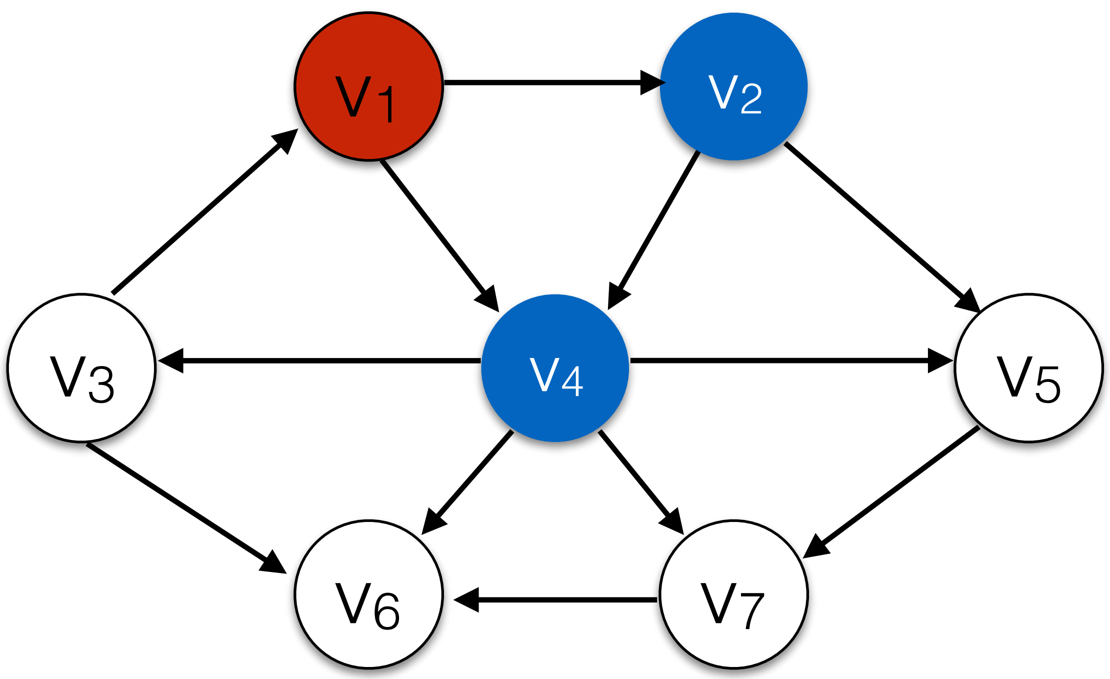

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v_4
v_2

Visited: $\{v_1\}$

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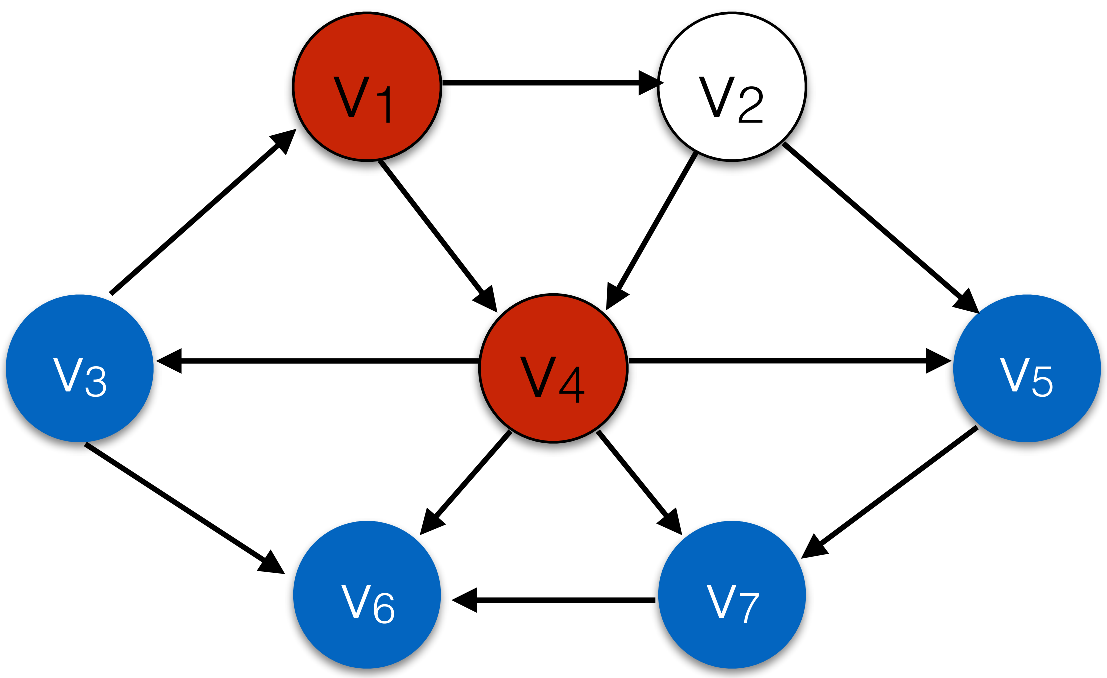

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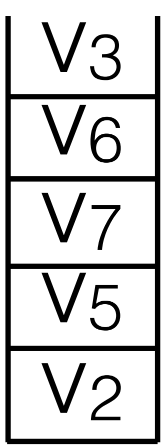

Visited: $\{v_1, v_4\}$

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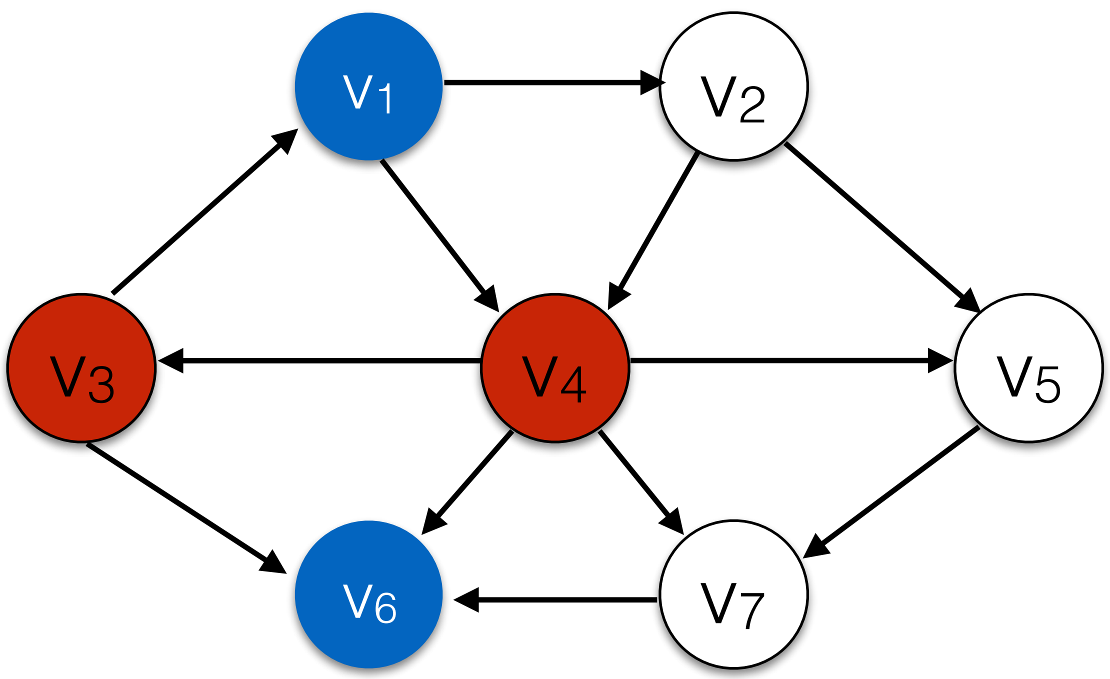

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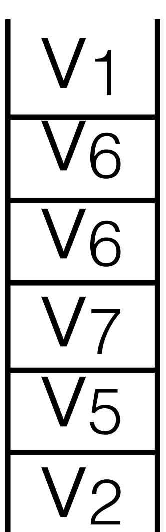

Visited: $\{v_1, v_4, v_3\}$

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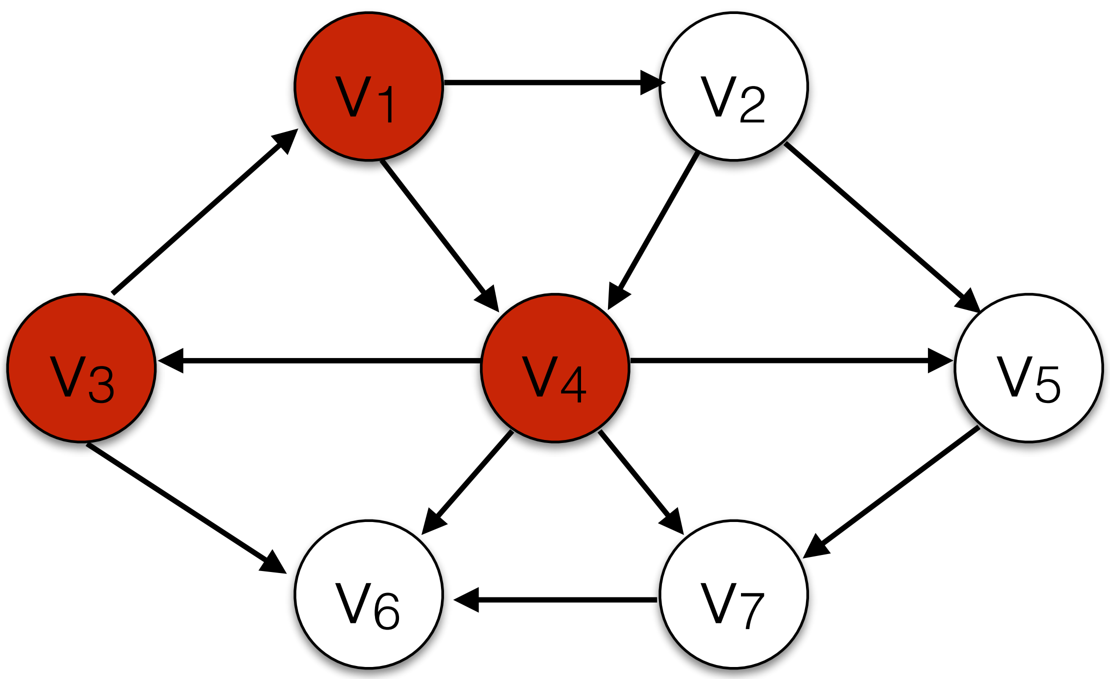

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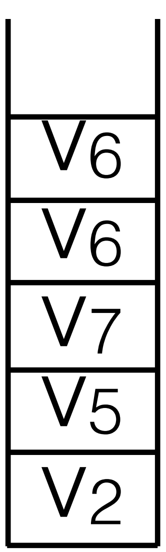

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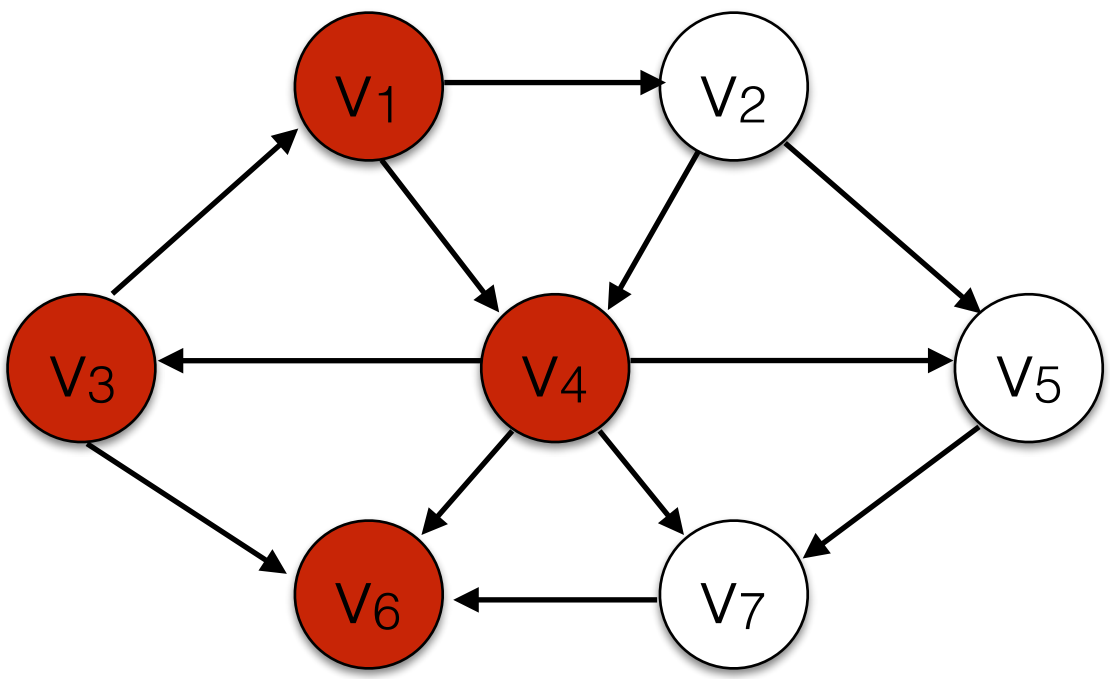

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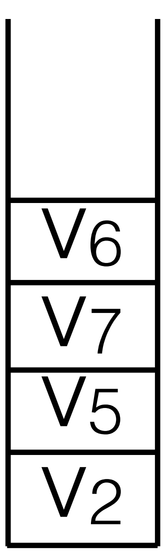

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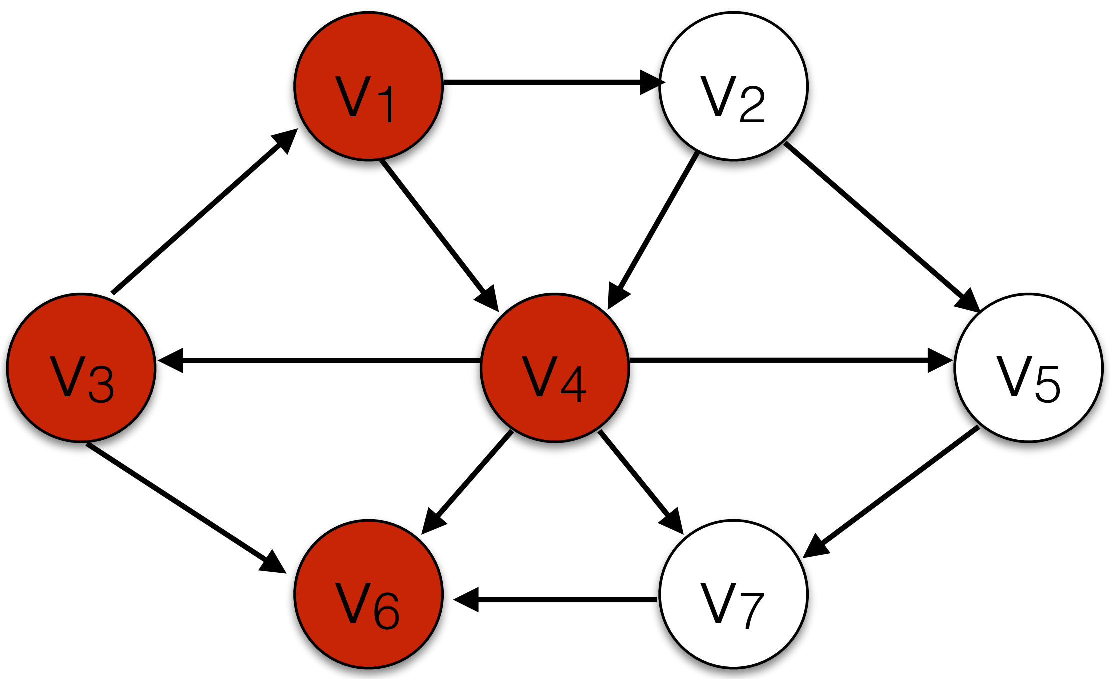

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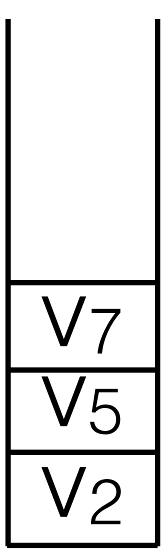

Visited: {v_1,v_4,v_3,$\mathbf{v_6}$}

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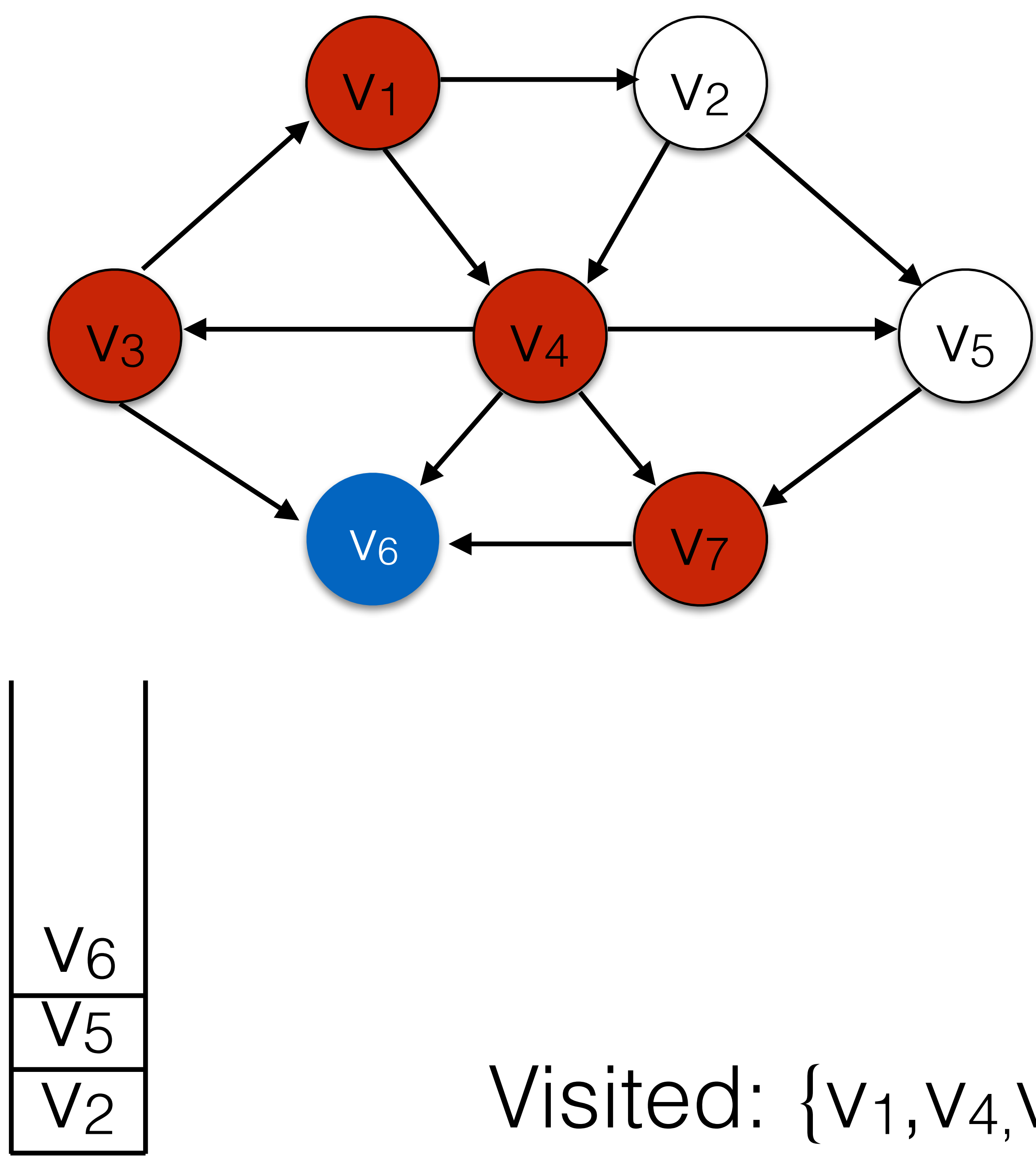

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Visited: $\{v_1, v_4, v_3, v_6, v_7\}$

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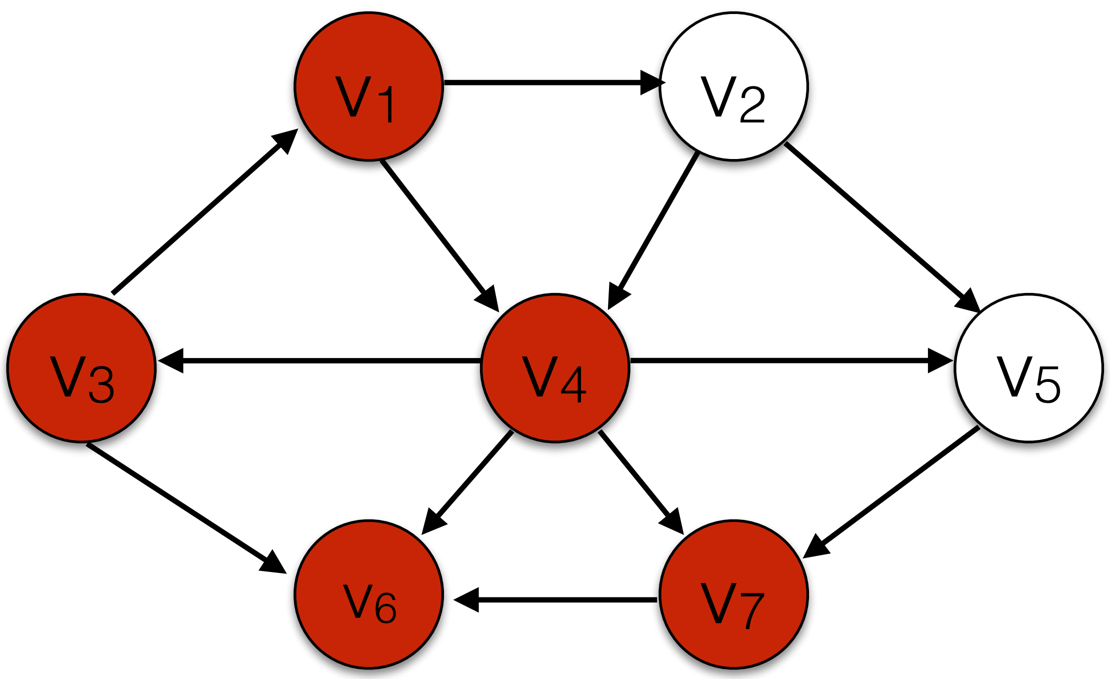

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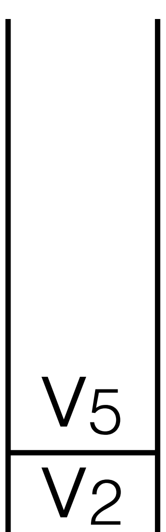

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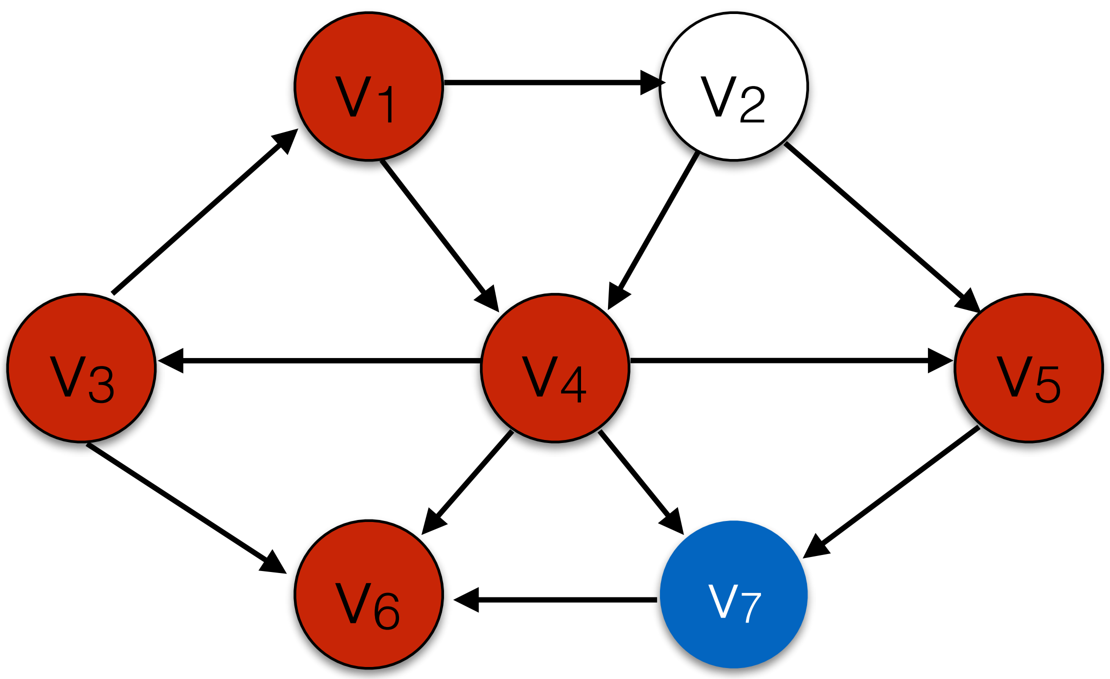

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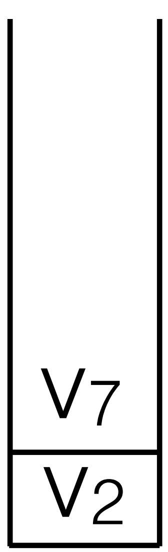

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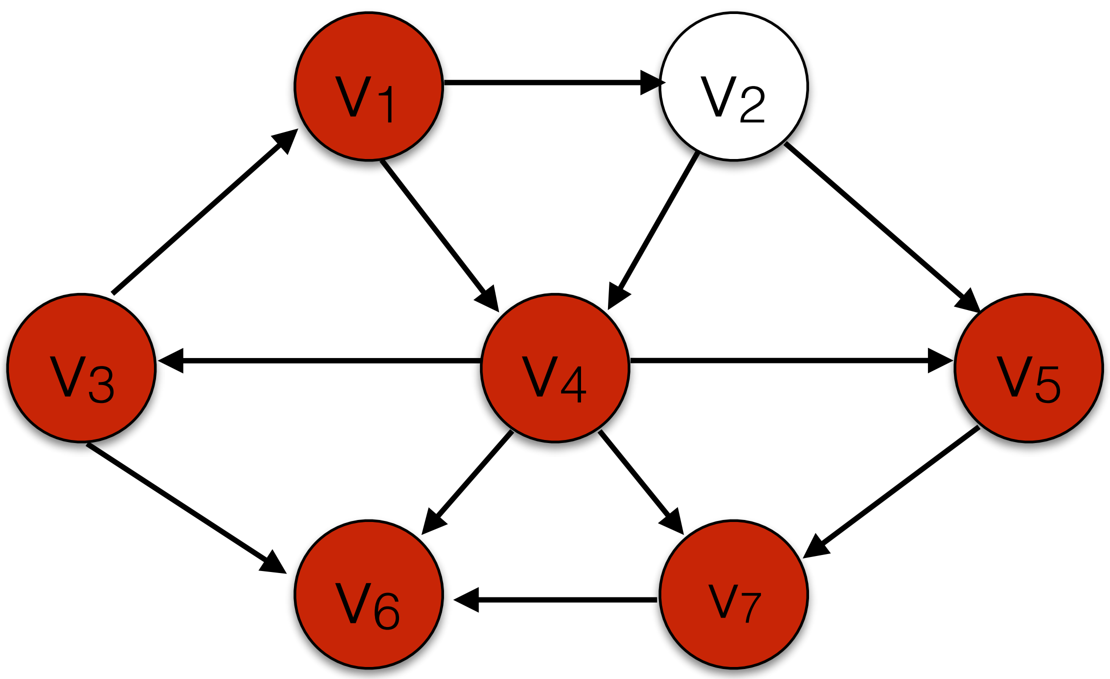

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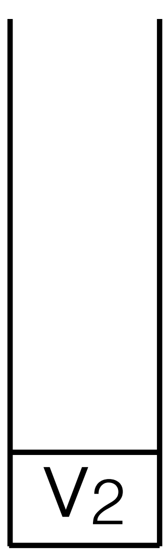

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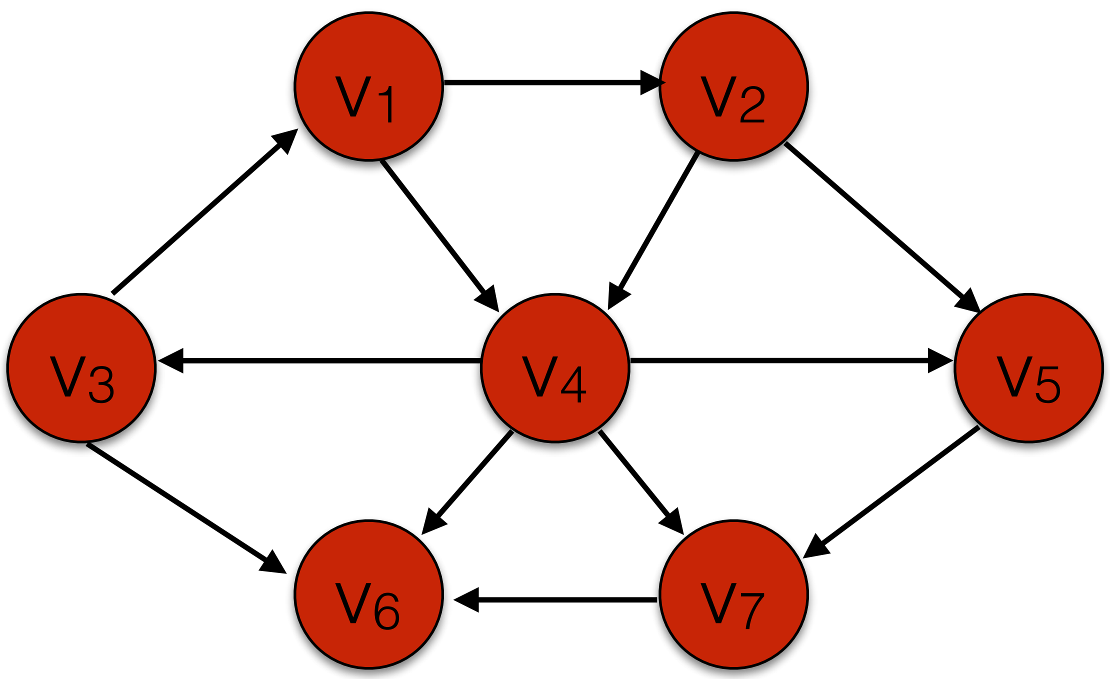

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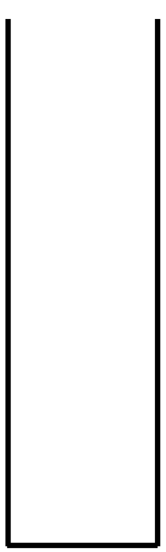

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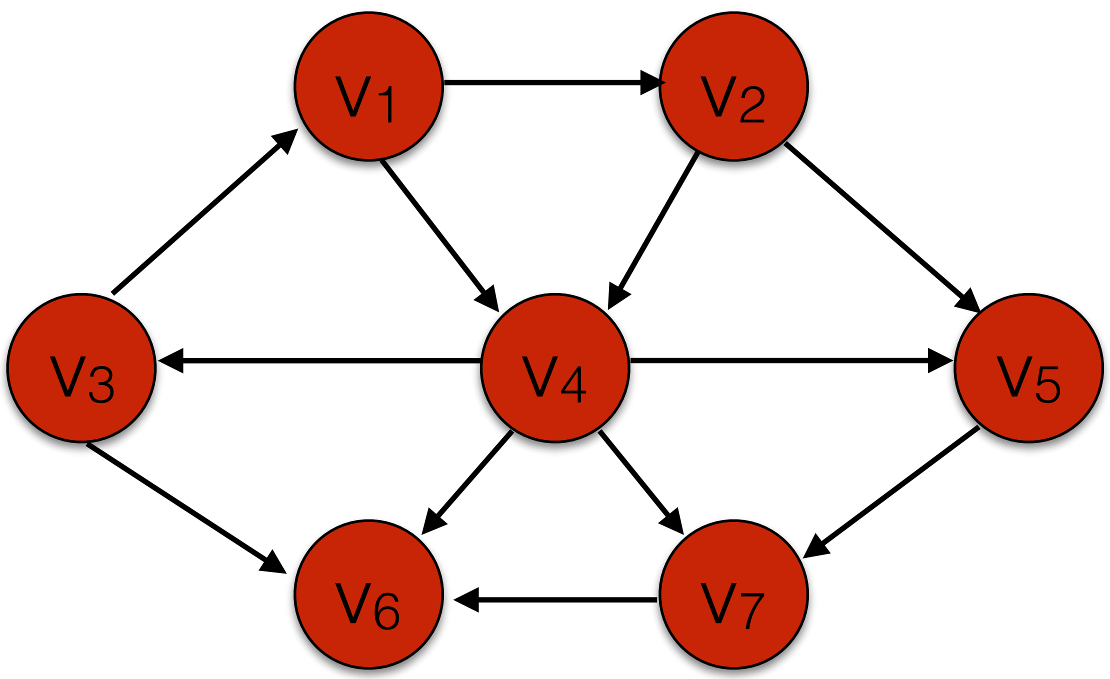

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Running time: O(|E|)

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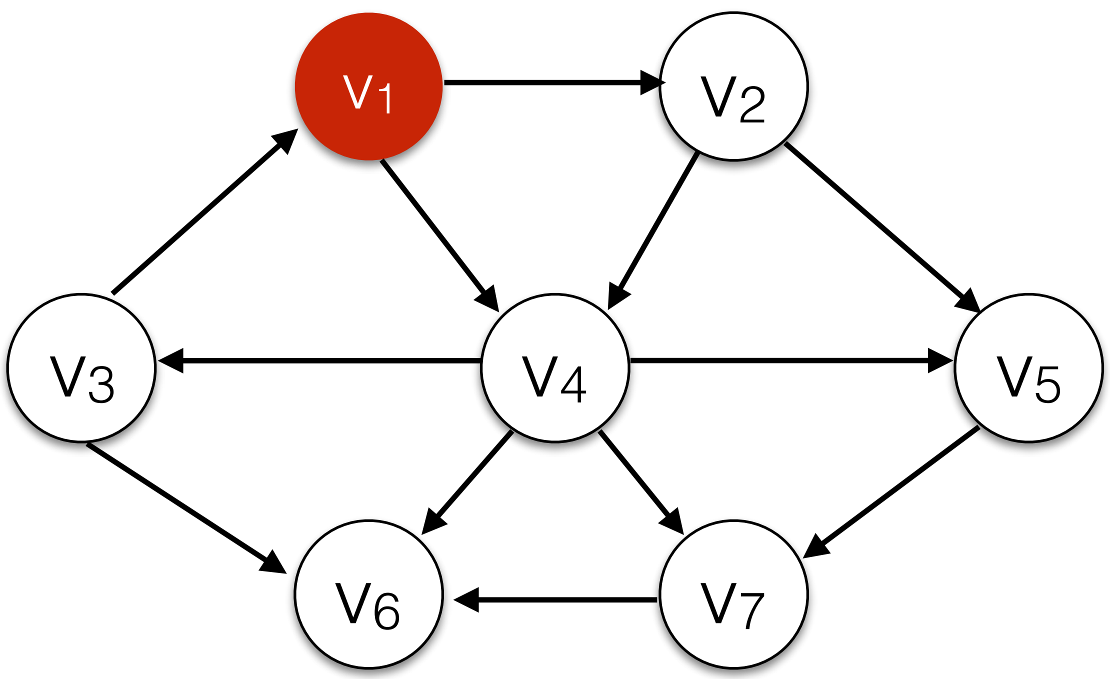
Recursive DFS (with *visited* marker kept on vertex objects)

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void dfs( Vertex v ) {
  v.visited = true;
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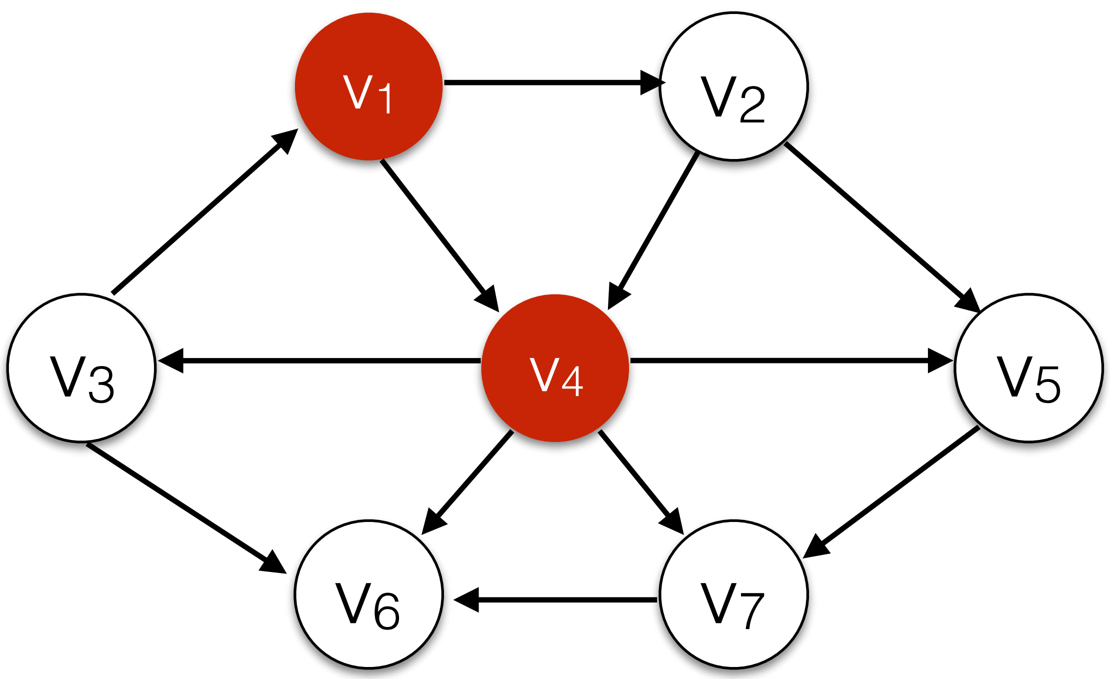
```

DFS Spanning Tree

V_1

Recursive DFS (with *visited* marker kept on vertex objects)

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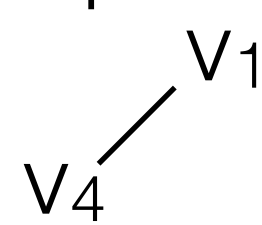

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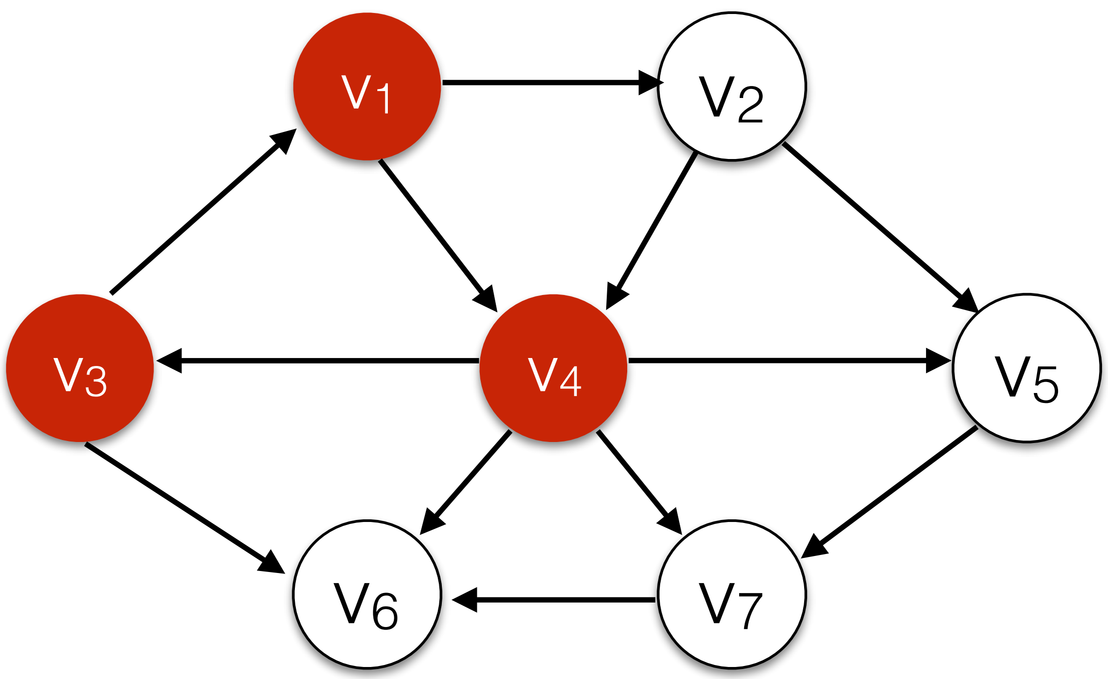


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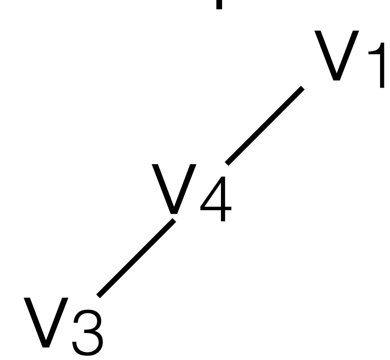

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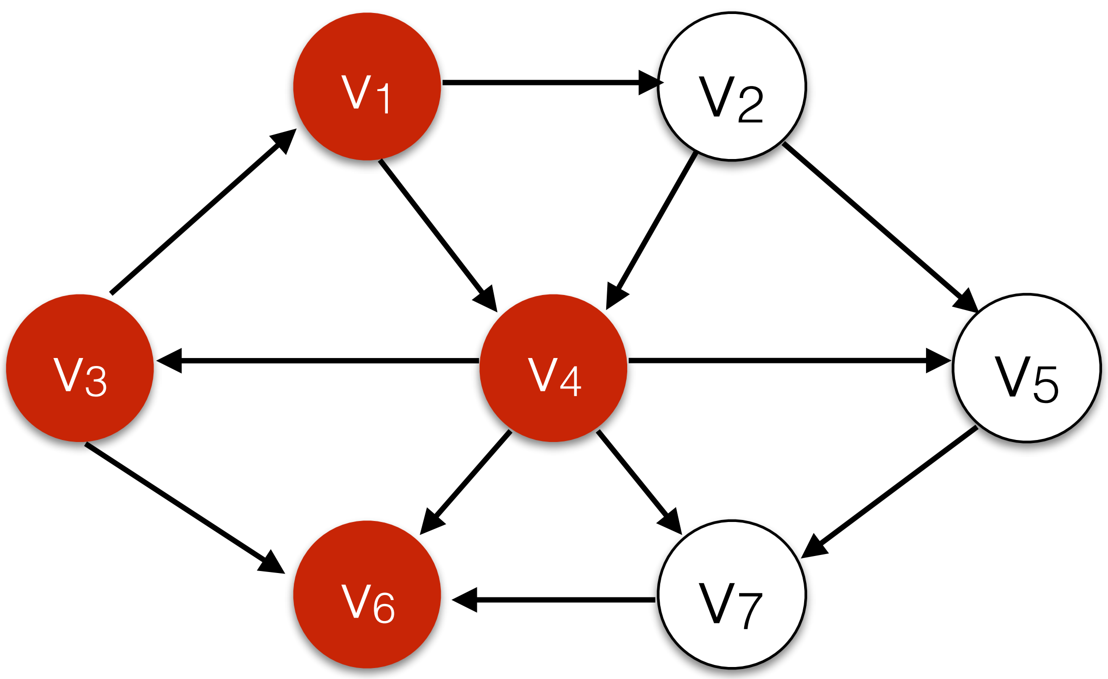


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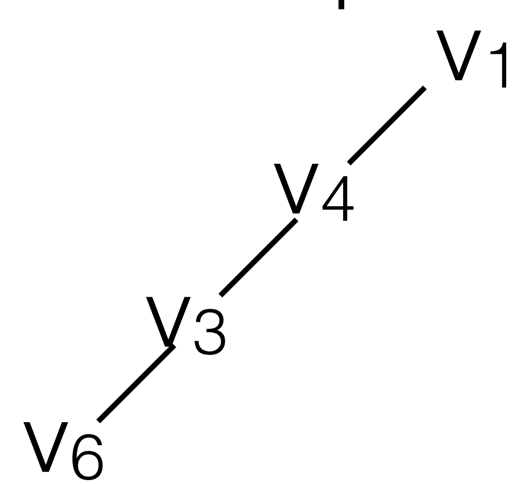
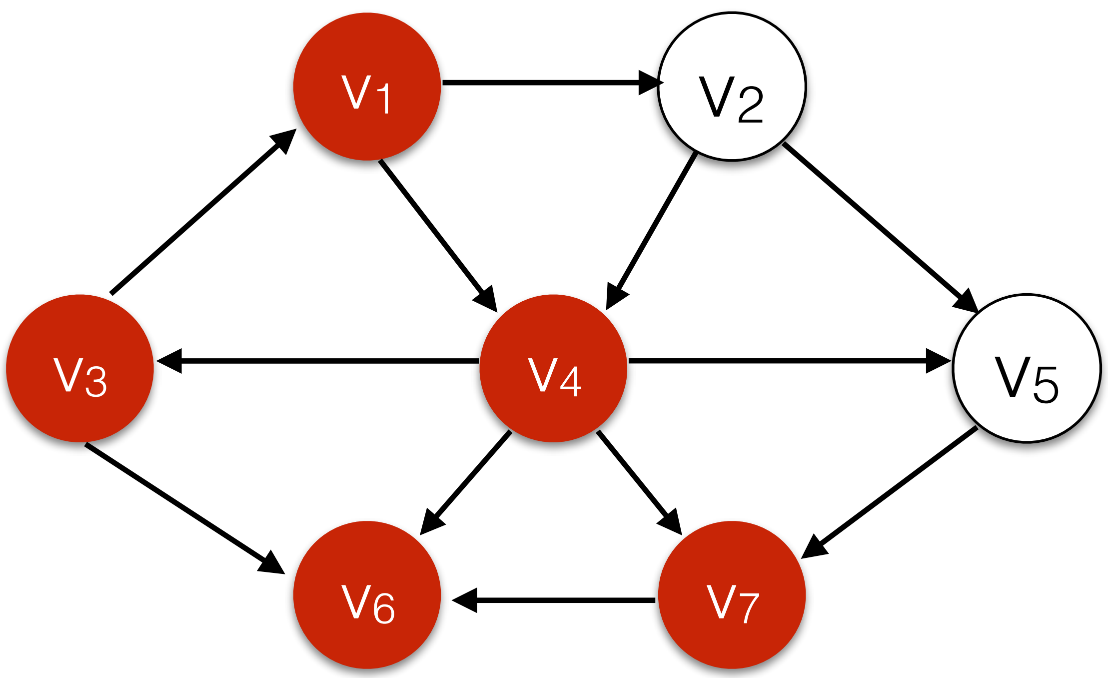
DFS Spanning Tree

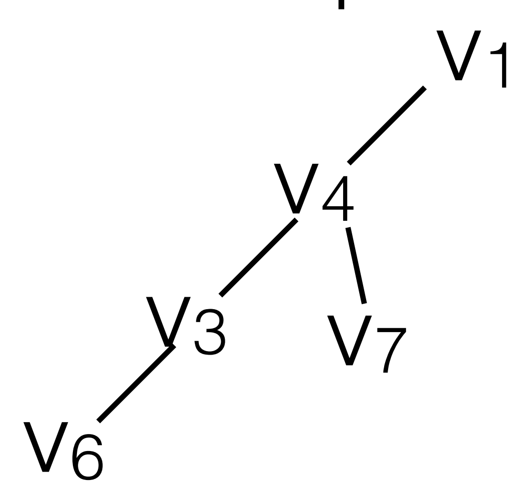


Recursive DFS (with *visited* marker kept on vertex objects)




```
void dfs( Vertex v ) {
  v.visited = true;
  for each Vertex w adjacent to v
    if( !w.visited )
      dfs( w );
}
```

DFS Spanning Tree



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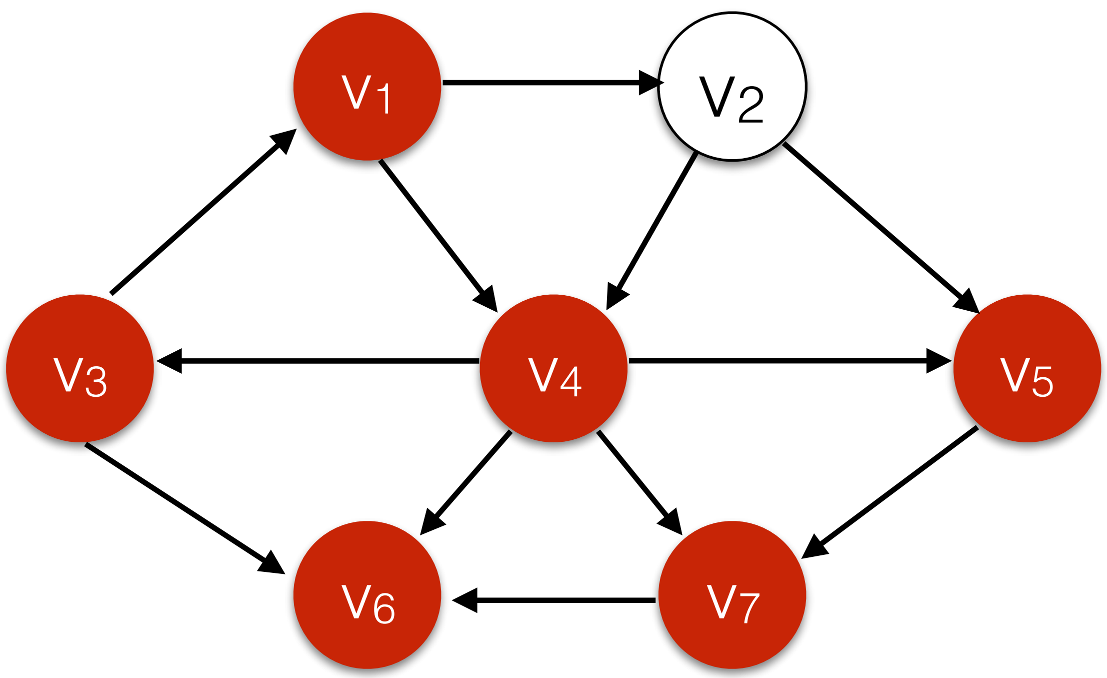


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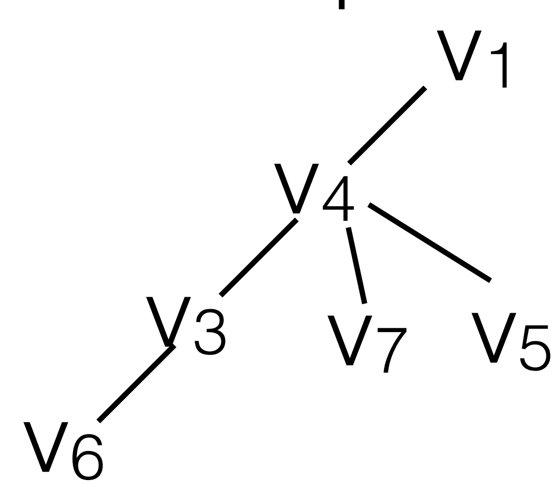
DFS Spanning Tree



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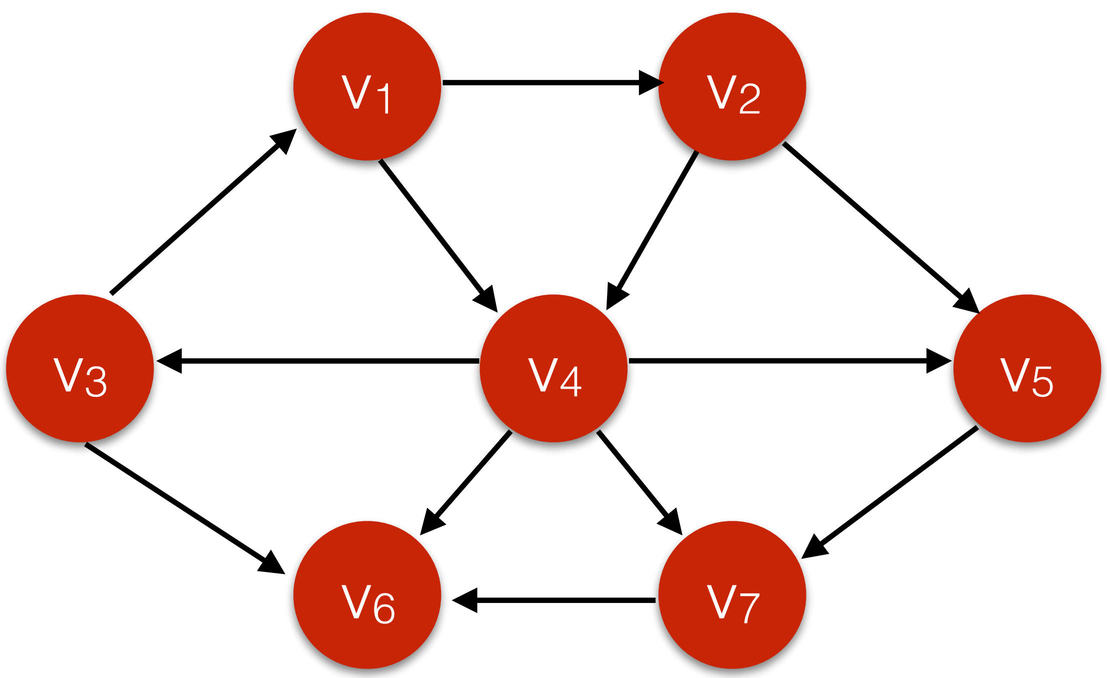


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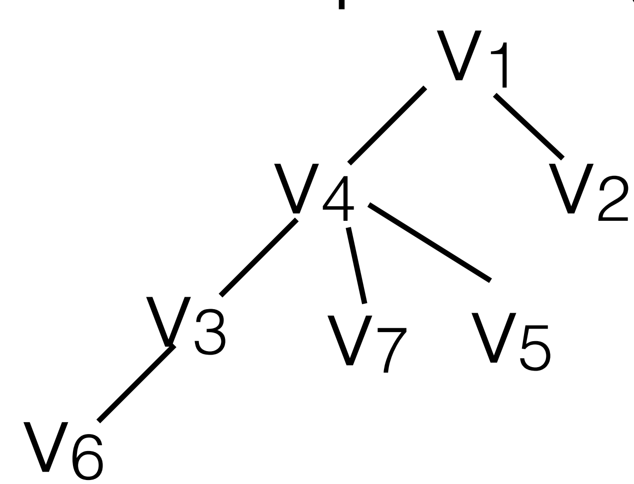
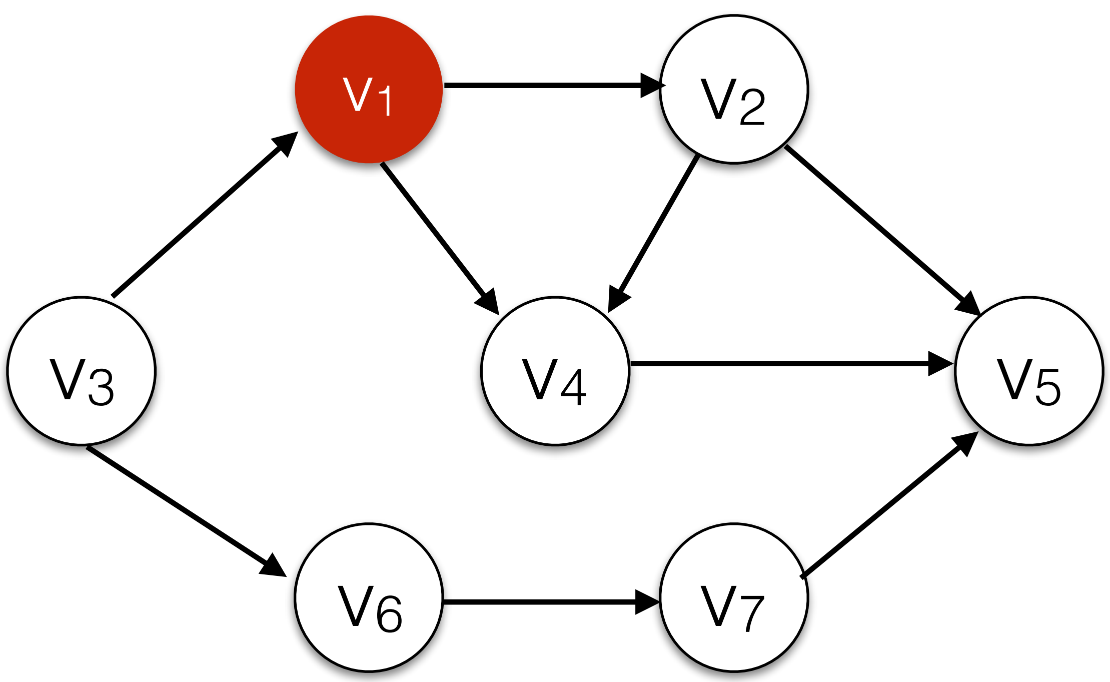
DFS Spanning Tree



DFS on the Entire Graph

- It is possible that not all vertices are reachable from a designated start vertex.



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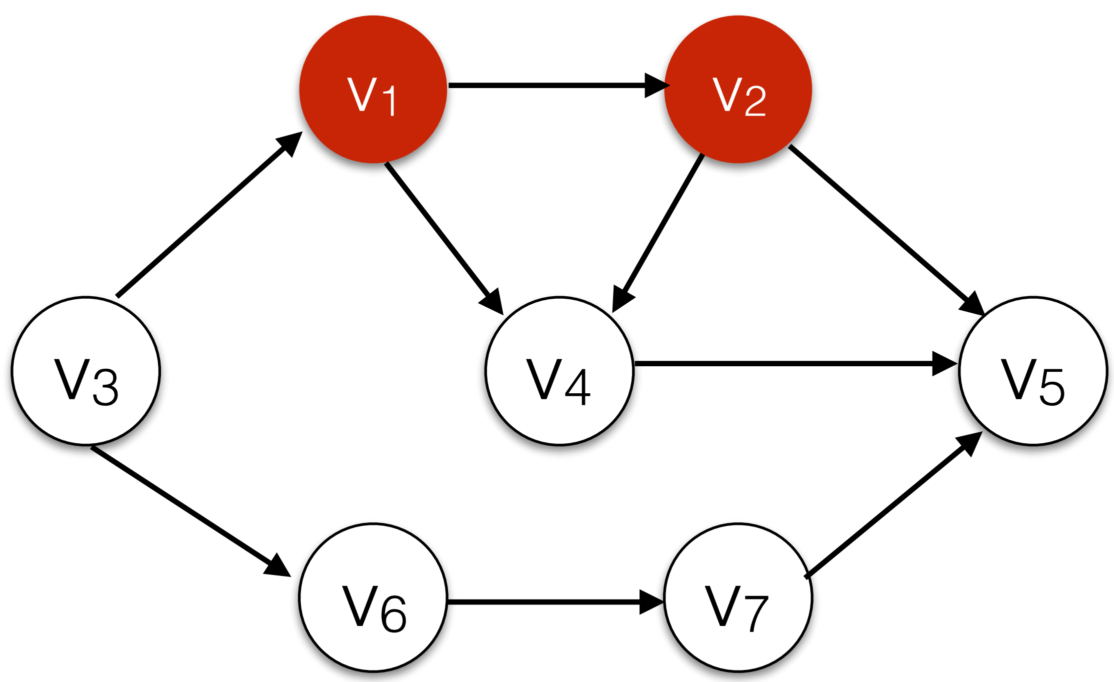

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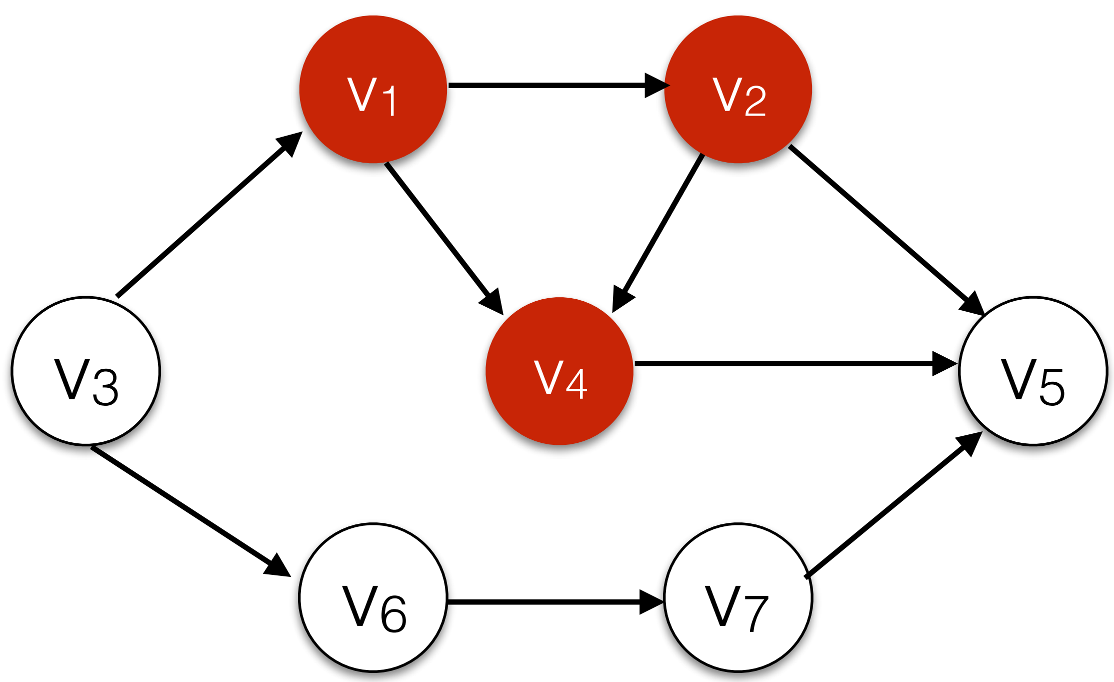

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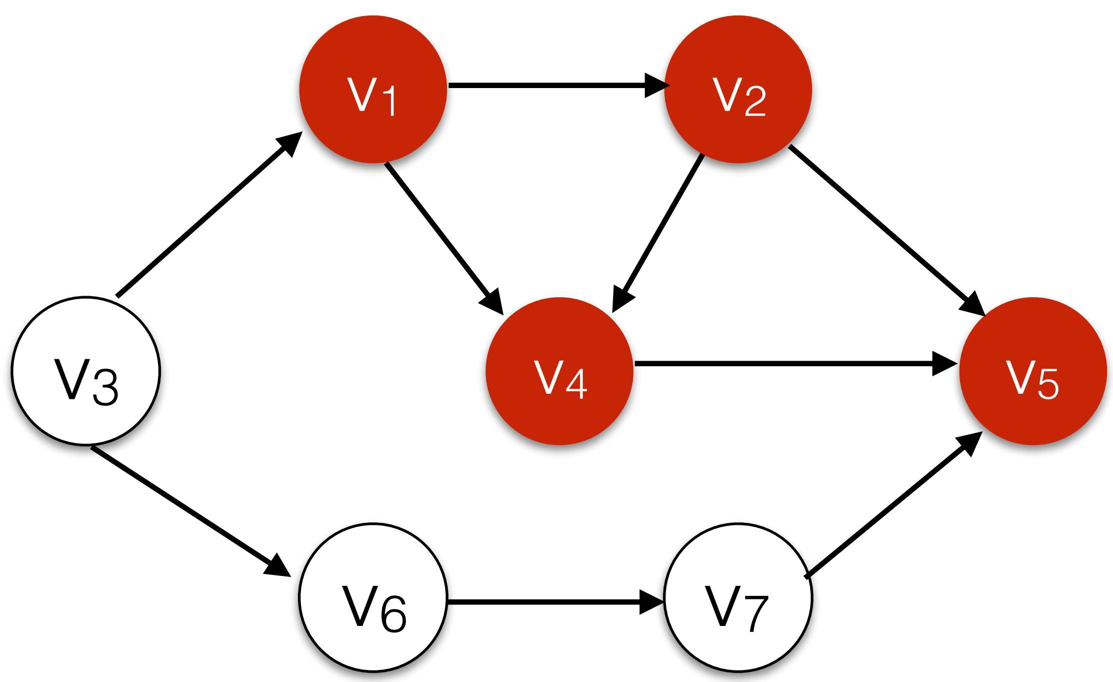
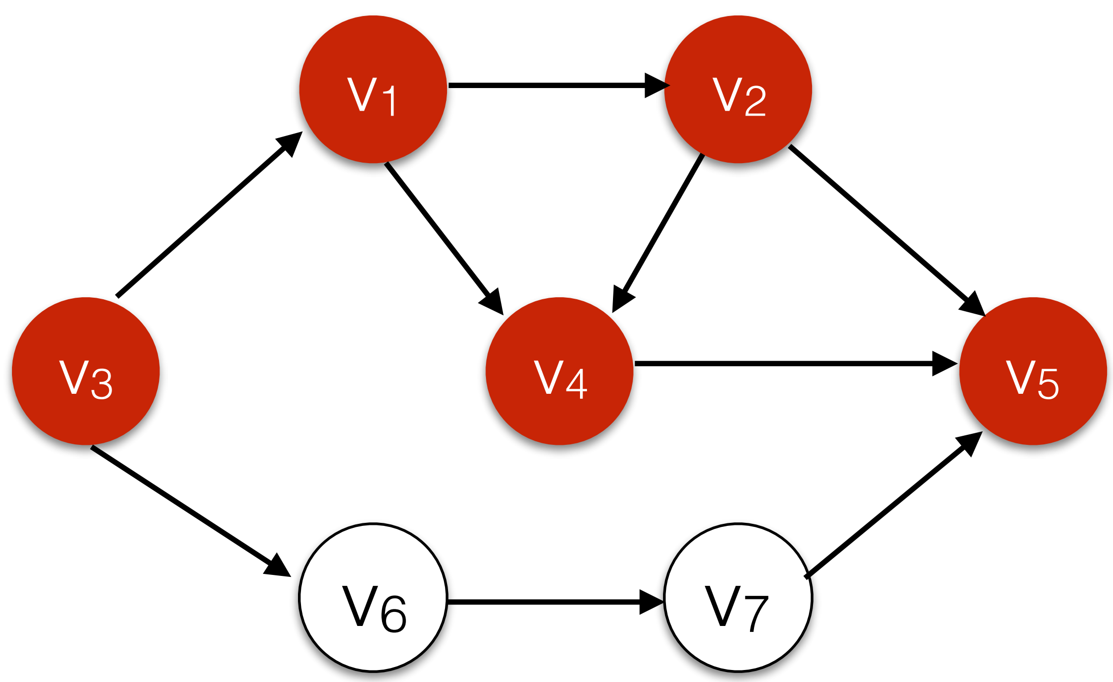


v_1:	v_2, v_4
v_2:	v_4, v_5
v_3:	v_1, v_6
v_4:	v_5
v_5:	
v_6:	v_7
v_7:	v_5

DFS on the Entire Graph

- It is possible that not all vertices are reachable from a designated start vertex.



V_1: V_2, V_4

V_2: V_4, V_5

V_3: V_1, V_6

V_4: V_5

V_5:

V_6: V_7

V_7: V_5

- If stack is empty or we reach top of recursion, scan through adjacency list until we find an unseen starting node.

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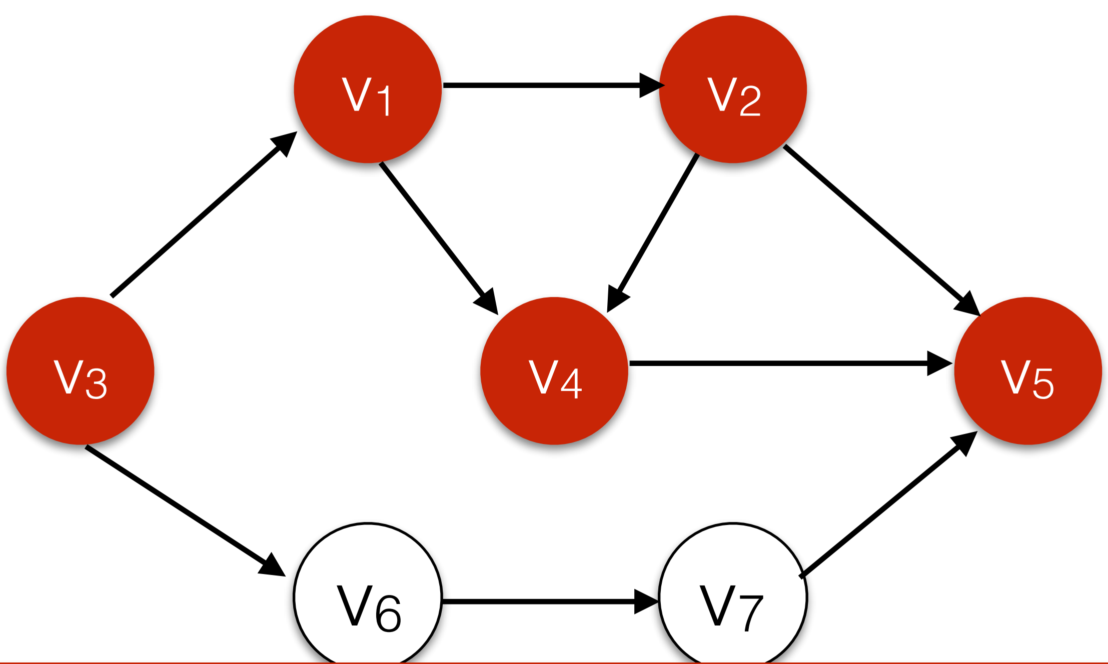


v_1: v_2, v_4

v_2: v_4, v_5

v_3: v_1, v_6

v_4: v_5

v_5:

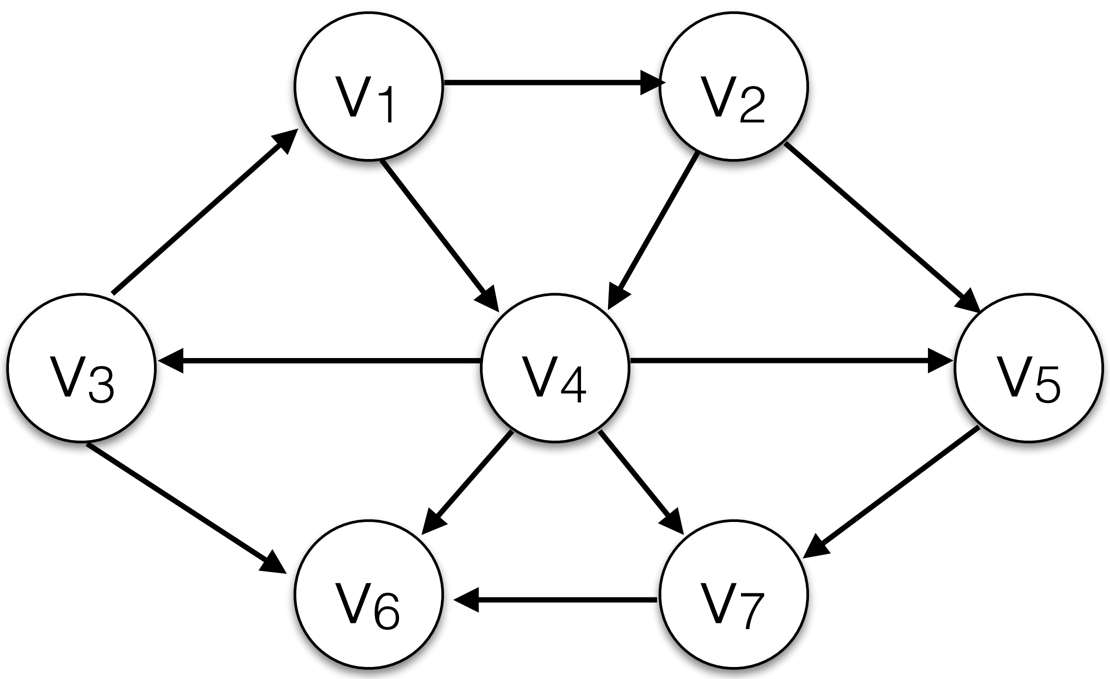
v_6: v_7

v_7: v_5

Running time for complete DFS traversal: $O(|V|+|E|)$

- If stack is empty or we reach top of recursion, scan through adjacency list until we find an unseen starting node.

Breadth-First Search (BFS)



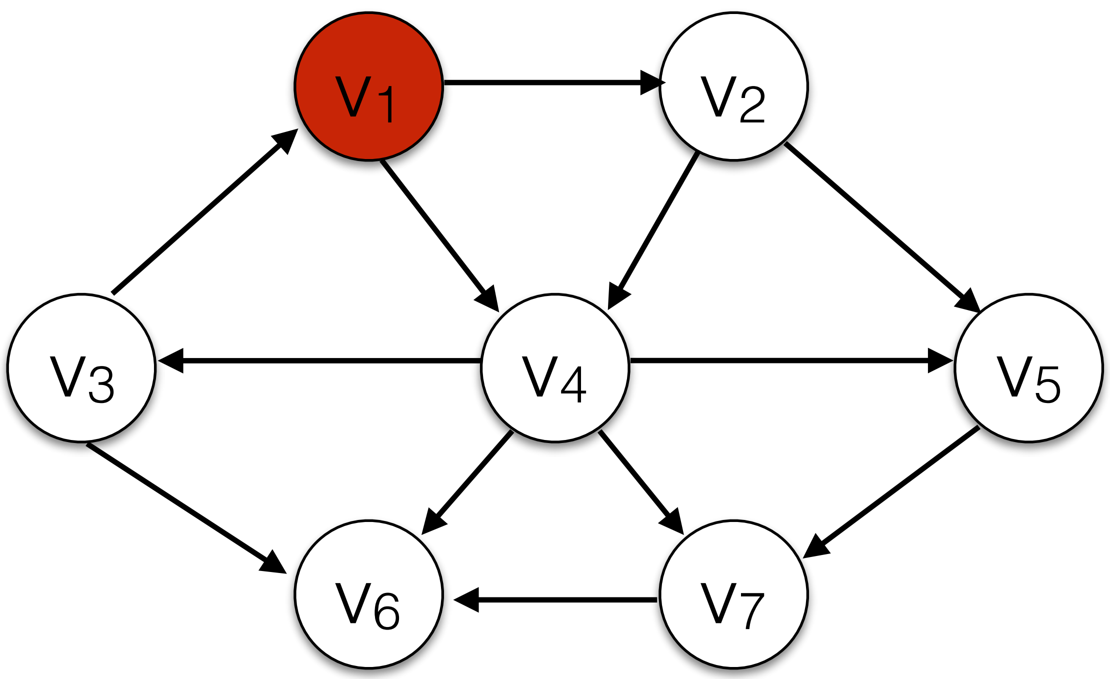
Use a **queue** and a set *visited*.

- Enqueue *s*
- Add *s* to *visited*
- While the queue is not empty:
  - *u* <- dequeue()
    - for each vertex *v* that is adjacent to *u*:
      - if *v* is not in *visited:*
        - Add *v* to visited.
        - enqueue(v).

Queue $\boxed{v_1}$

Visited: $\{v_1\}$

Breadth-First Search (BFS)



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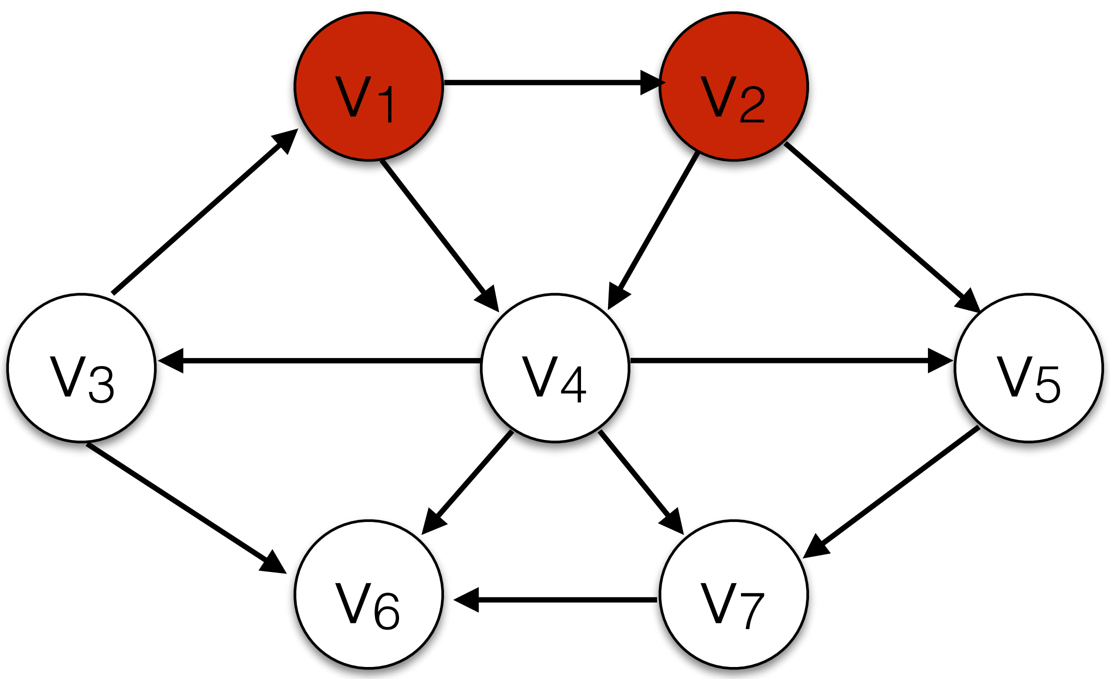
        - enqueue(v).

Queue v_2 v_4

Visited: $\{v_1, v_2, v_4\}$

Breadth-First Search (BFS)



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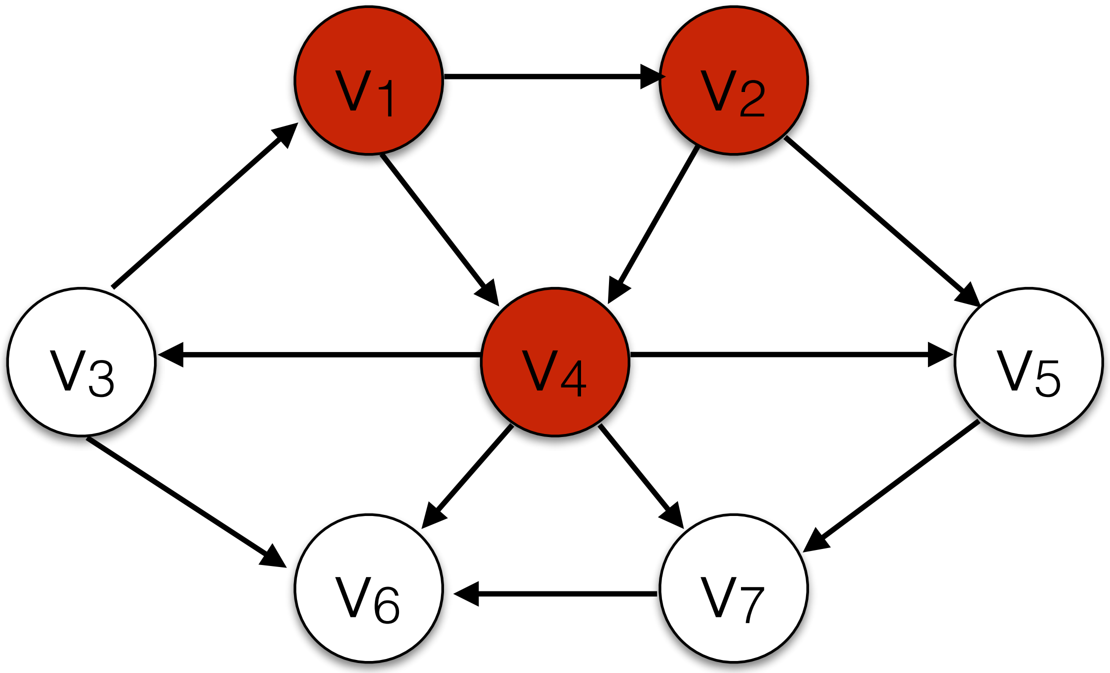
Queue

v_4	v_5

Visited: $\{v_1, v_2, v_4, v_5\}$

Breadth-First Search (BFS)



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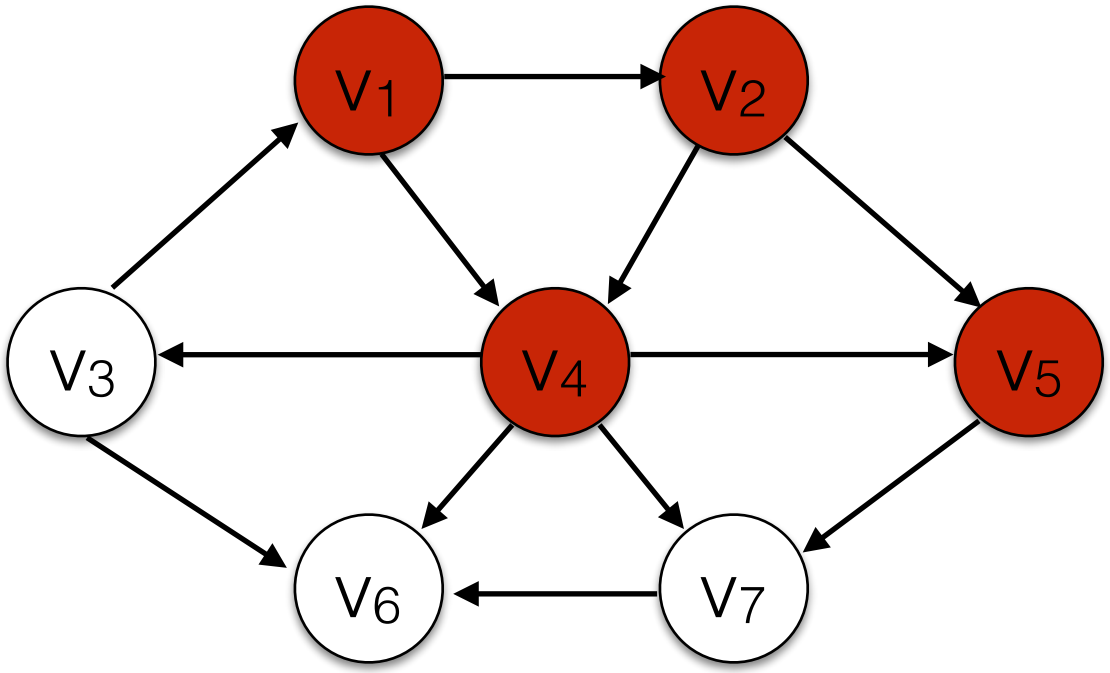
        - enqueue(v).

Queue

v_5	v_7	v_6	v_3

Visited: $\{v_1, v_2, v_4, v_5, v_7, v_6, v_3\}$

Breadth-First Search (BFS)



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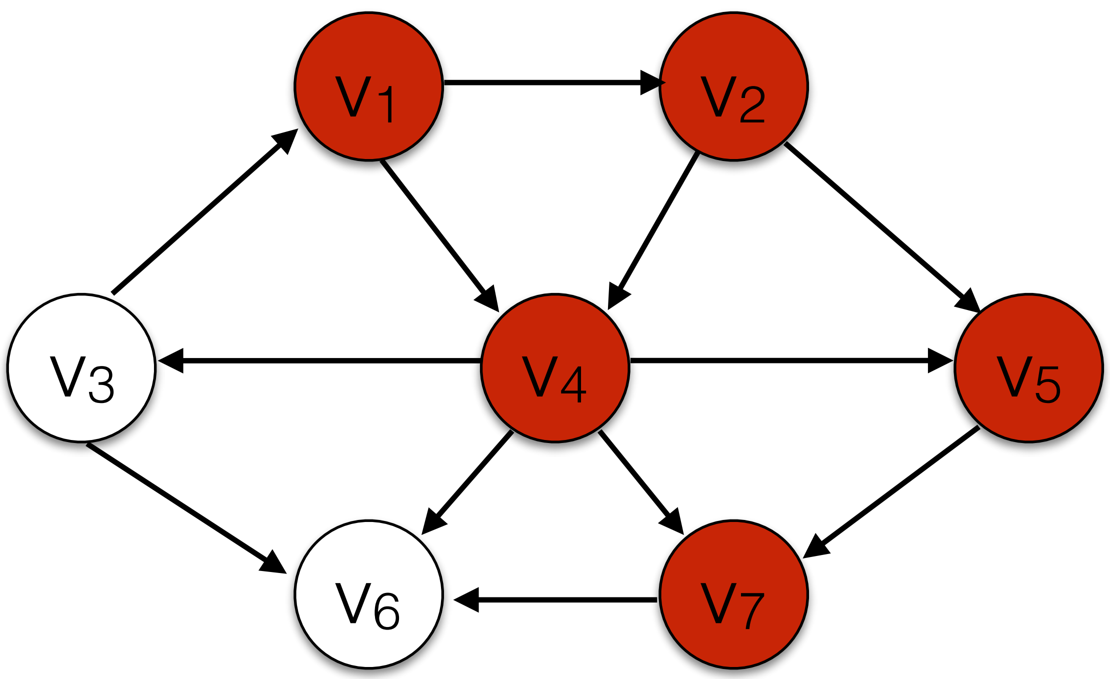
Queue

v_7	v_6	v_3

Visited:$\{v_1, v_2, v_4, v_5, v_7, v_6, v_3\}$

Breadth-First Search (BFS)



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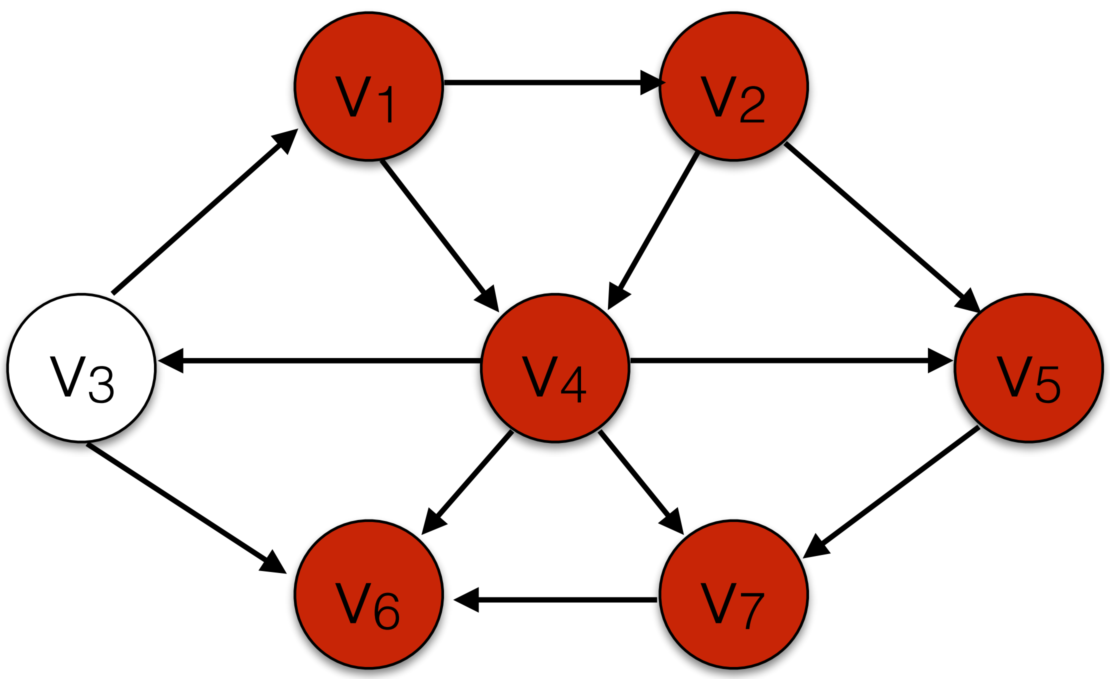
Queue | v_6 | v_3 |

Visited:$\{v_1,v_2,v_4,v_5,v_7,v_6,v_3\}$

Breadth-First Search (BFS)



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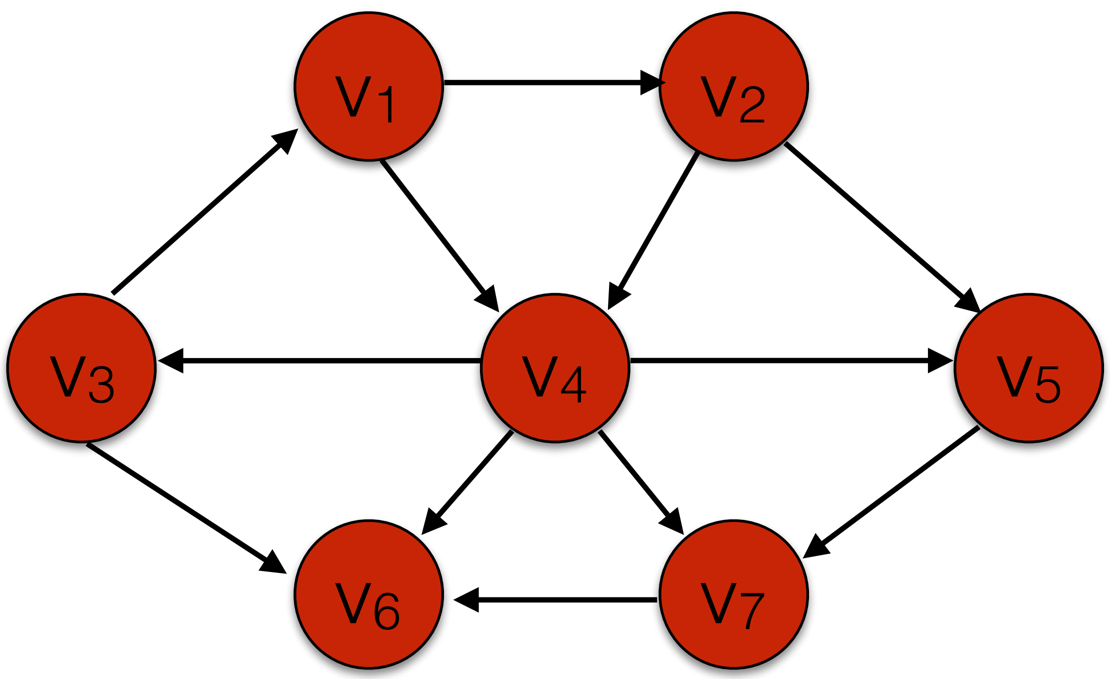
Queue $\boxed{v_3}$

Visited: $\{v_1, v_2, v_4, v_5, v_7, v_6, v_3\}$

Breadth-First Search (BFS)



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Queue

Visited:{v_1,v_2,v_4,v_5,v_7,v_6,v_3}

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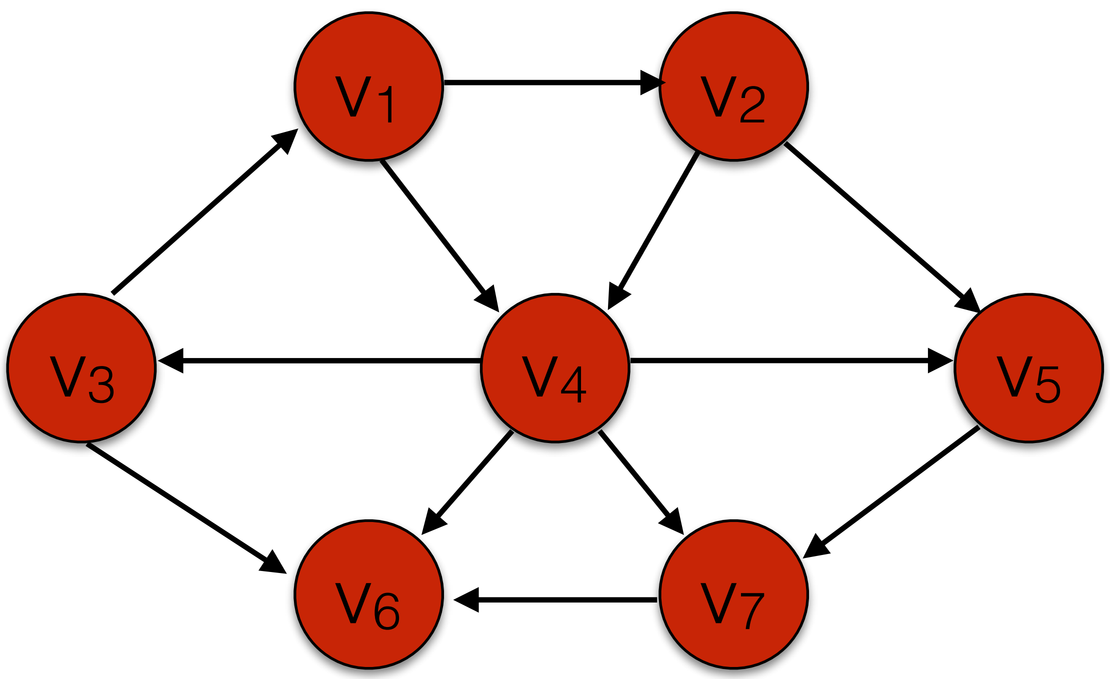

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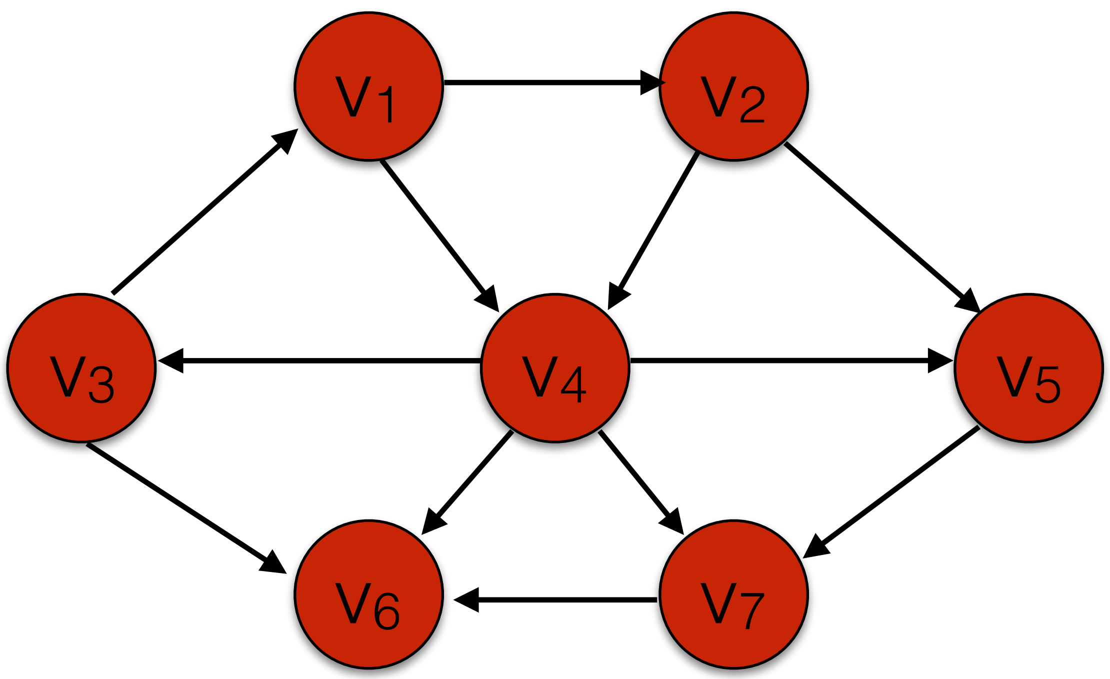
Running time (to traverse the entire graph): O(|V|+|E|)

Queue

Visited:{v_1,v_2,v_4,v_5,v_7,v_6,v_3}

Breadth-First Search (BFS)



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BFS will traverse the entire graph even without a visited set.
DFS can get stuck in a loop.

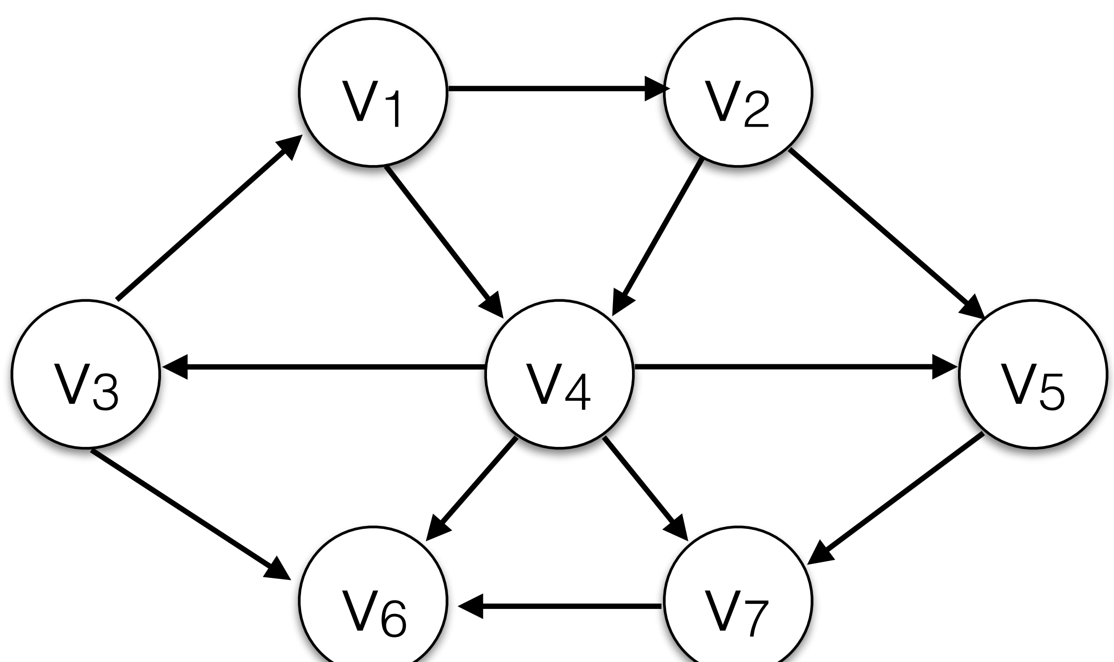
Running time (to traverse the entire graph): $O(|V|+|E|)$

Queue

Visited: $\{v_1, v_2, v_4, v_5, v_7, v_6, v_3\}$

Finding Shortest Paths

- Goal: Find the shortest path between two vertices s and t.



What is the shortest path between v_3 and v_7?

Finding Shortest Paths

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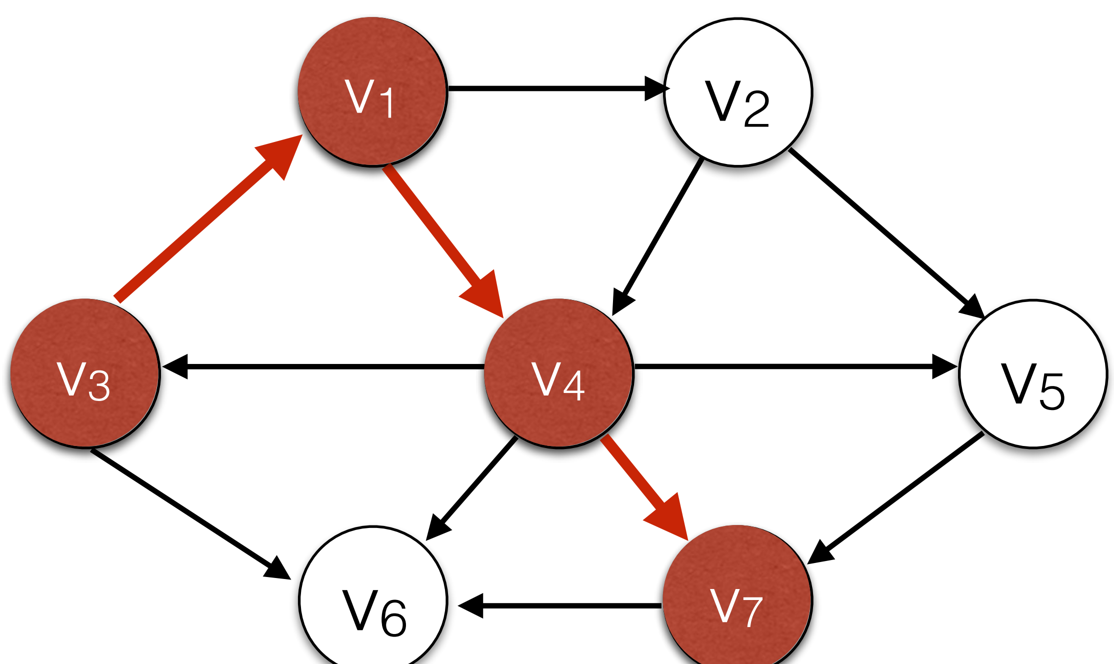
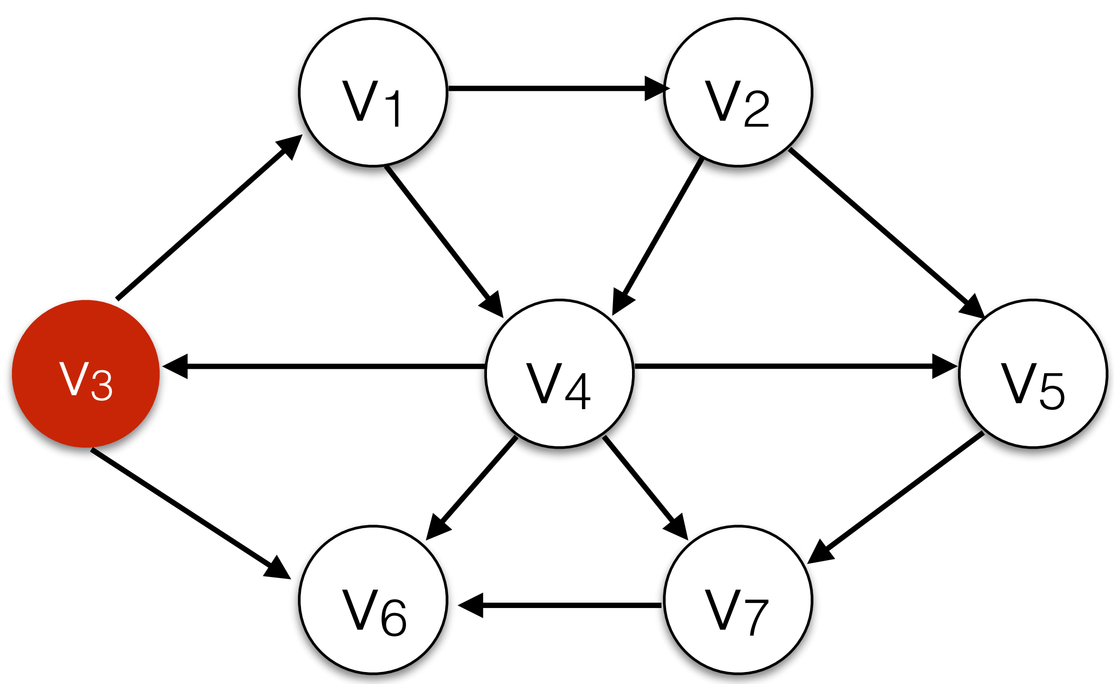


length 3

What is the shortest path between v_3 and v_7?

Finding Shortest Paths

- Goal: Find the shortest path between two vertices s and t.
- It turns out that finding the shortest path between s and ALL other vertices is just as easy. This problem is called **single-source shortest paths**.



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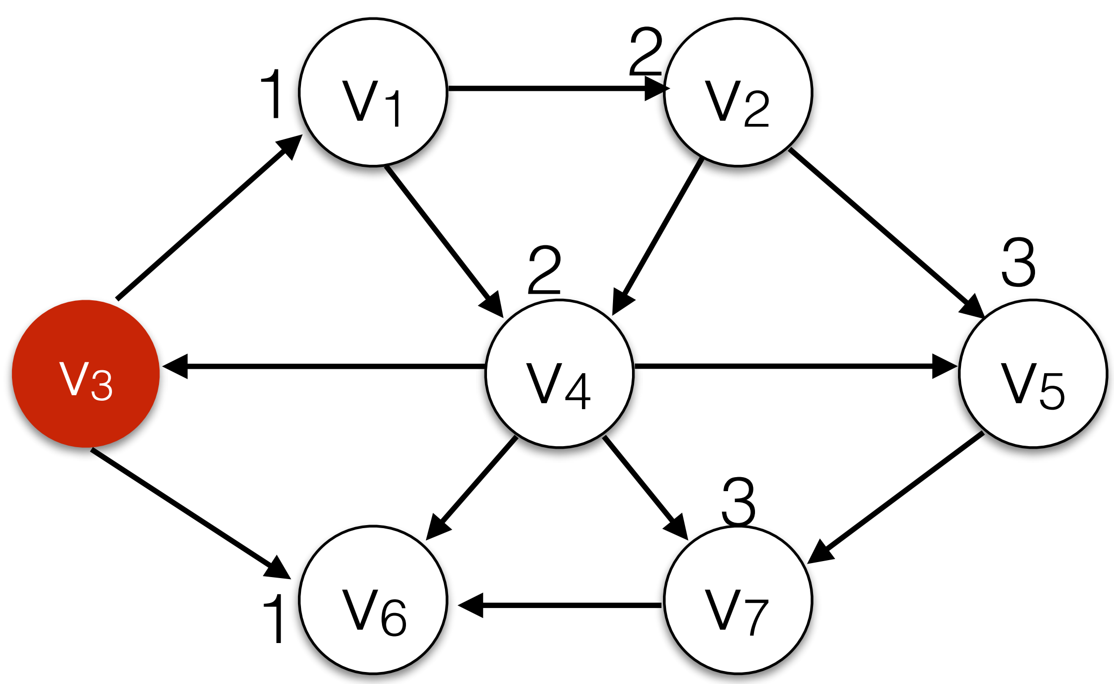

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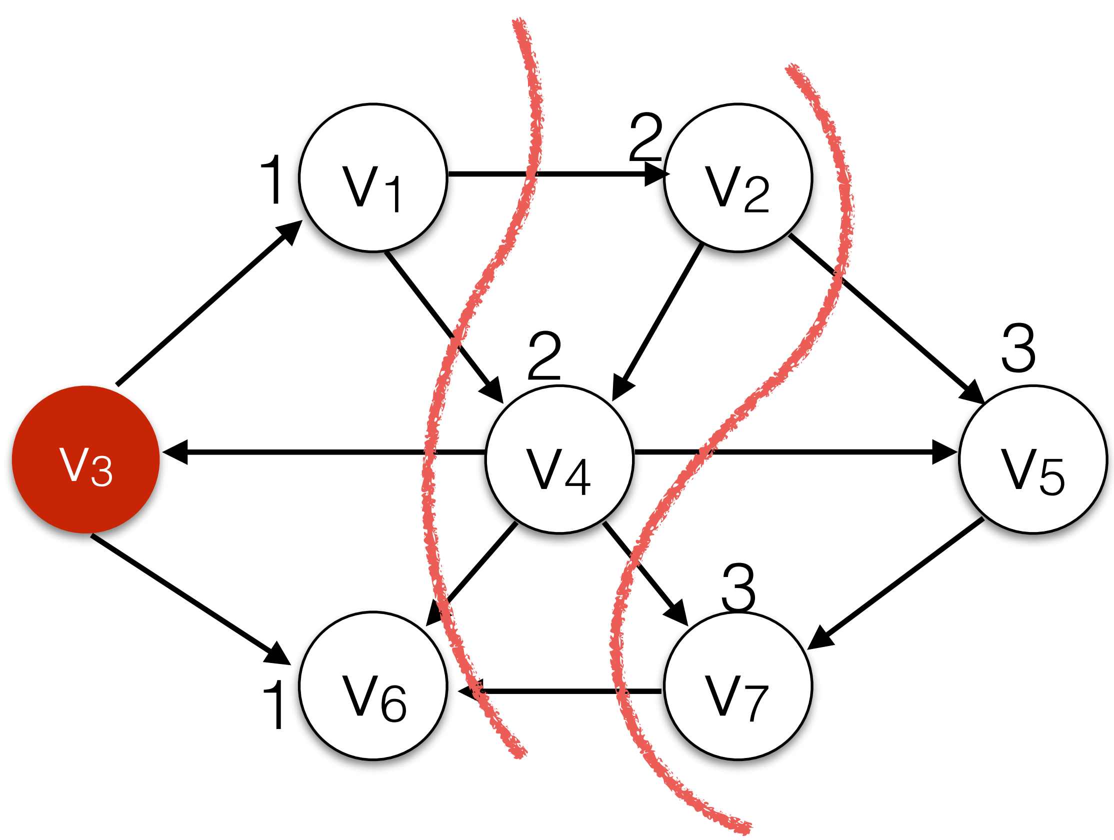
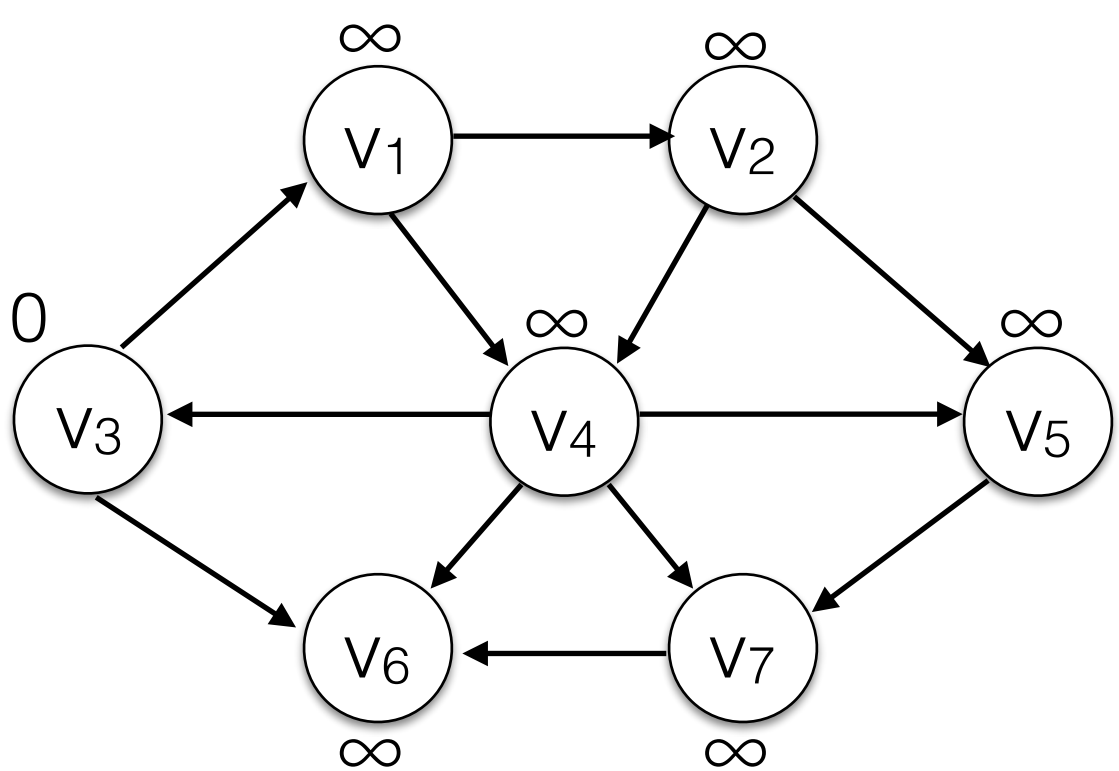


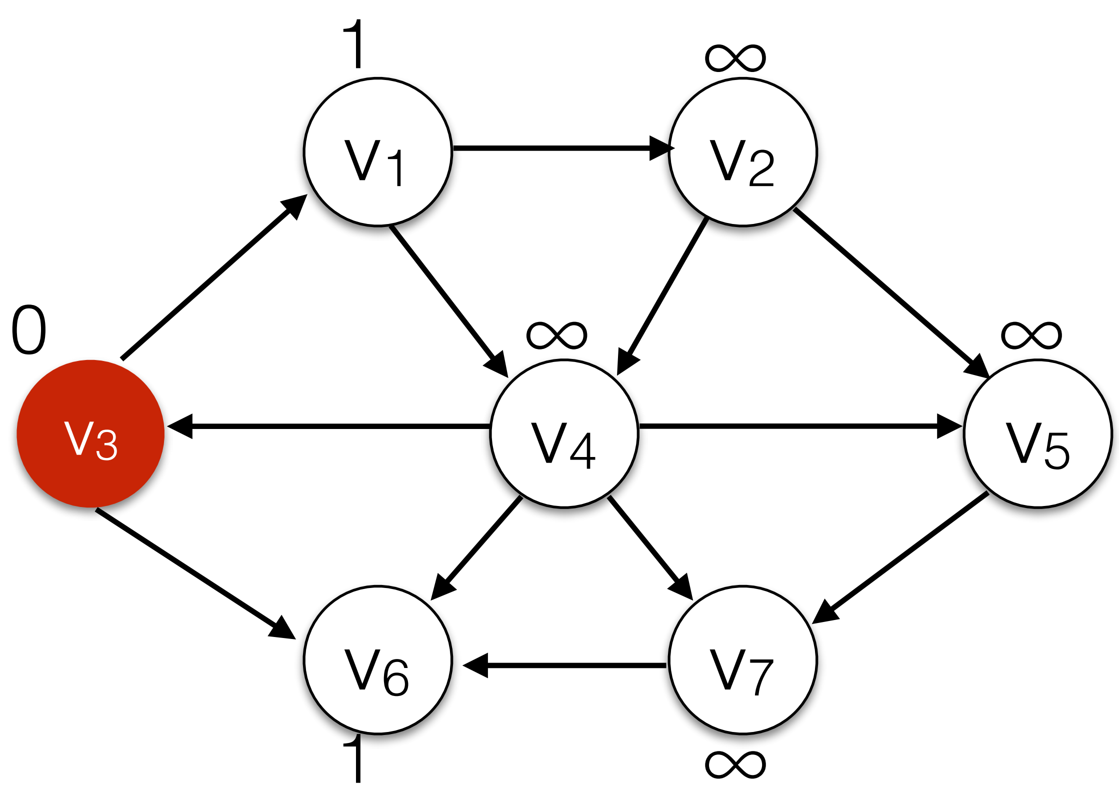
Finding Shortest Paths with BFS



- s.distance = 0
- for all v ∈ V set v.distance = ∞
- enqueue s
- While the queue is not empty:
  - *u* <- dequeue()
    - for each vertex *v* that is adjacent to u:
      - if v.distance == ∞
        - v.distance = u.distance + 1
        - enqueue(v)

Queue $\boxed{v_3}$

Finding Shortest Paths with BFS



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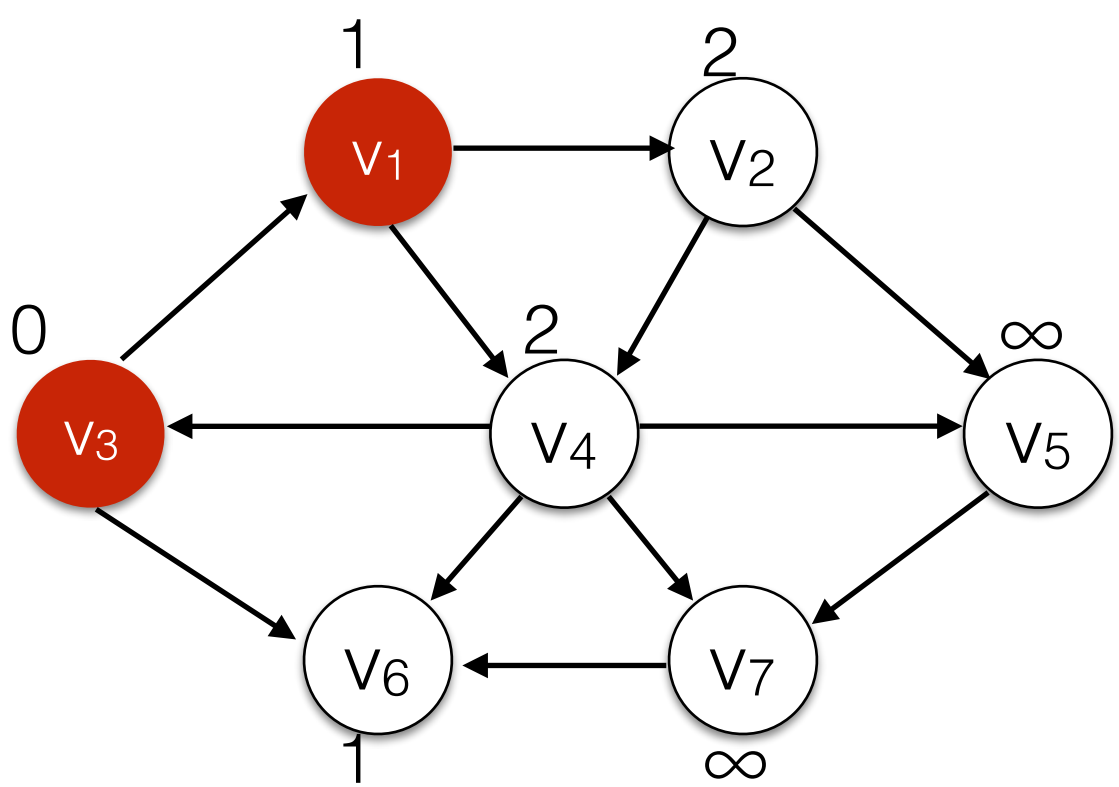
        - enqueue(v)

Queue

V1	V6

Finding Shortest Paths with BFS



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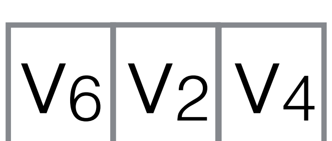
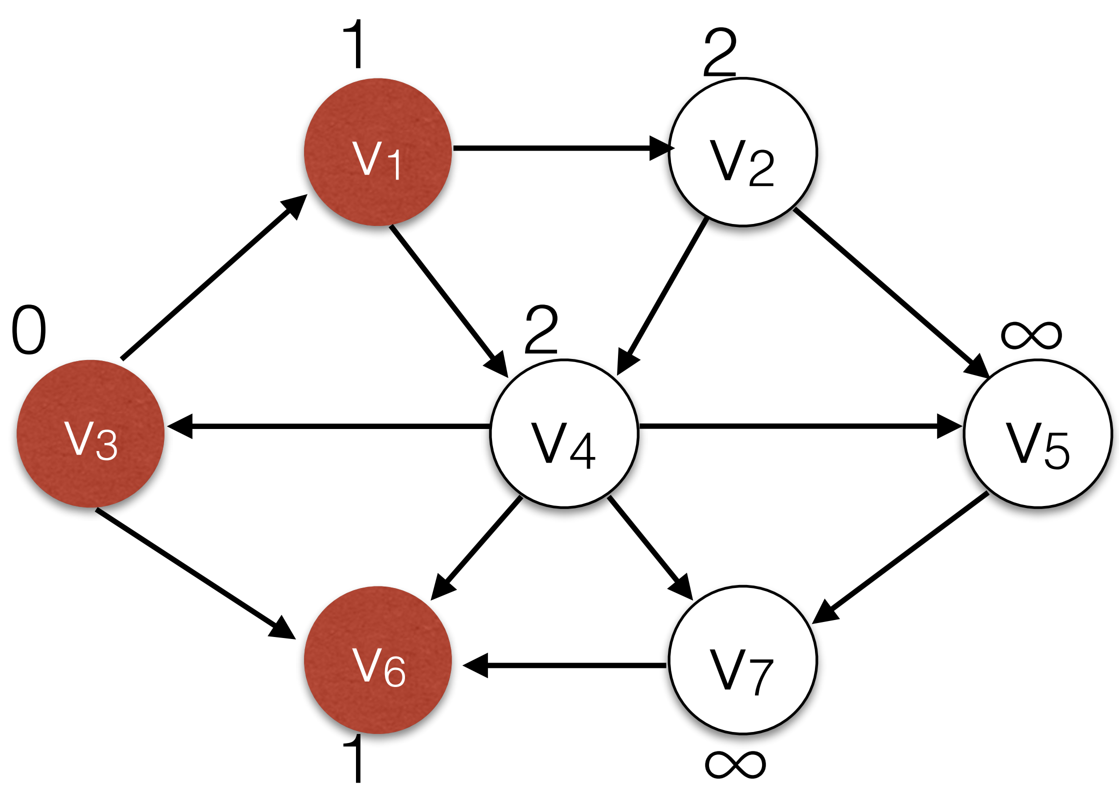
Queue



Finding Shortest Paths with BFS



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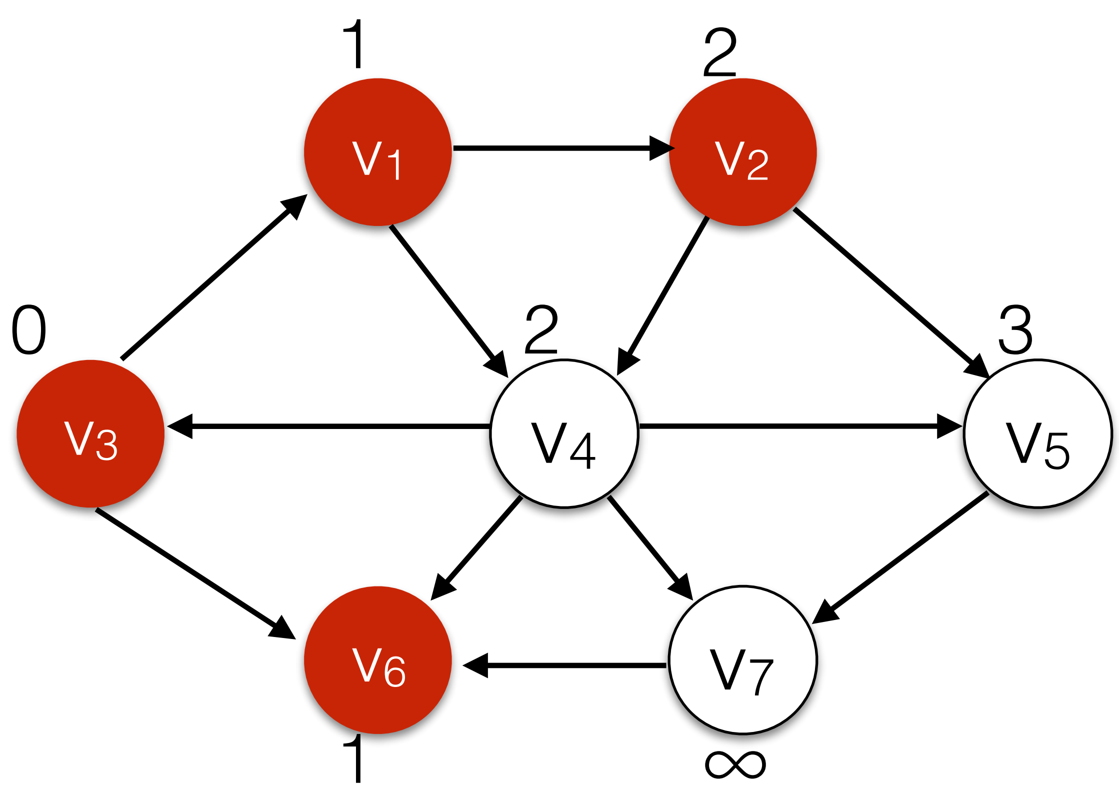
        - enqueue(v)

Queue

v_2	v_4

Finding Shortest Paths with BFS



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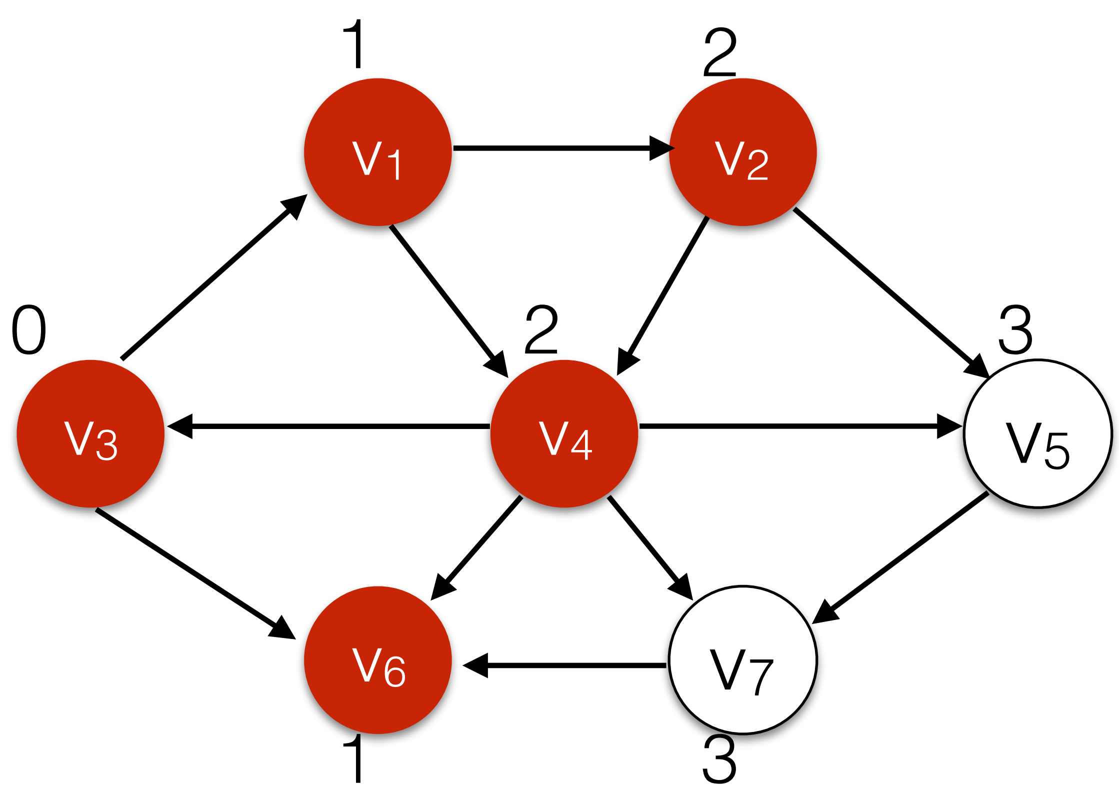
- enqueue s
- While the queue is not empty:
  - *u* <- dequeue()
    - for each vertex *v* that is adjacent to u:
      - if v.distance == ∞
        - v.distance = u.distance + 1
        - enqueue(v)

Queue

V4	V5

Finding Shortest Paths with BFS



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- for all v ∈ V set v.distance = ∞
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- While the queue is not empty:
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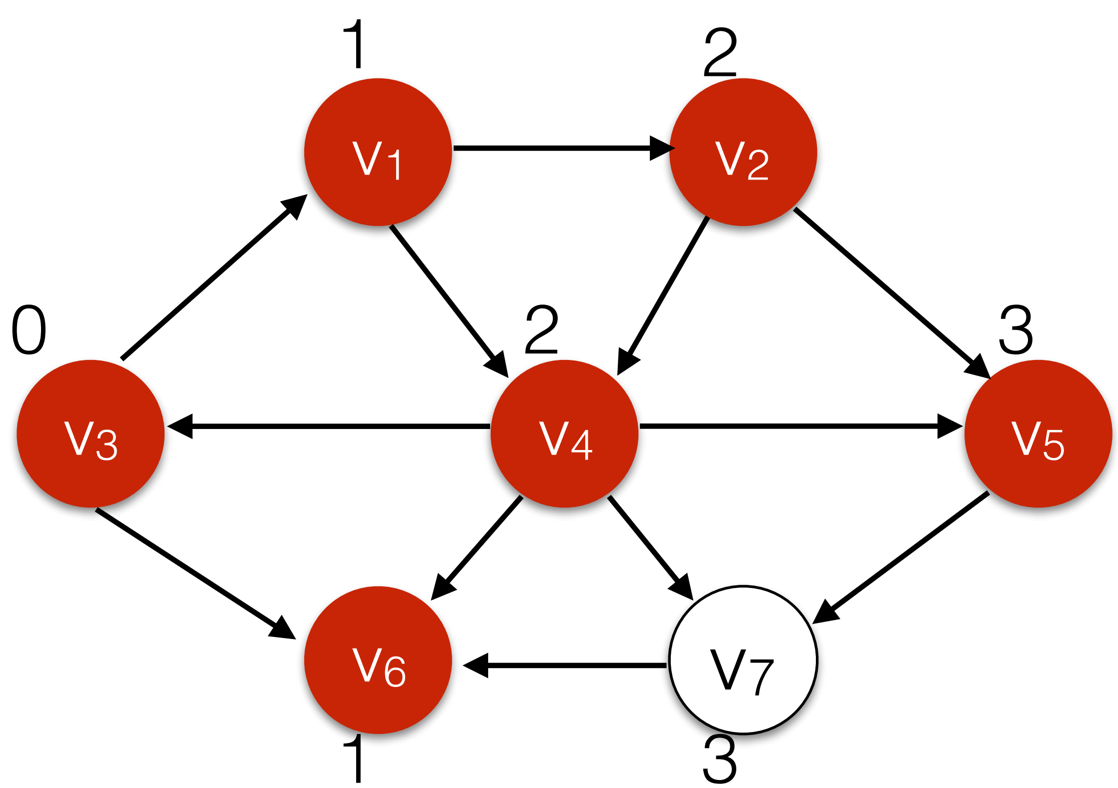
            - if v.distance == ∞
                - v.distance = u.distance + 1
                - enqueue(v)

Queue

V_5	V_7

Finding Shortest Paths with BFS



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- for all v ∈ V set v.distance = ∞
- enqueue s
- While the queue is not empty:
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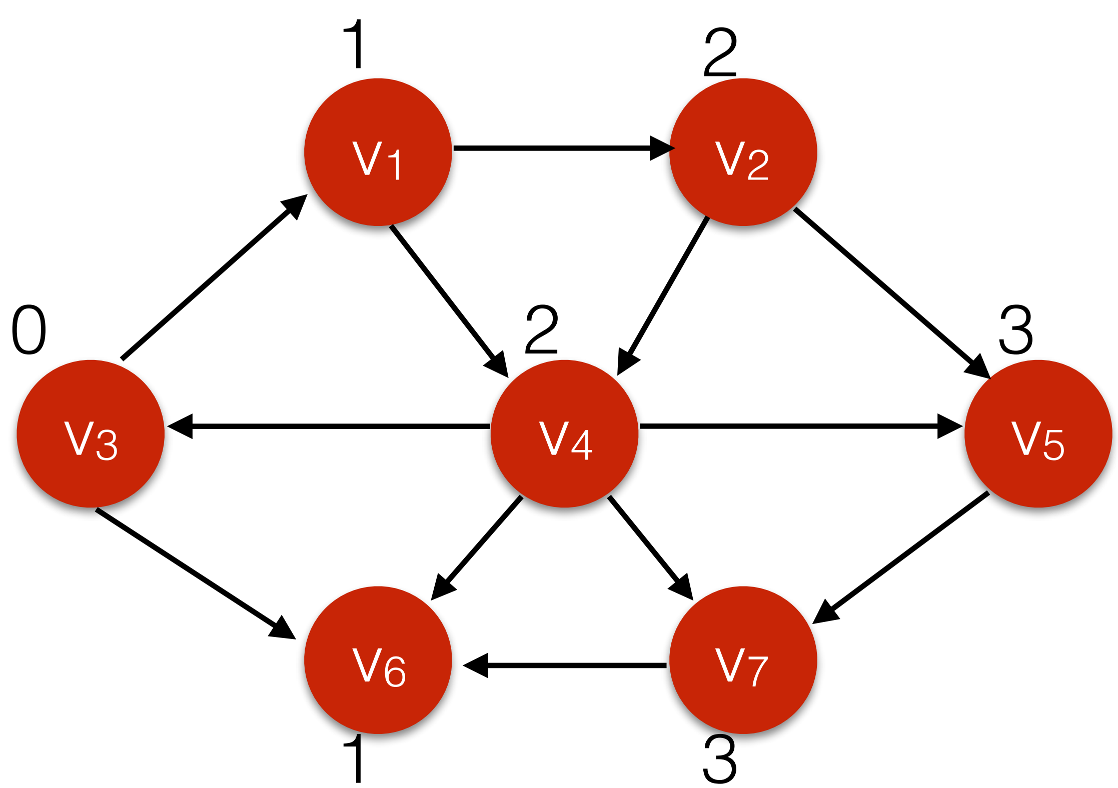
    - for each vertex *v* that is adjacent to u:
      - if v.distance == ∞
        - v.distance = u.distance + 1
        - enqueue(v)

Queue V_7

Finding Shortest Paths with BFS



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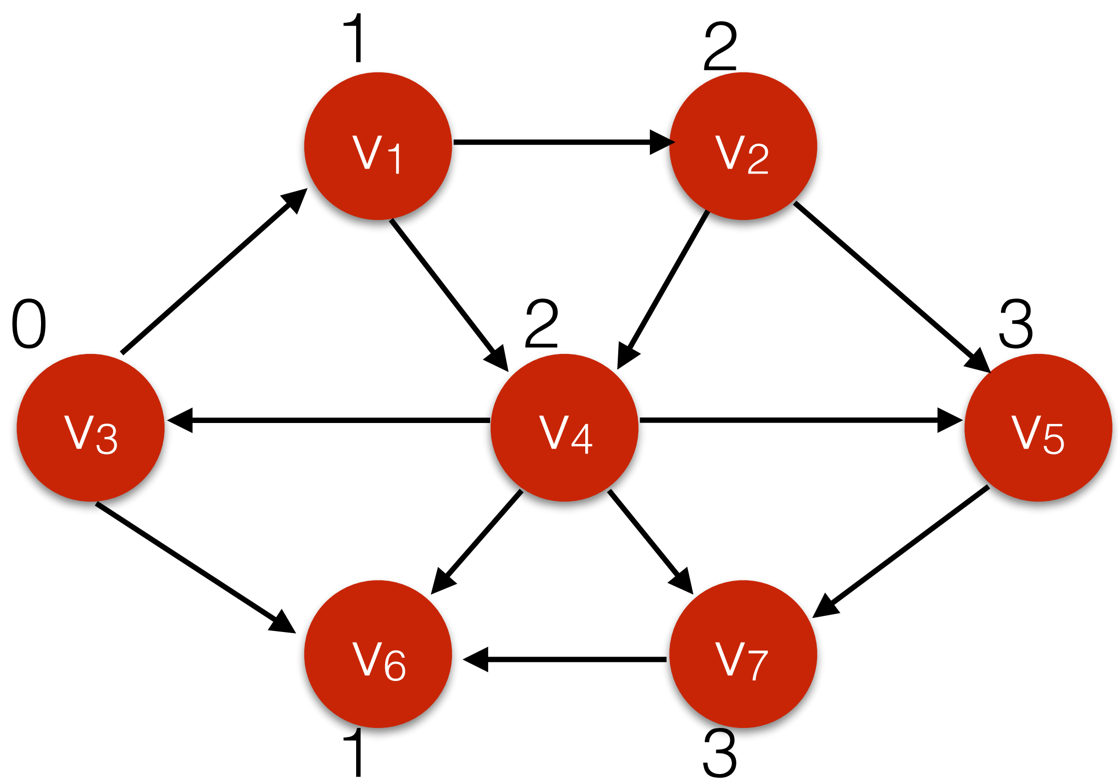
  - *u* <- dequeue()
    - for each vertex *v* that is adjacent to u:
      - if v.distance == ∞
        - v.distance = u.distance + 1
        - enqueue(u)

Queue

Finding Shortest Paths with BFS



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            - if v.distance == ∞

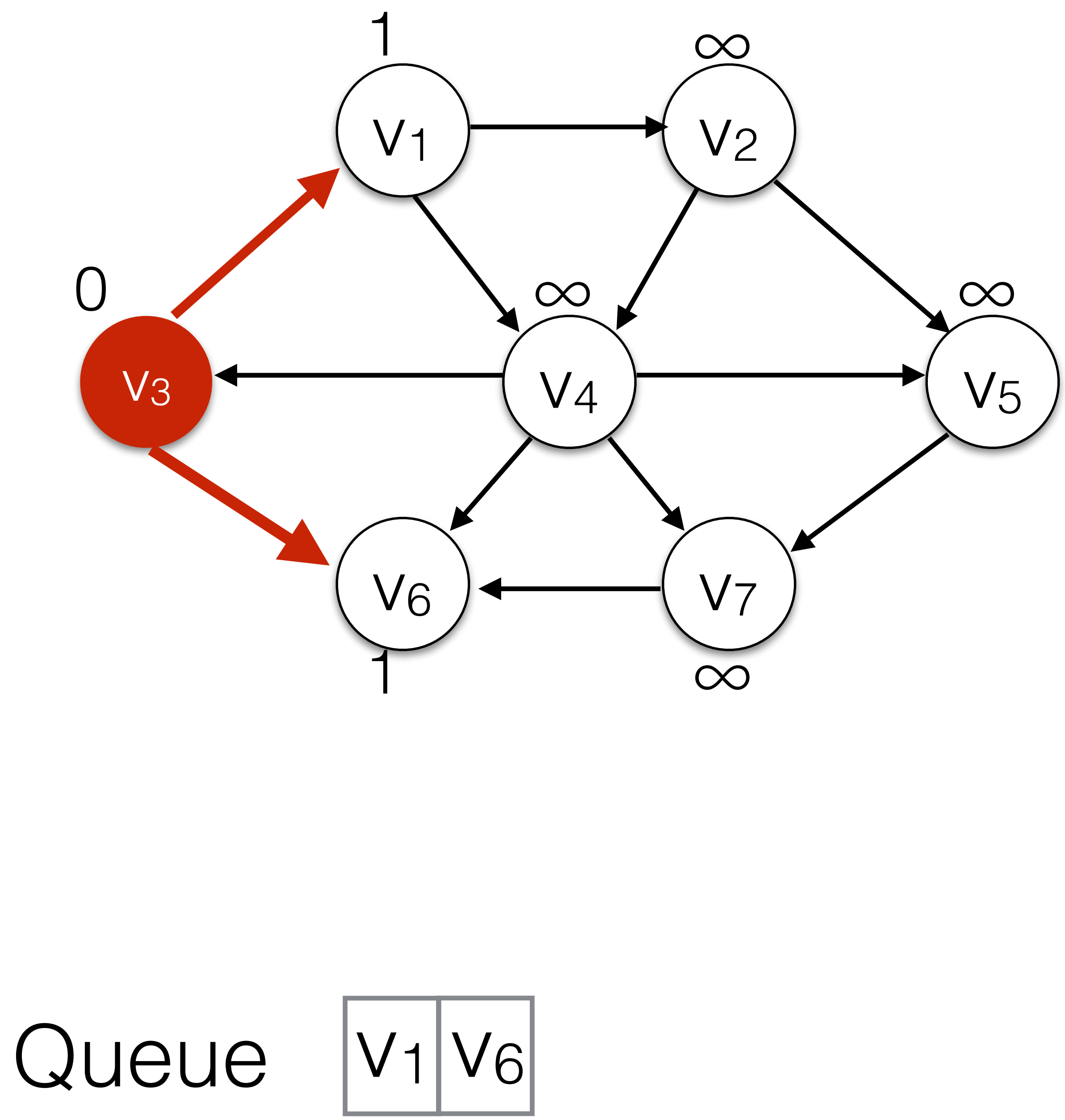
ance + 1

This is just BFS. Running time: O(|V|+|E|)

Queue

Finding Shortest Paths with BFS - Back pointers

Maintain pointers to the previous node on the shortest path.



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    - for each vertex *v* that is adjacent to u:
      - if v.distance == ∞
        - **v.prev = u**
        - v.distance = u.distance + 1
        - enqueue(v)

Finding Shortest Paths with BFS - Back pointers

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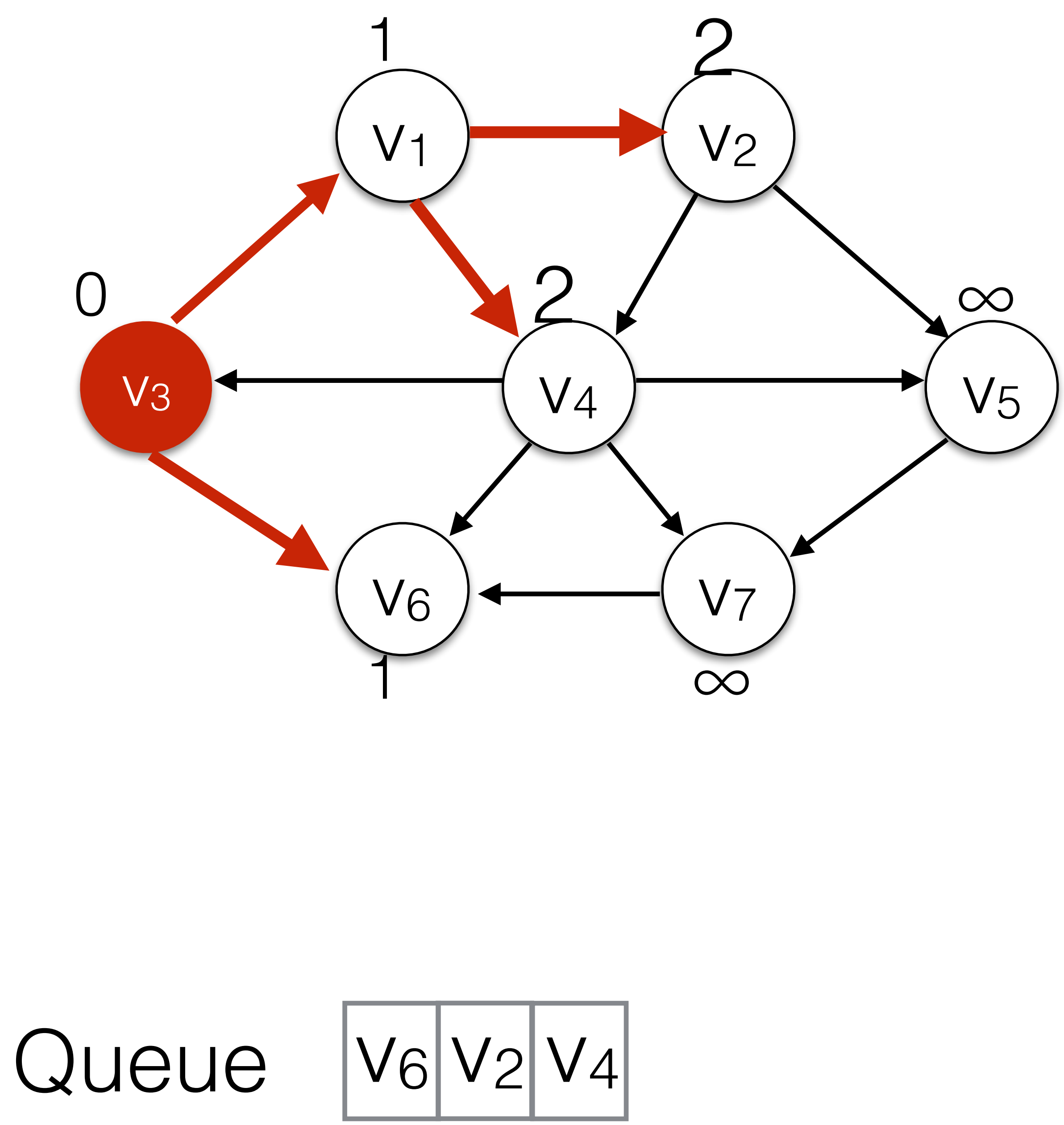

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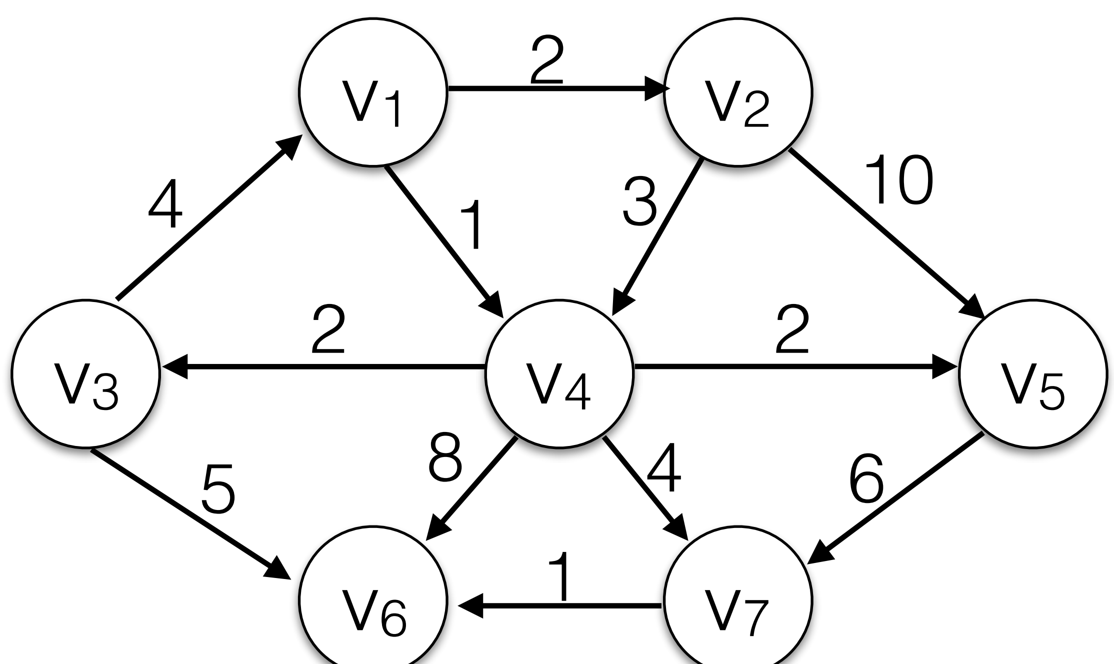
            - if v.distance == ∞
                - **v.prev = u**
                - v.distance = u.distance + 1
                - enqueue(v)

Weighted Shortest Paths

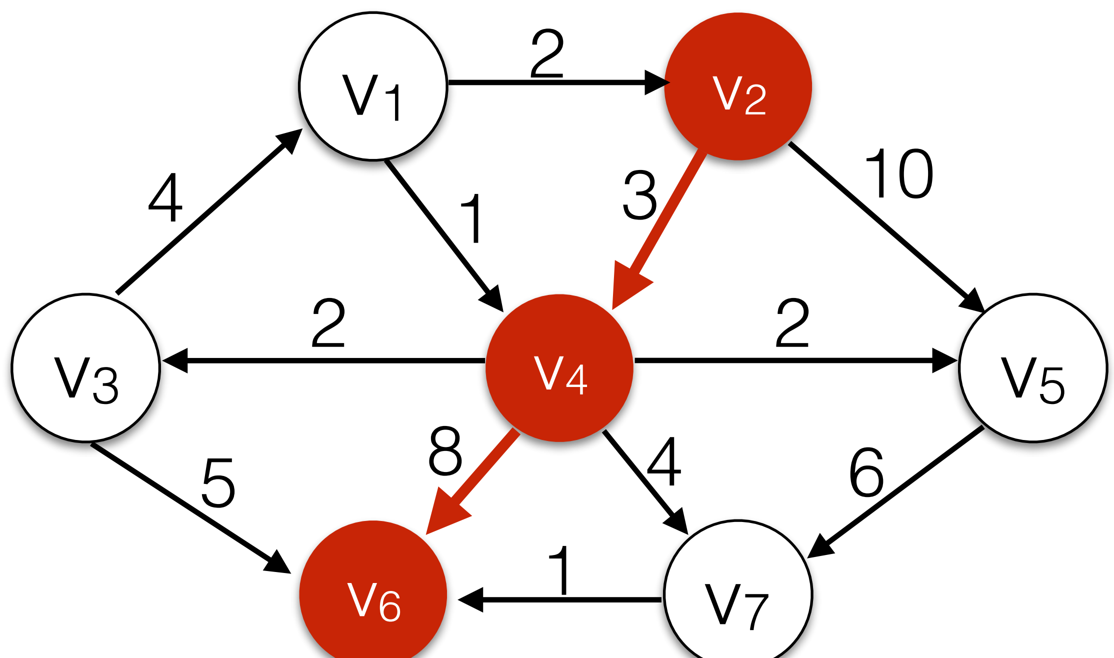
- Goal: Find the shortest path between two vertices s and t.



What is the shortest path between v_2 and v_6?

Weighted Shortest Paths

- Goal: Find the shortest path between two vertices s and t.
- Normal BFS will find this path.

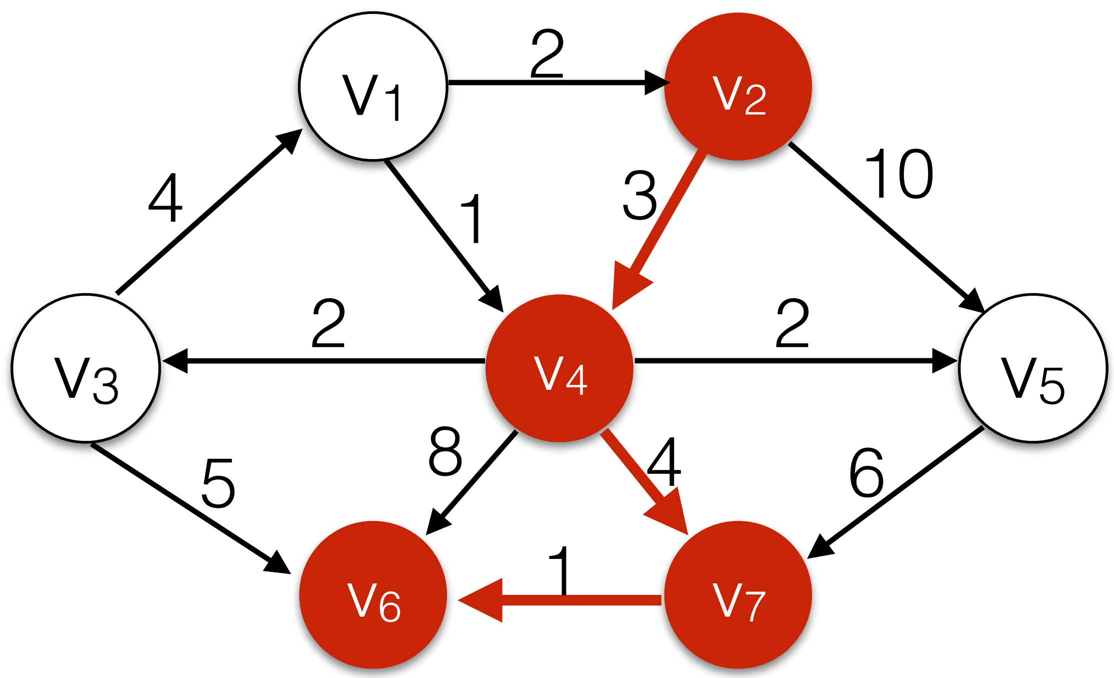


length 2
cost 11

What is the shortest path between v_2 and v_6?

Weighted Shortest Paths

- Goal: Find the shortest path between two vertices s and t.
- This path is shorter.

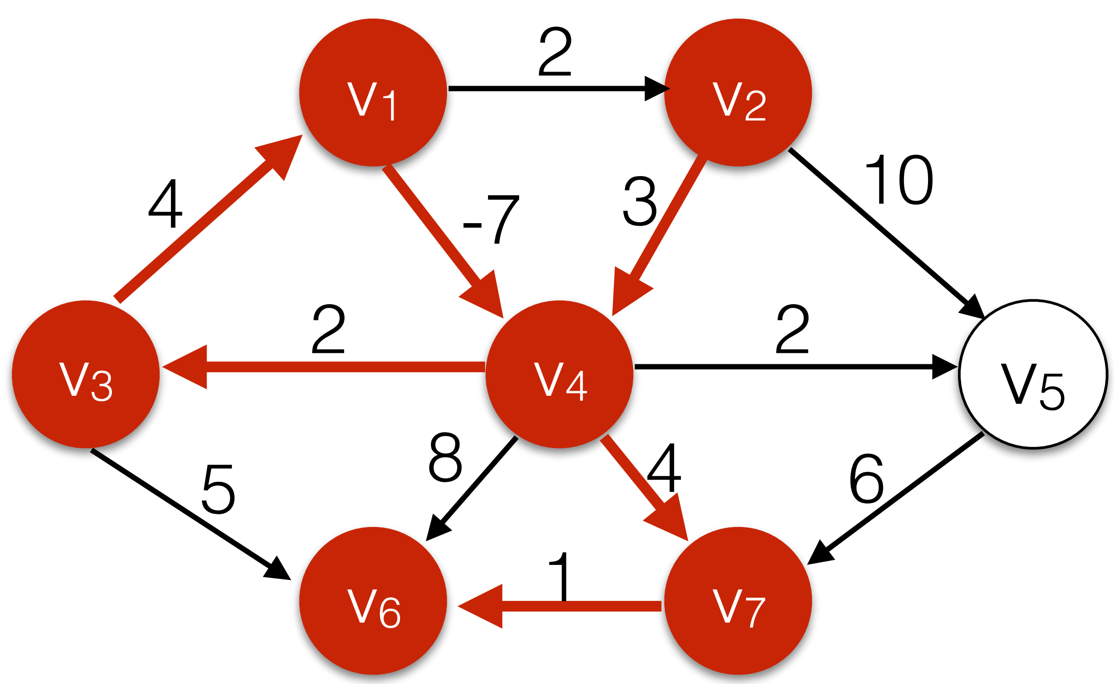


length 3
cost 8

What is the shortest path between v_2 and v_6?

Negative Weights

- We normally expect the shortest path to be simple.
- Edges with Negative Weights can lead to negative cycles.
- The concept of "shortest path" is then not clearly defined.



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Dijkstra’s Algorithm for Weighted Shortest Path

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- Keep nodes on a **priority queue** and always expand the vertex with the lowest cost annotation first! ← This is a **greedy** algorithm
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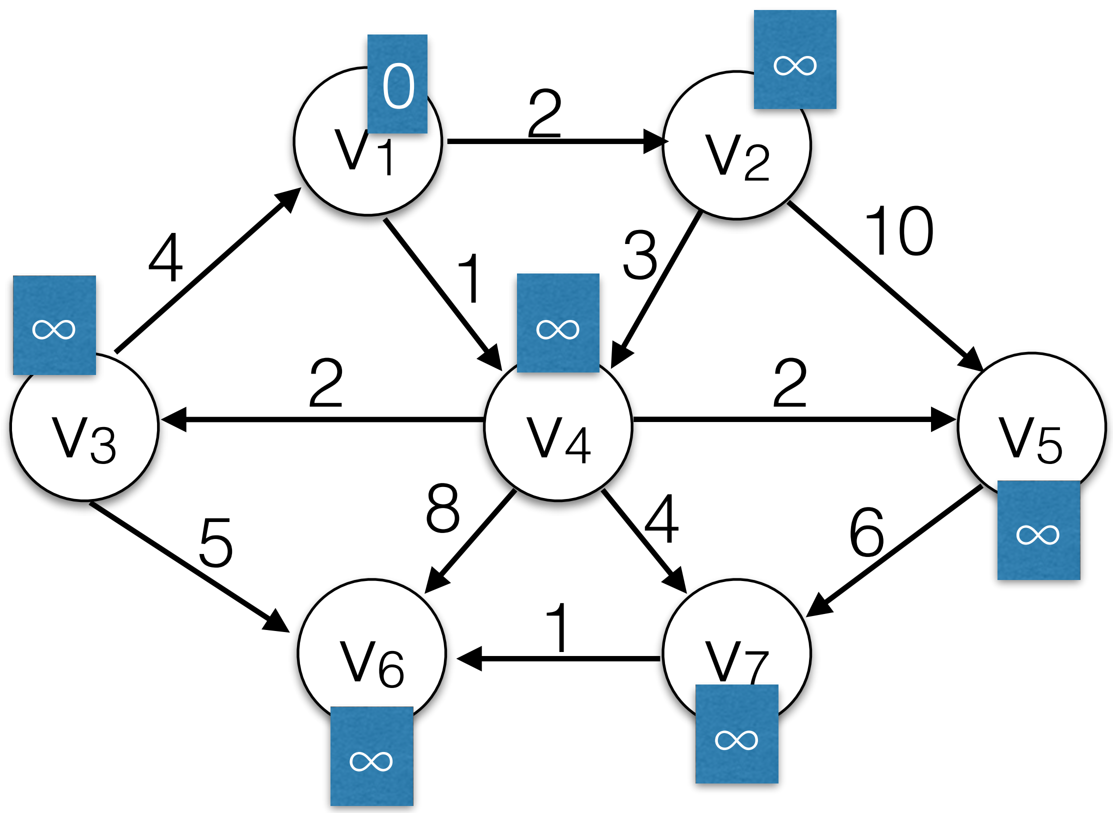
Dijkstra's Algorithm



Use a **Priority Queue** *q*

- for all v ∈ V

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- While *q* is not empty:
  - u <- *q*.deleteMin()
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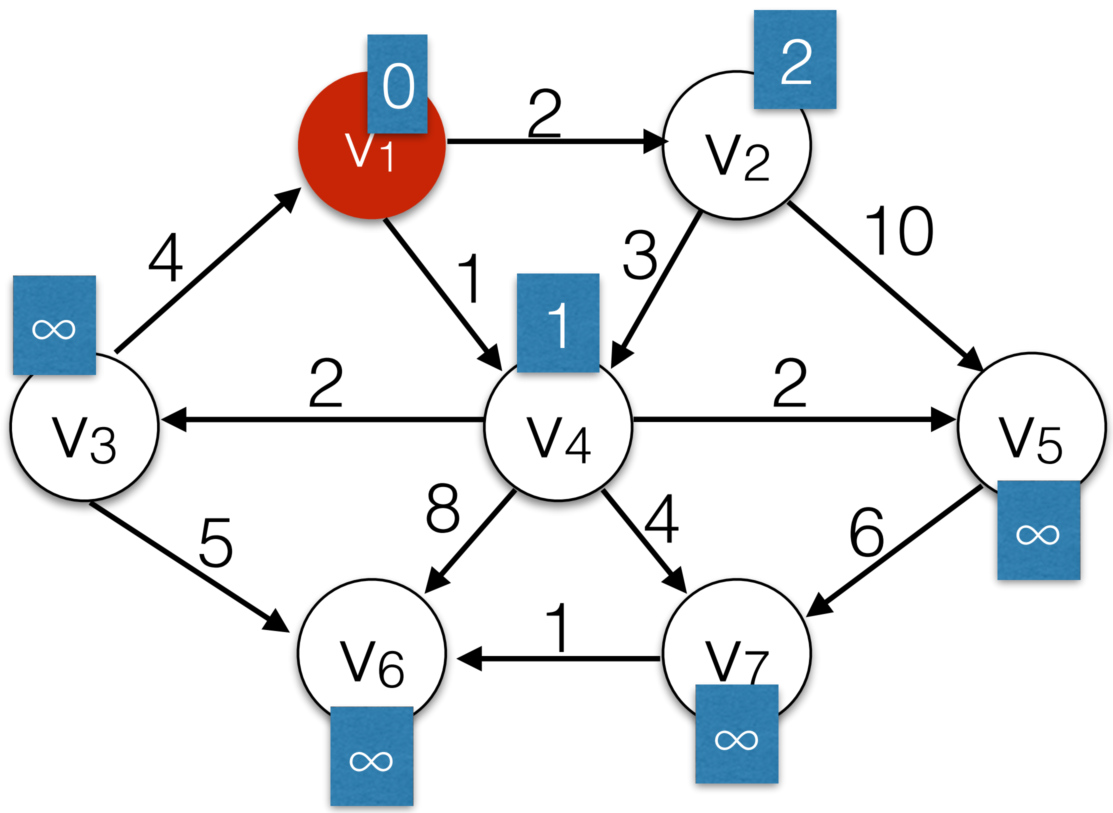

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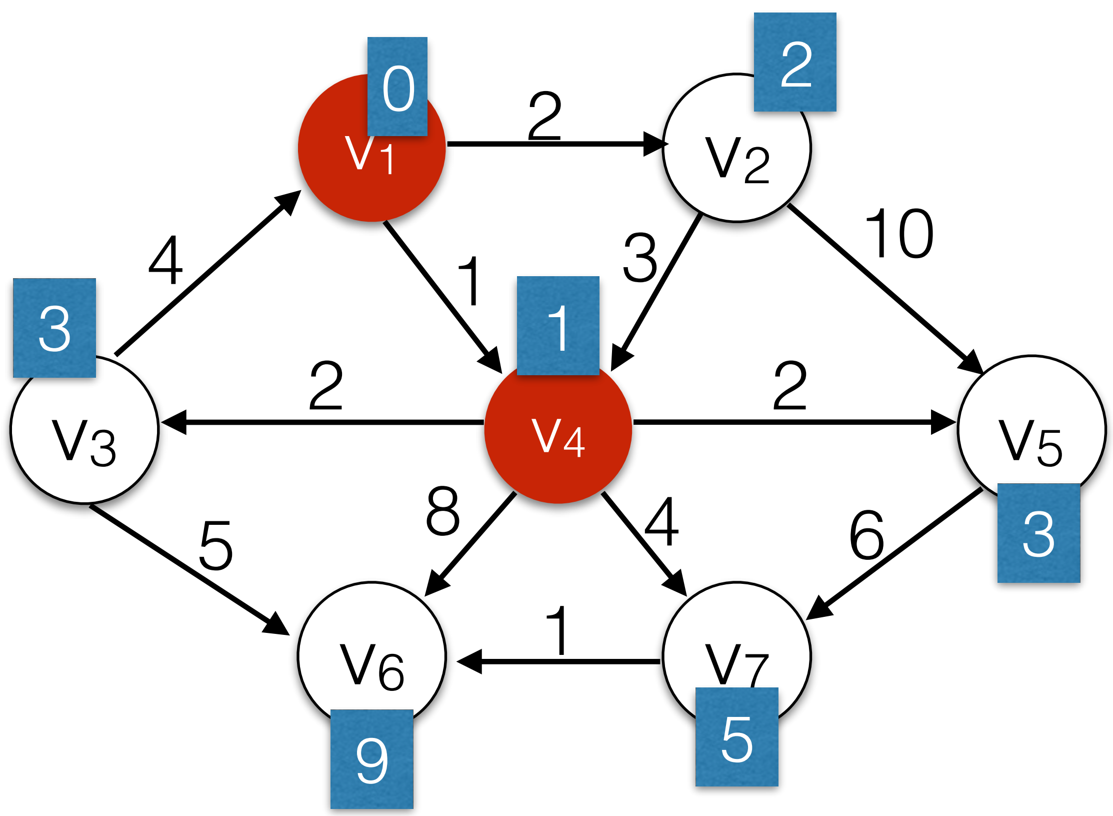

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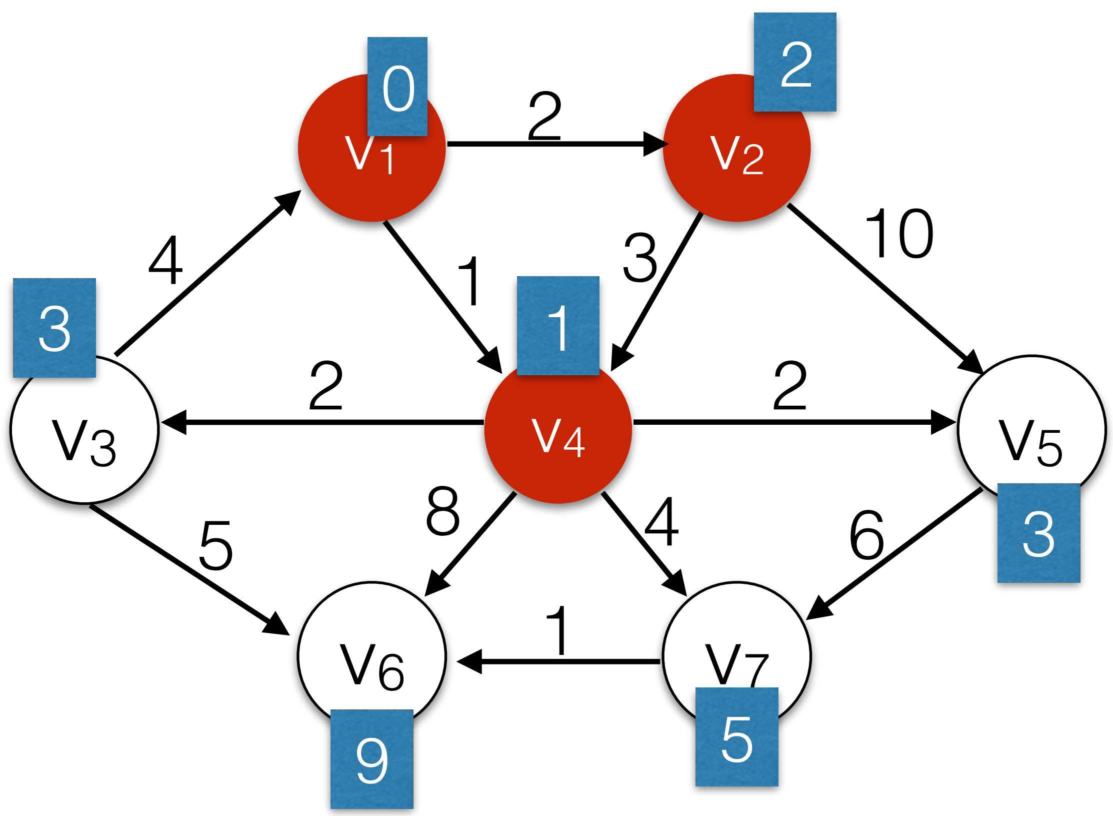

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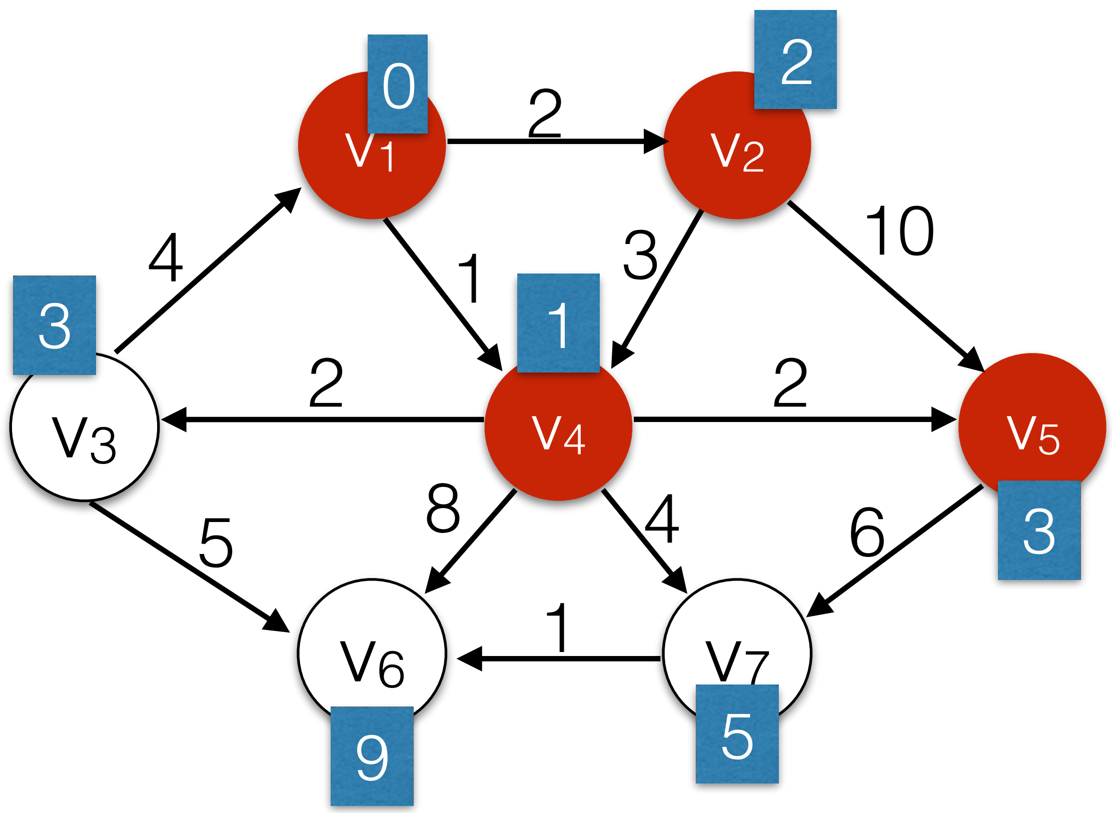

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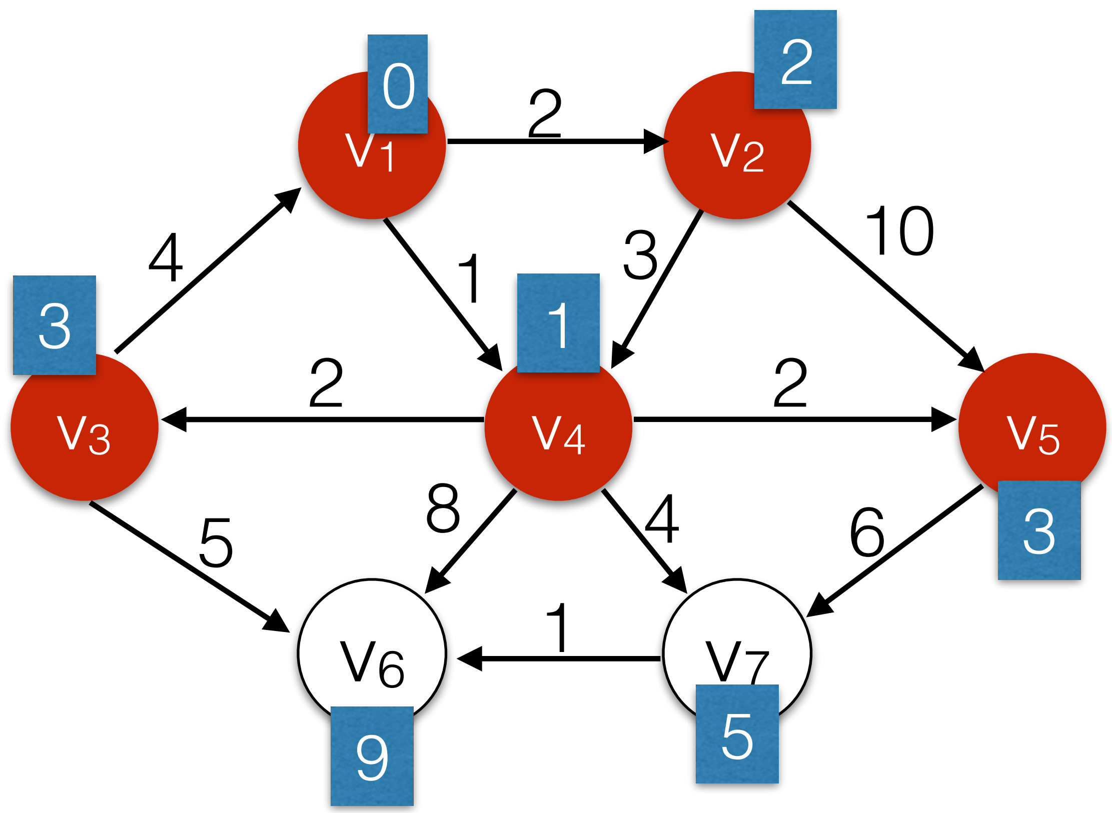

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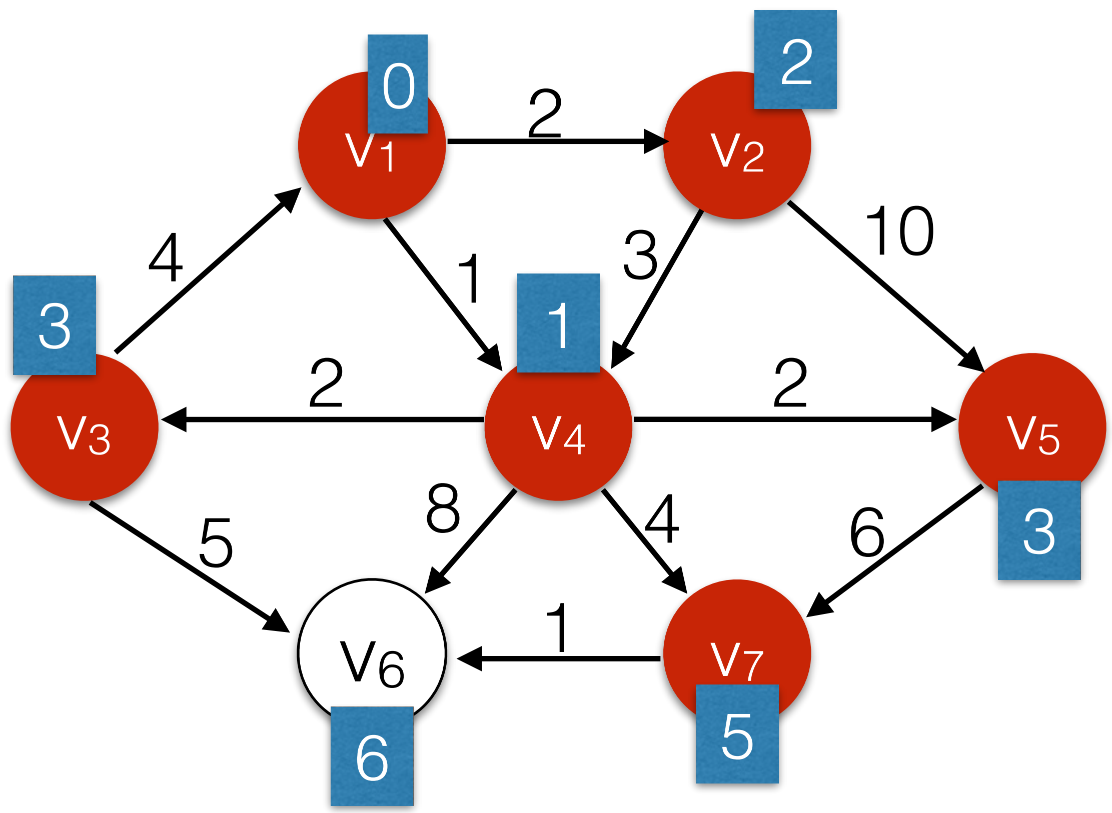

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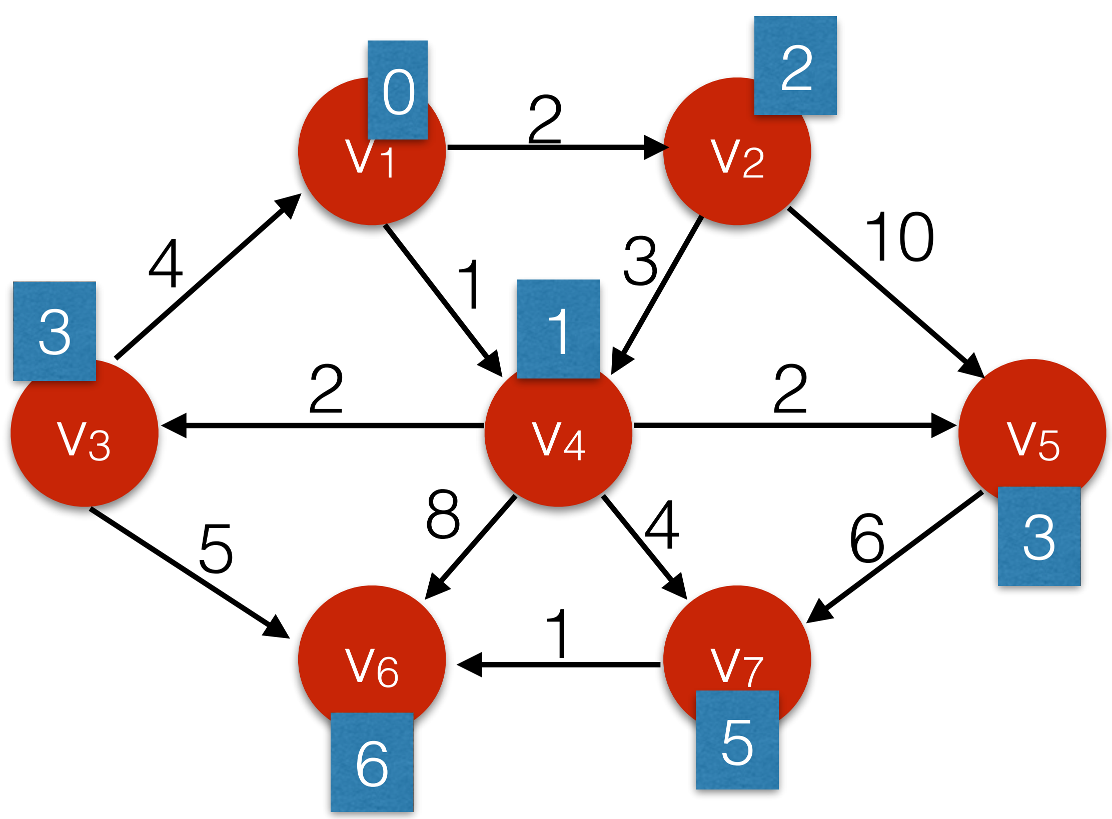

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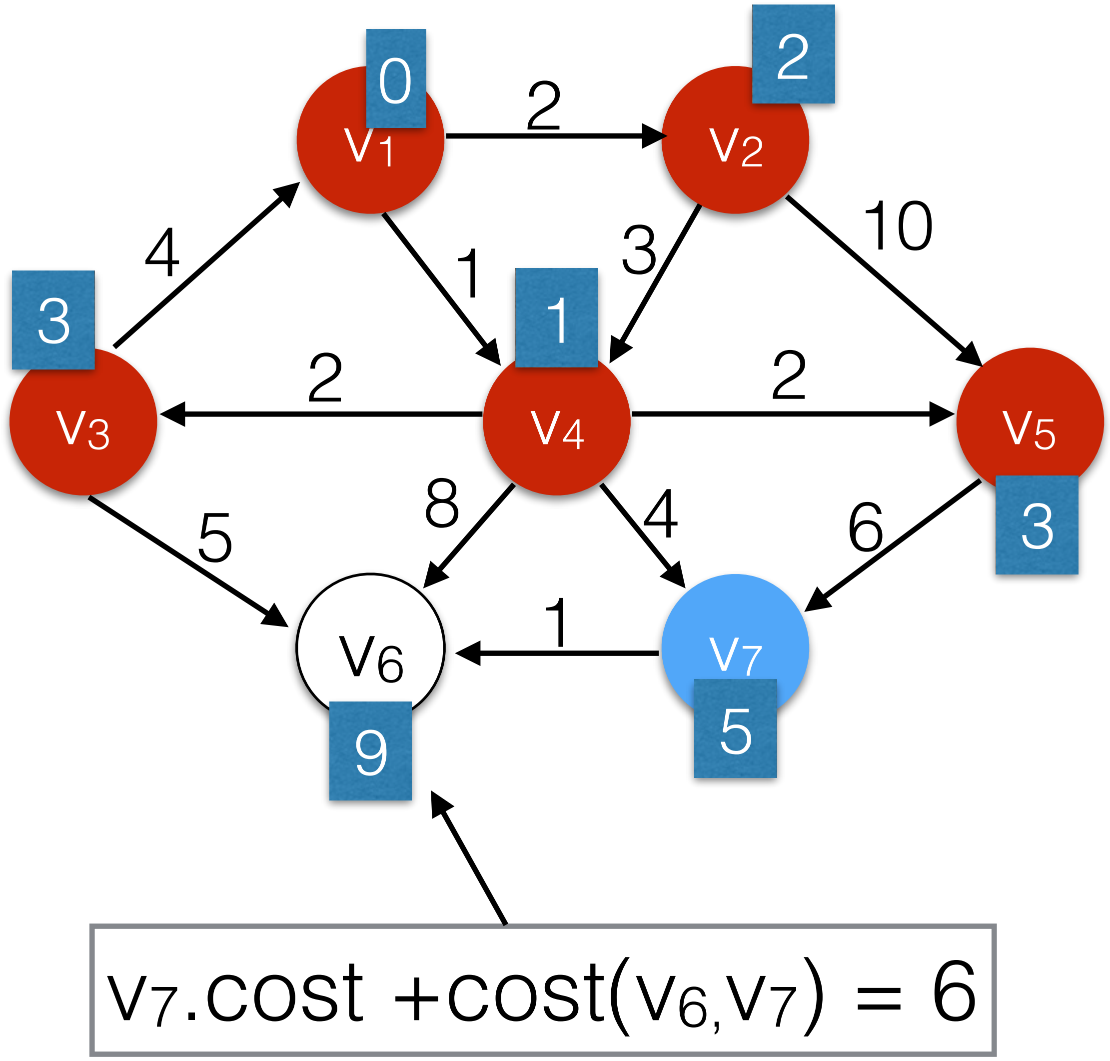

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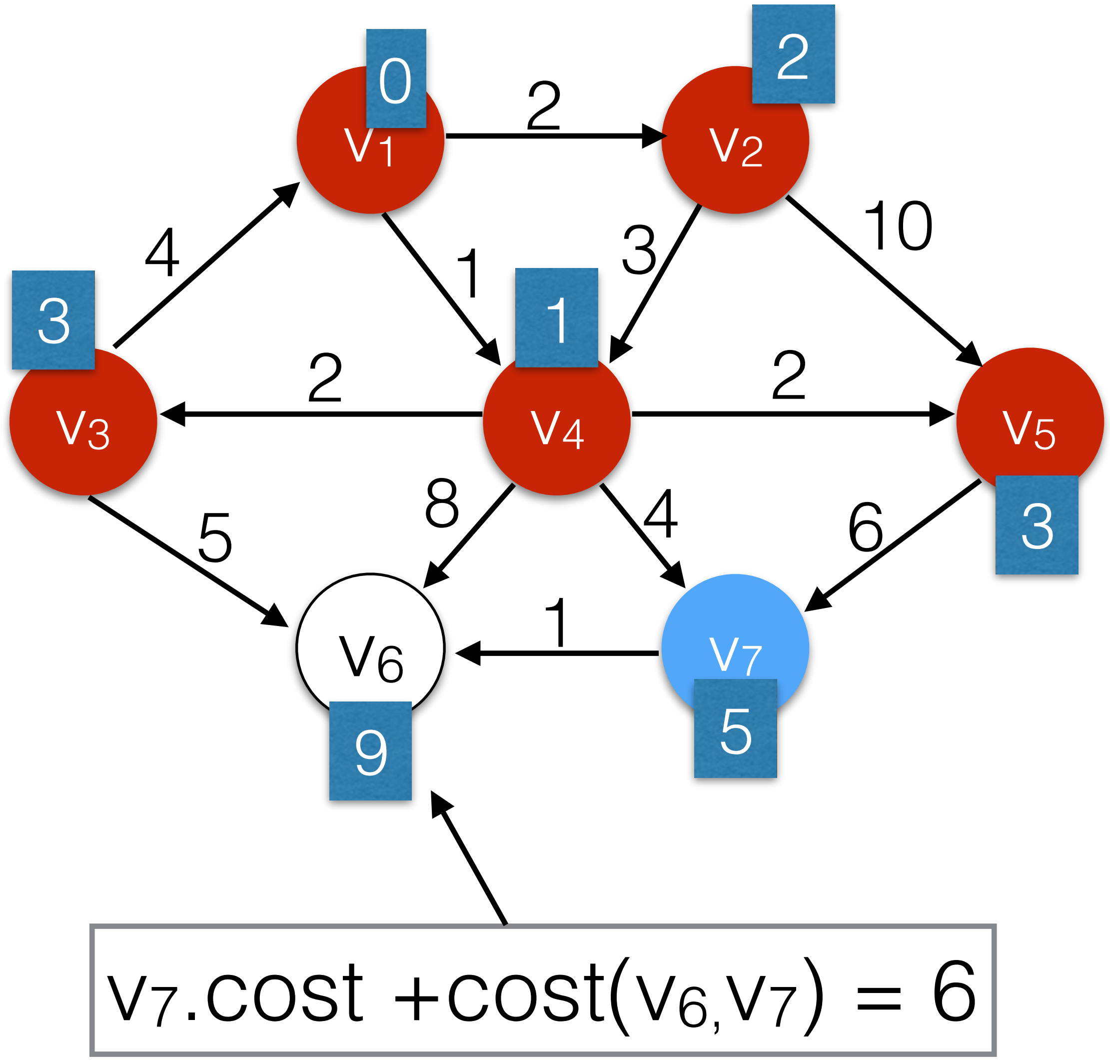

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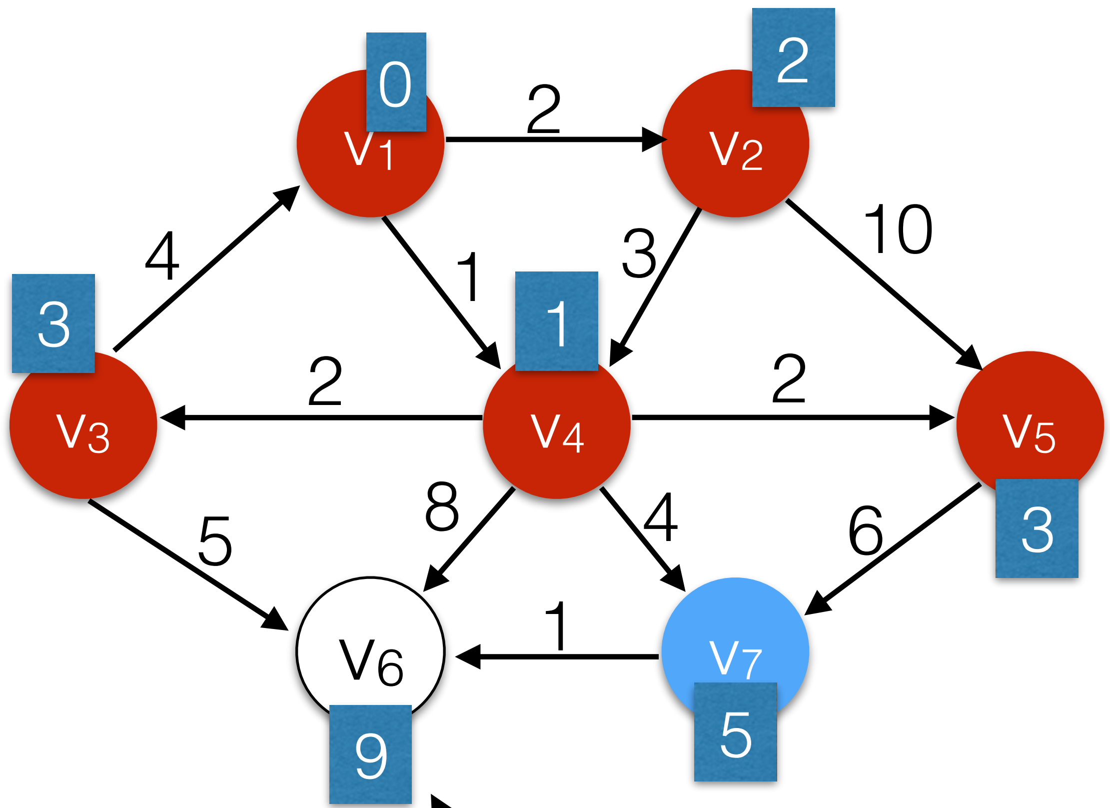
- v_7 is already in *q*, and has not been visited.
- does insert(v_7) create a new entry in the *q* or update the existing one?
- if q is a heap, updating the cost will change v_7 everywhere in the heap and might make the heap invalid.

Dijkstra's Algorithm - Fixed



v_7.cost +cost(v_6,v_7) = 6

- Keep a separate cost object in the queue that isn't updated.
- Ignore duplicate entries for vertices.

Use a **Priority Queue** *q*

- for all v ∈ V
  set v.cost = ∞, set v.visited = false
- s.cost = 0
- q.insert(**(0, s)**)
- While *q* is not empty:
  - **(costu, u)** <- *q*.deleteMin()
  - **if not u.visited:**
    - u.visited = true
    - for each edge (*u*,*v*):
      - if not v.visited:
        - if (**costu** + cost(u,v) < v.cost)
          - v.cost = u.cost + cost(u,v)
          - v.prev = u
          - q.insert(**(v.cost, v)**)

Dijkstra’s Running Time

- There are |E| insert and deleteMin operations.
- The maximum size of the priority queue is O(|E|). Each insert takes O(log |E|)

O(|E| log |E|)

Use a Priority Queue *q*

- for all v ∈ V

  set v.cost = ∞, set v.visited = false
- s.cost = 0
- q.insert((0, s))
- While *q* is not empty:
  - (costu, u) <- *q*.deleteMin()
  - if not u.visited:
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$O(|E| \log |E|)$
$= O(|E| \log |V|)$

because $|E| \leq |V|^2$,
and therefore
$\log |E| \leq 2 \log |V|$

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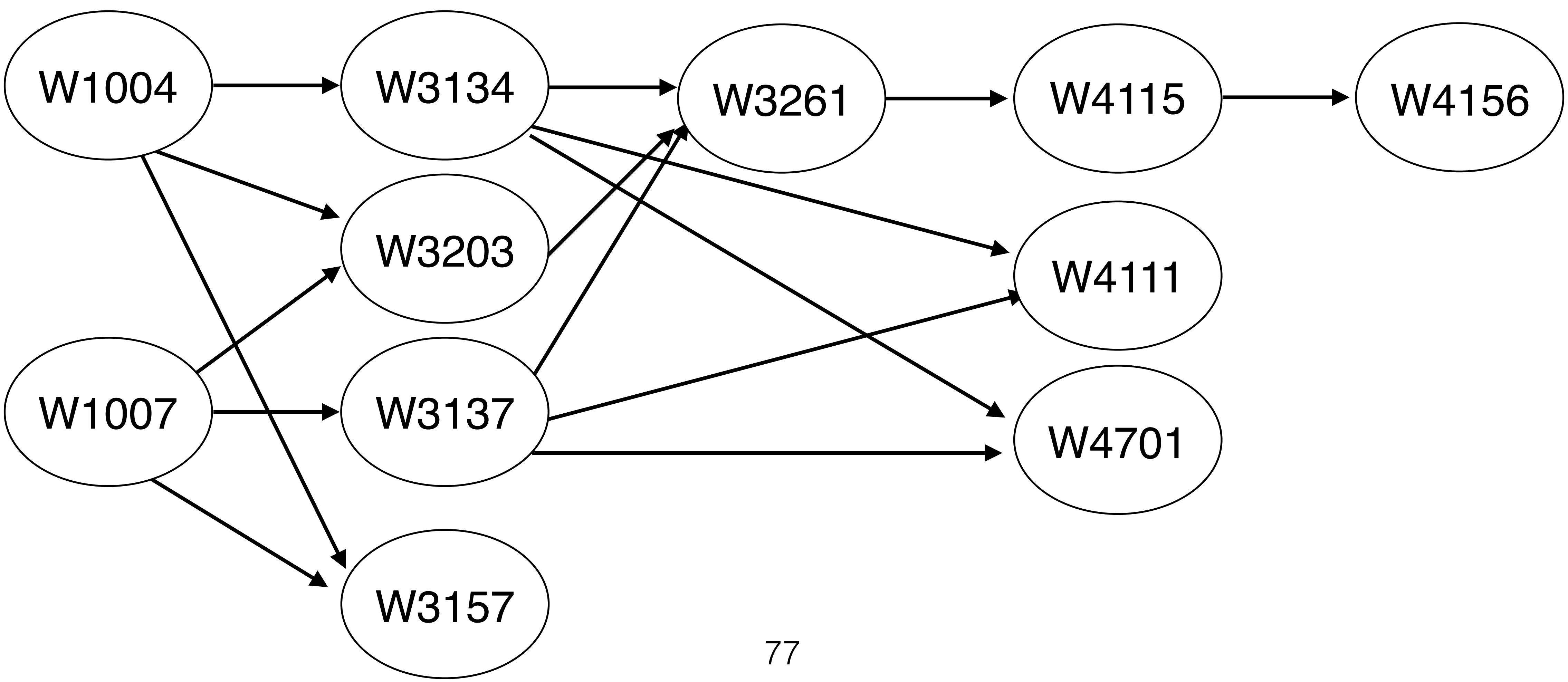
Topological Sort in DAGs

A topological sort of a DAG is an ordering of its vertices such that if there is a path from *u* to *w*, *u* appears before *w* in the ordering.



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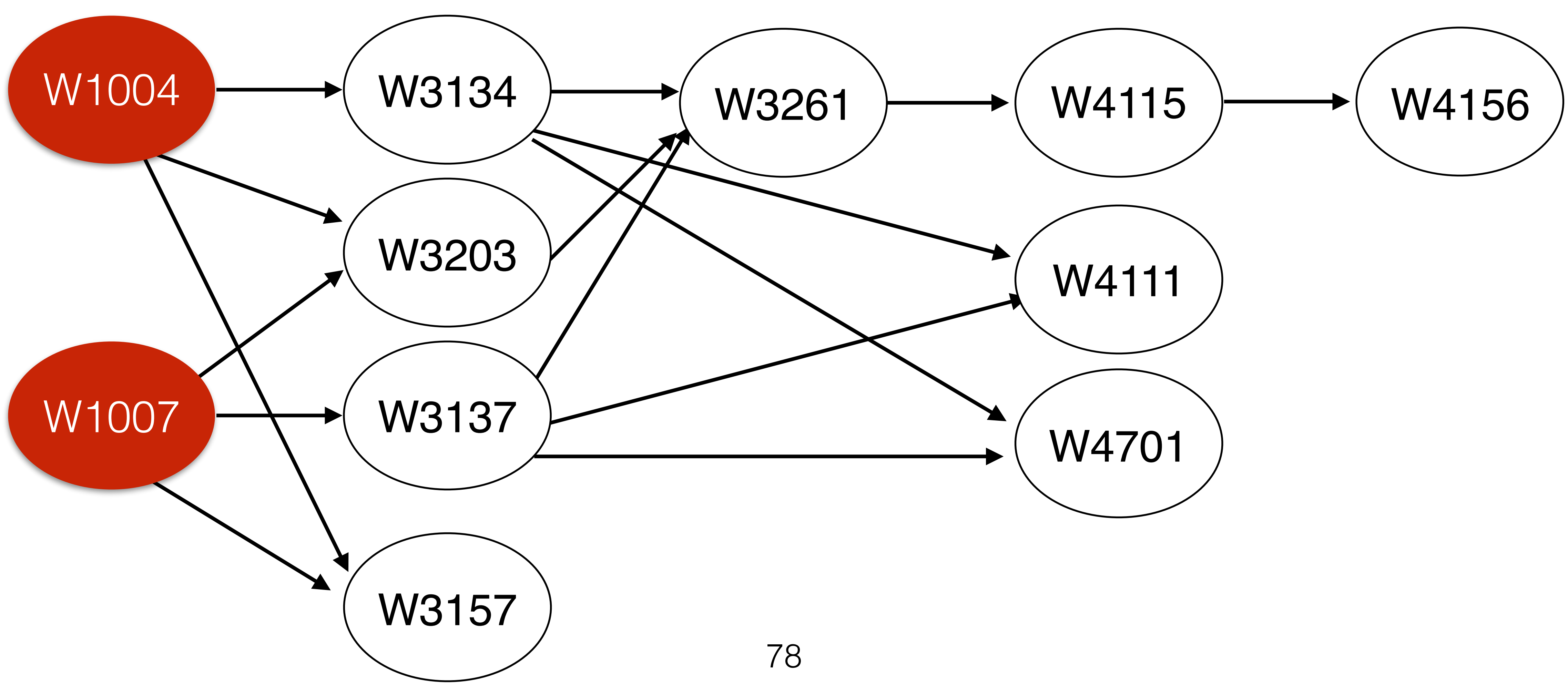
W1007 W1004



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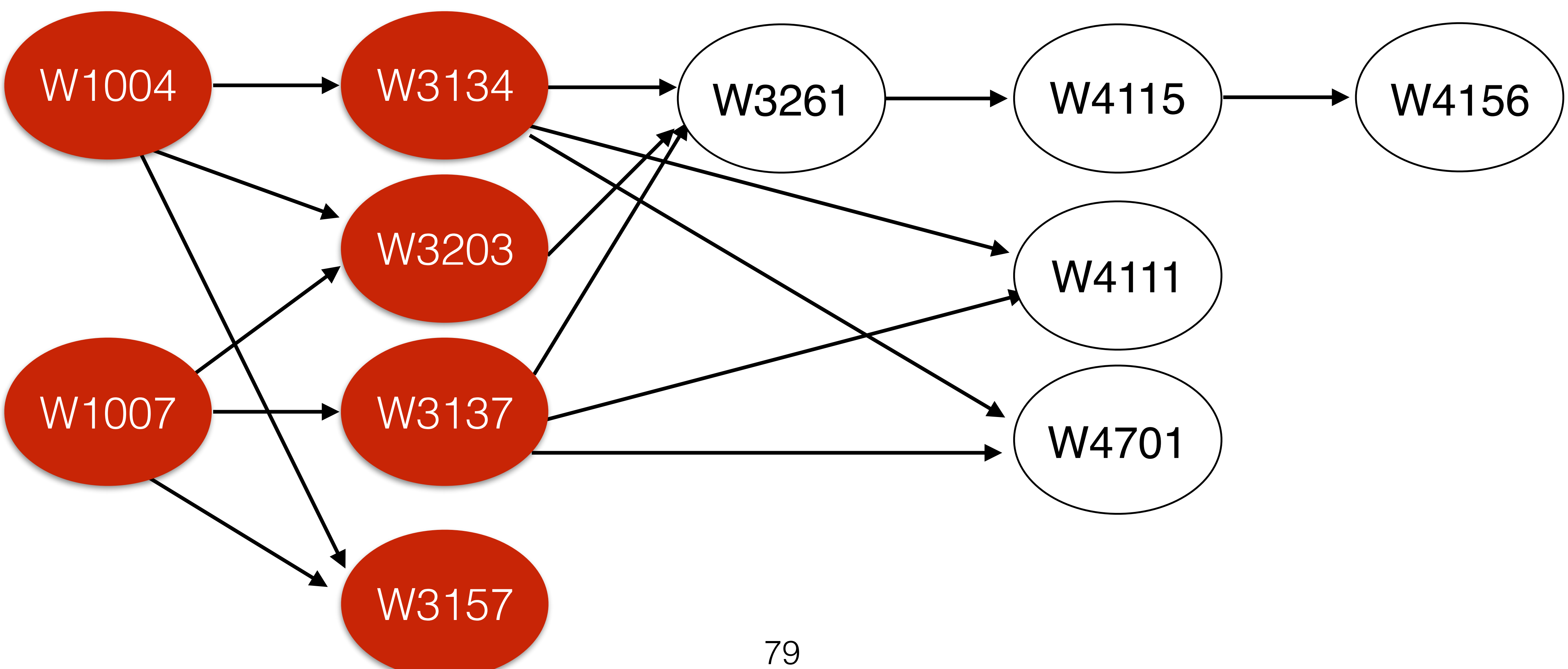

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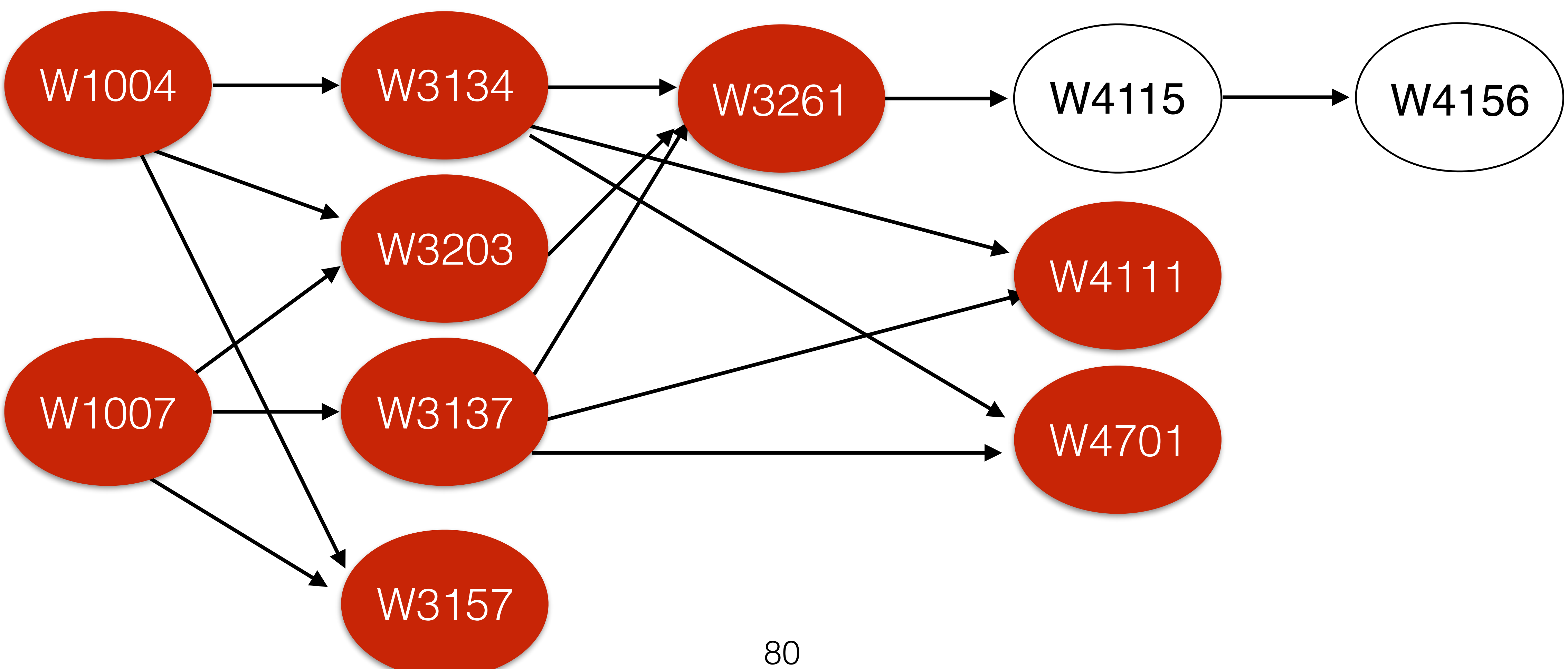

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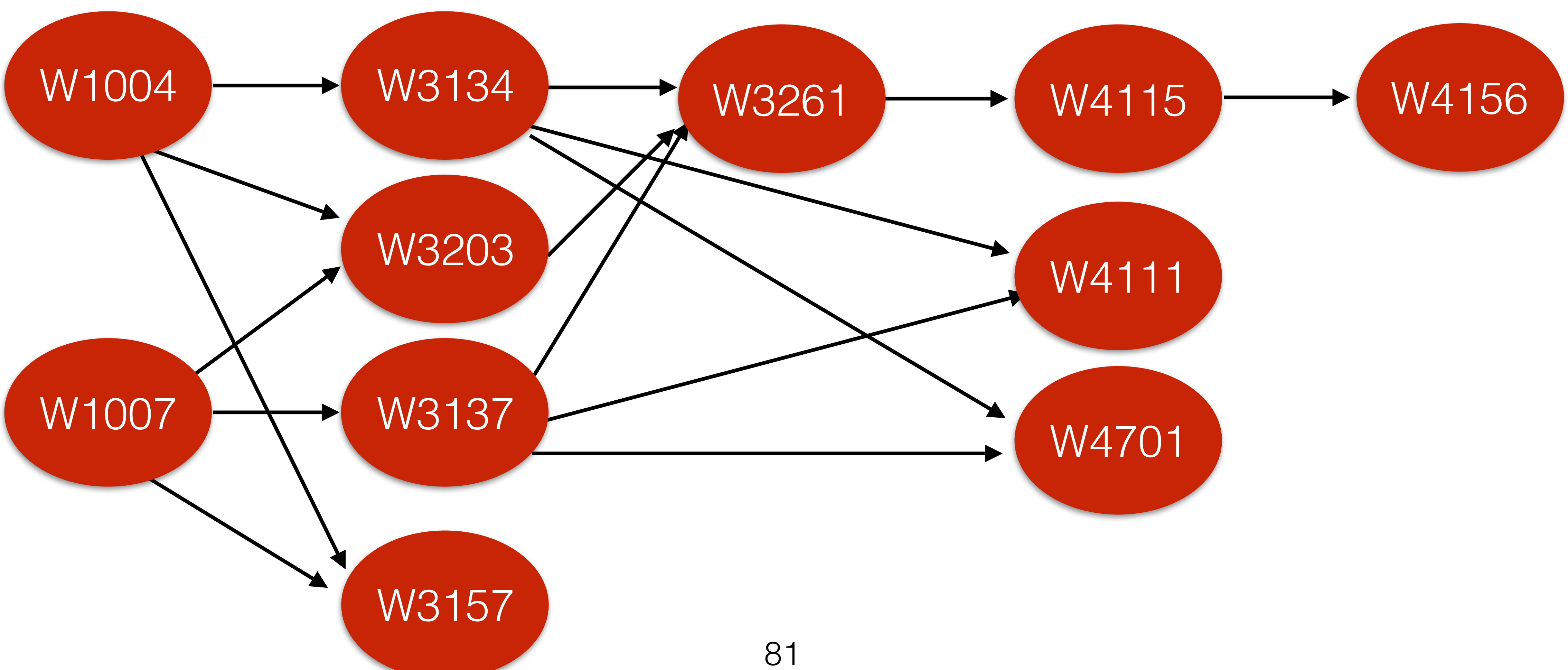
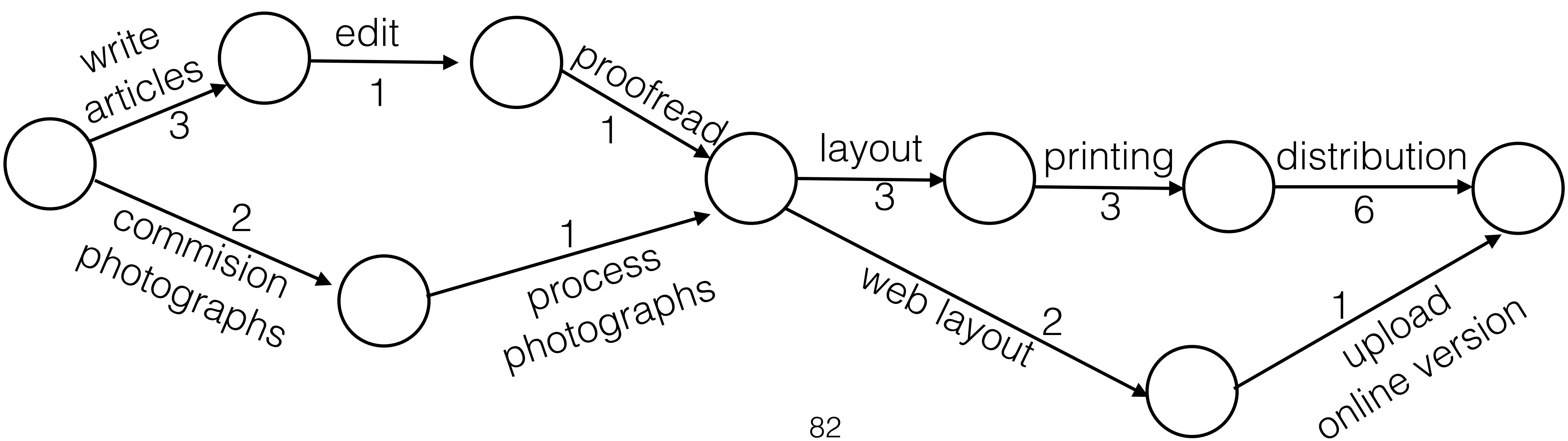


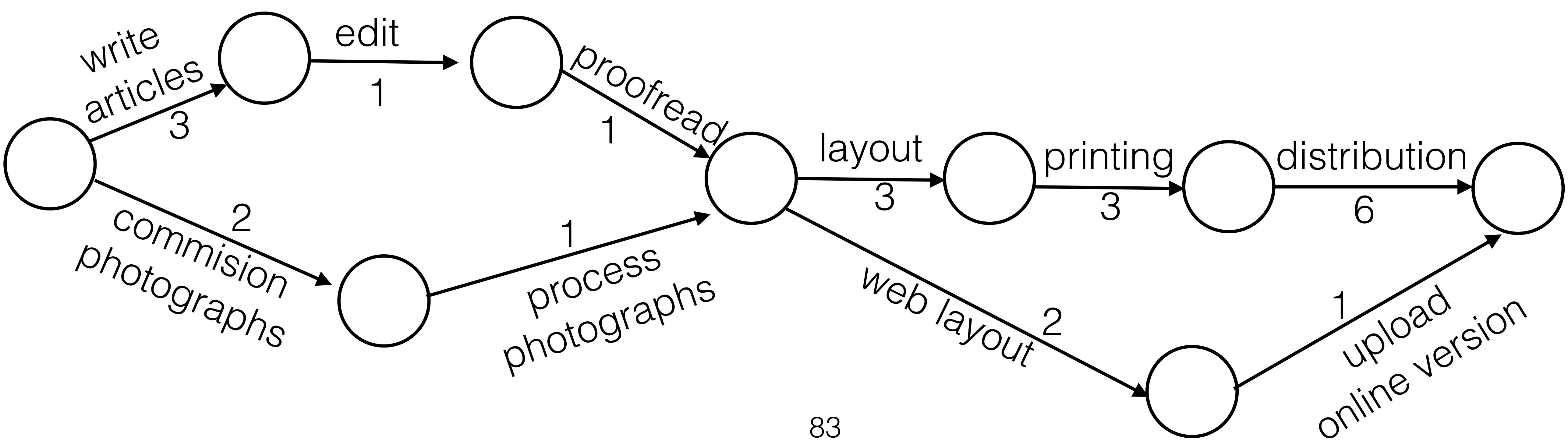
Application: Critical Path Analysis

- An **Event-Node Graph** is a DAG in which
  - Edges represent tasks, weight represents the time it takes to complete the task.
  - Vertices represent the event of completing a set of tasks.



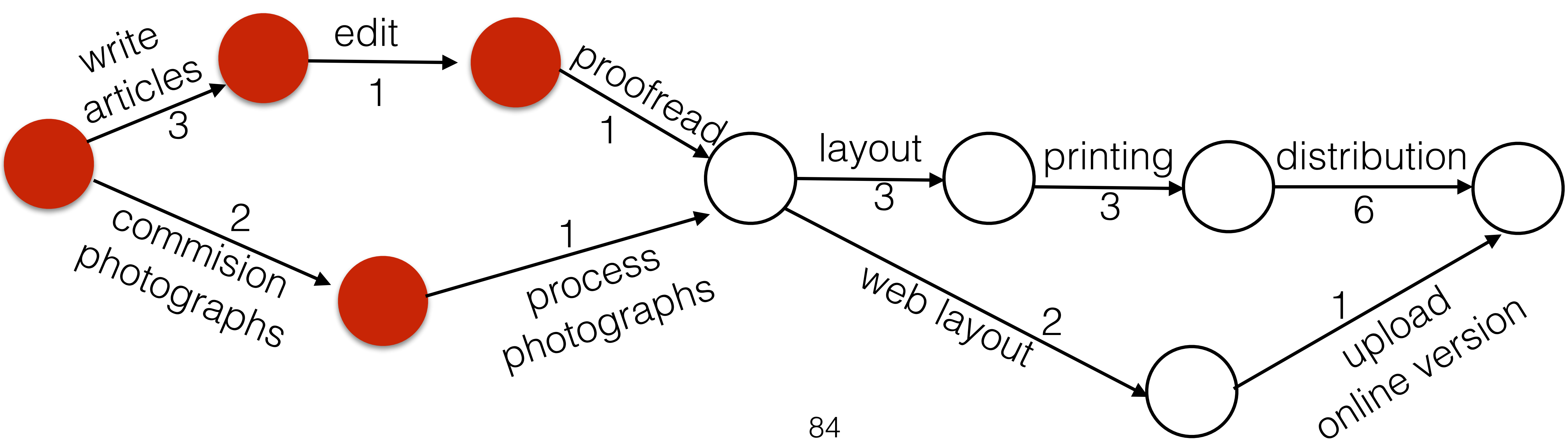
Application: Critical Path Analysis

- We are interested in the earliest completion time. (Earliest time we can reach the final event).
- This is equivalent to finding the *longest* path through the DAG (why does this not work with cycles?).



Application: Critical Path Analysis

- If an event has more than one incoming event, all tasks have to be finished before other tasks can proceed.



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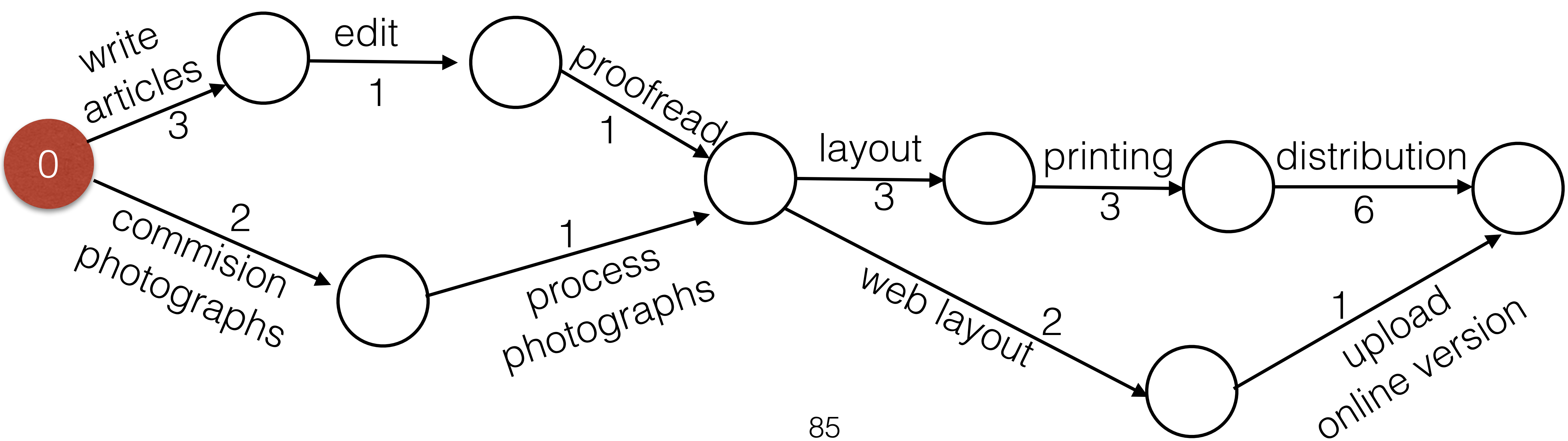
- Basic idea: Compute the earliest completion time for each event.
- Can use Dijkstra's algorithm O(|E| log |V|).
- We now try to find the longest path.



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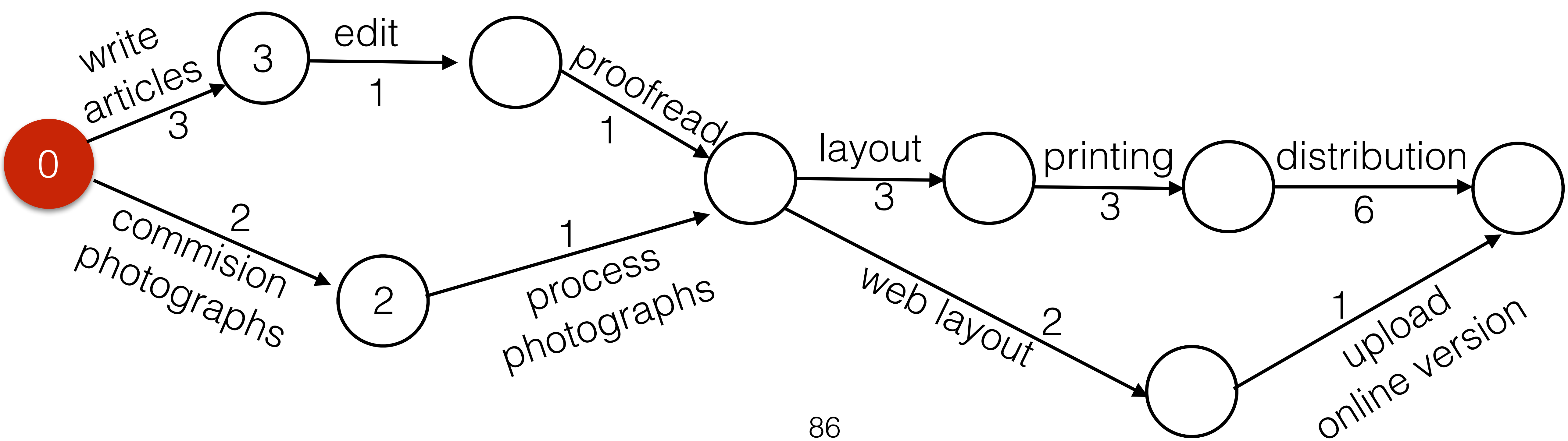

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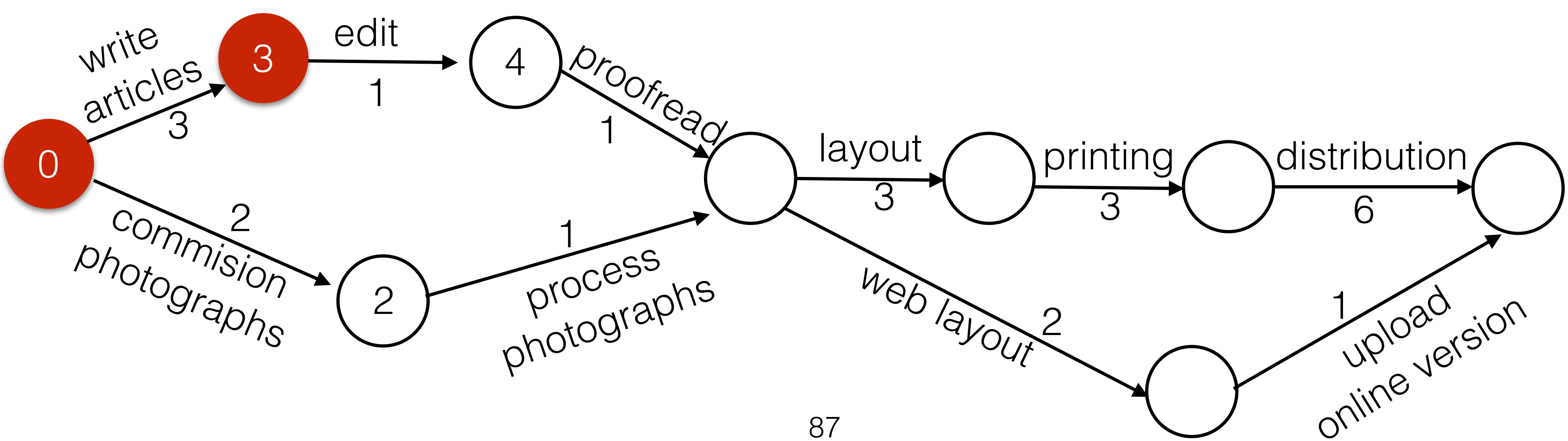

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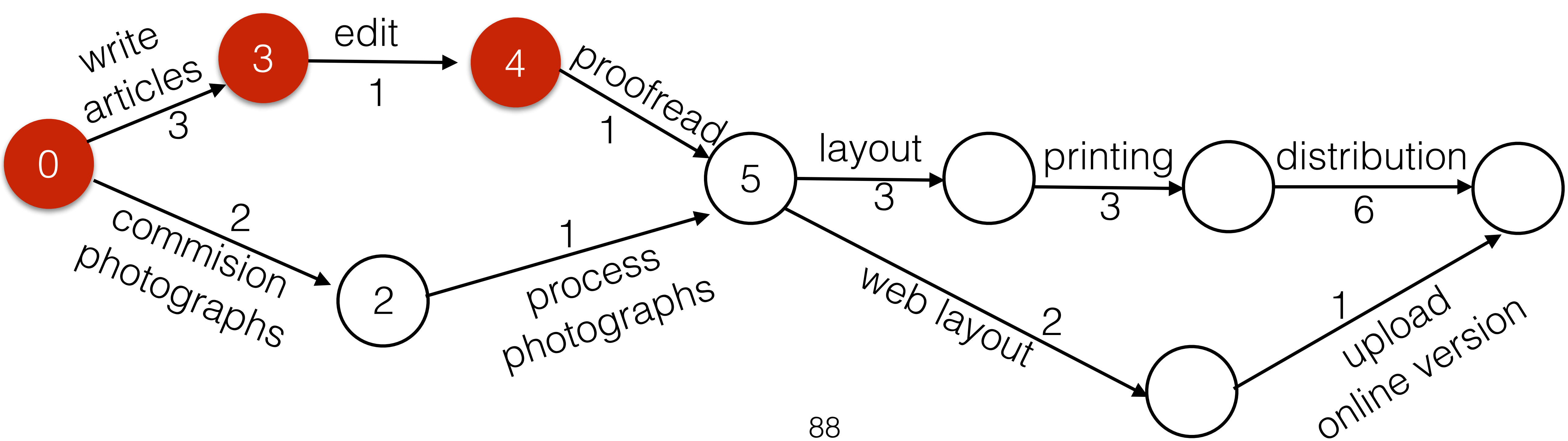

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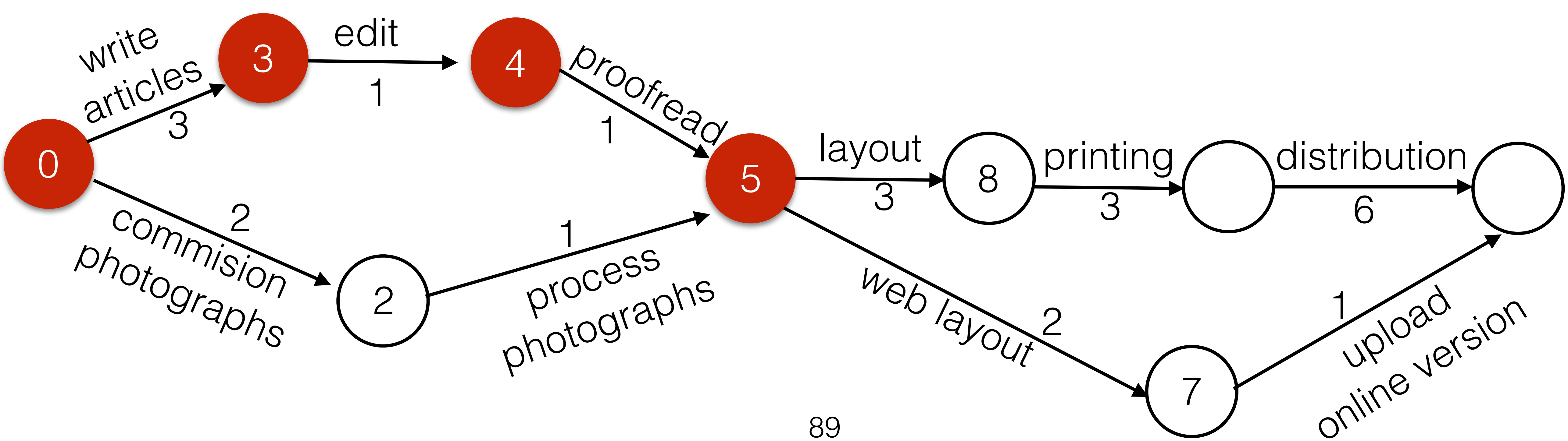

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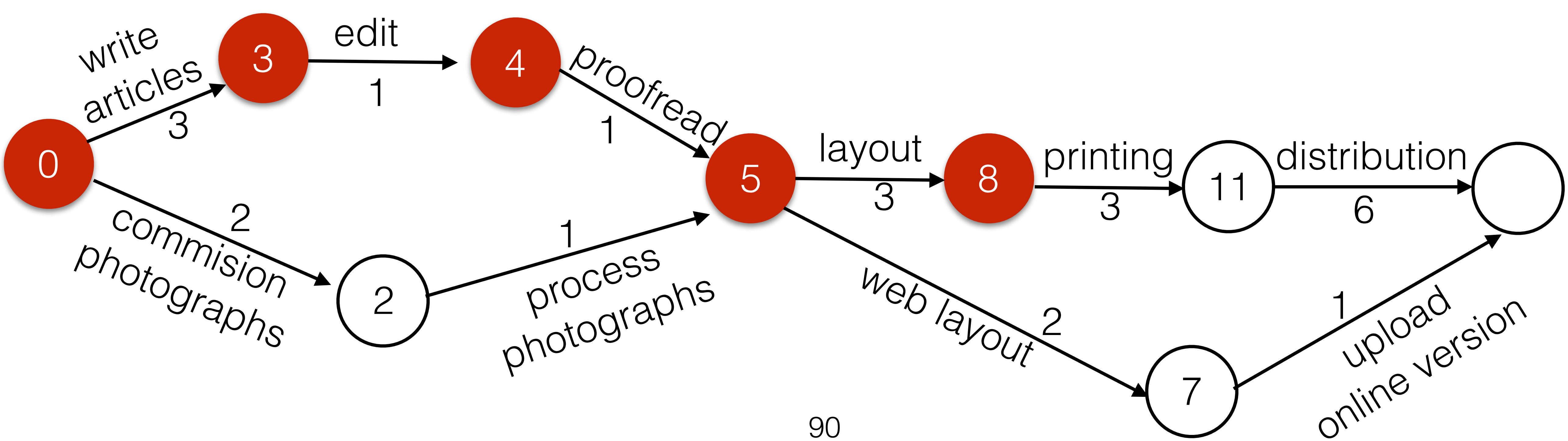

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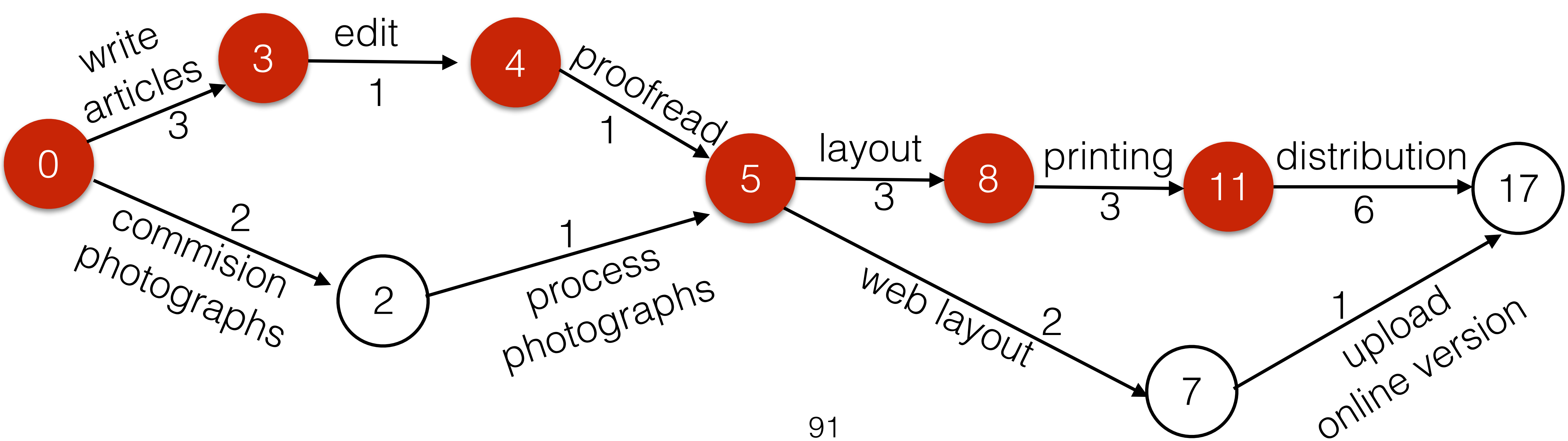

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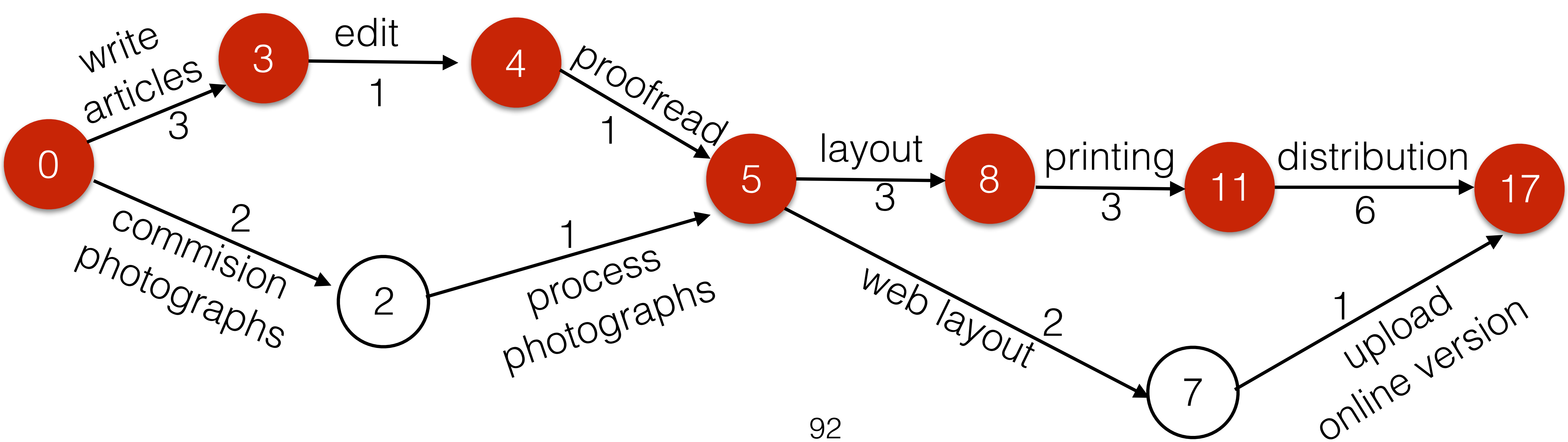

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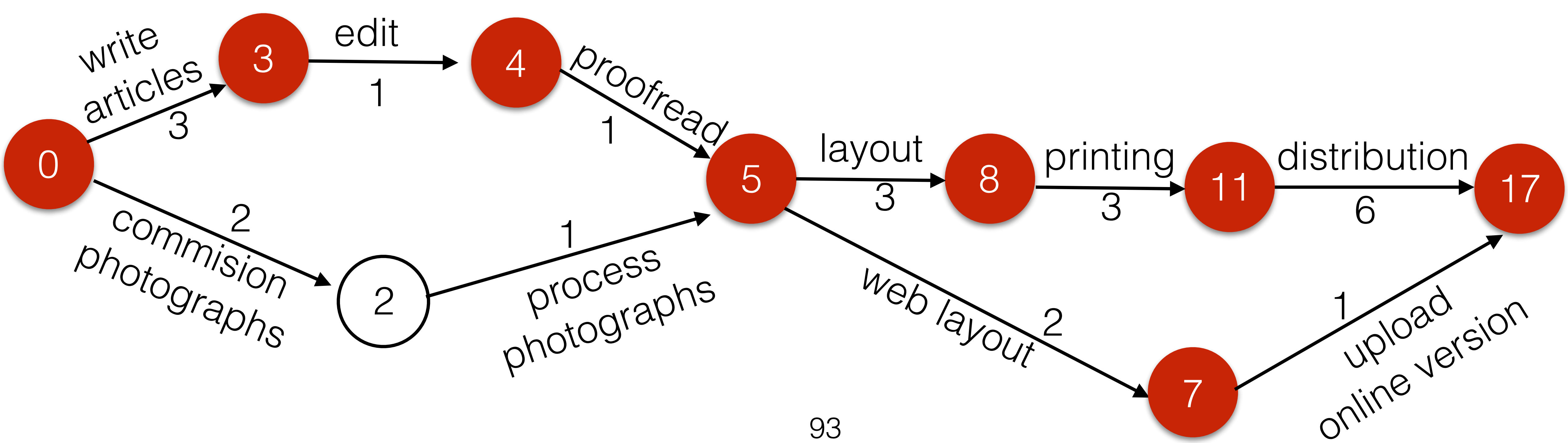

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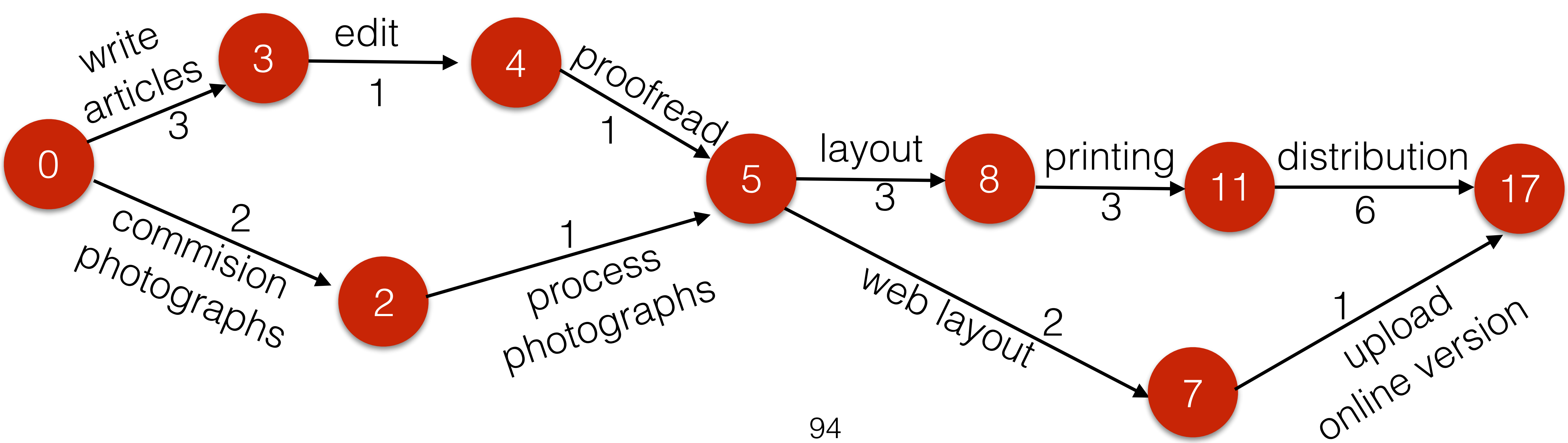
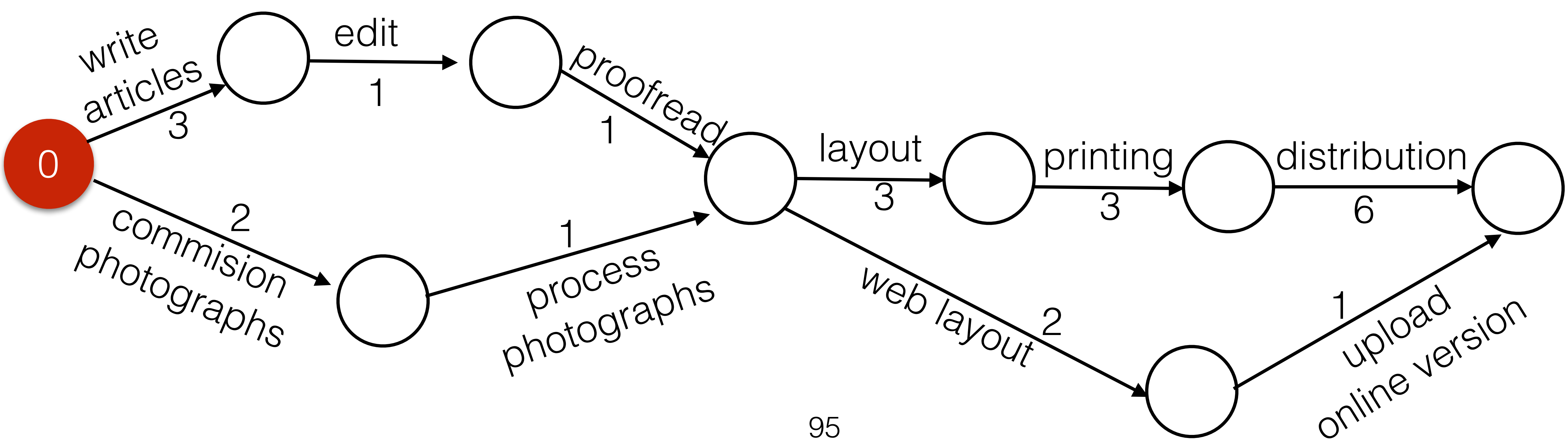


Application: Critical Path Analysis

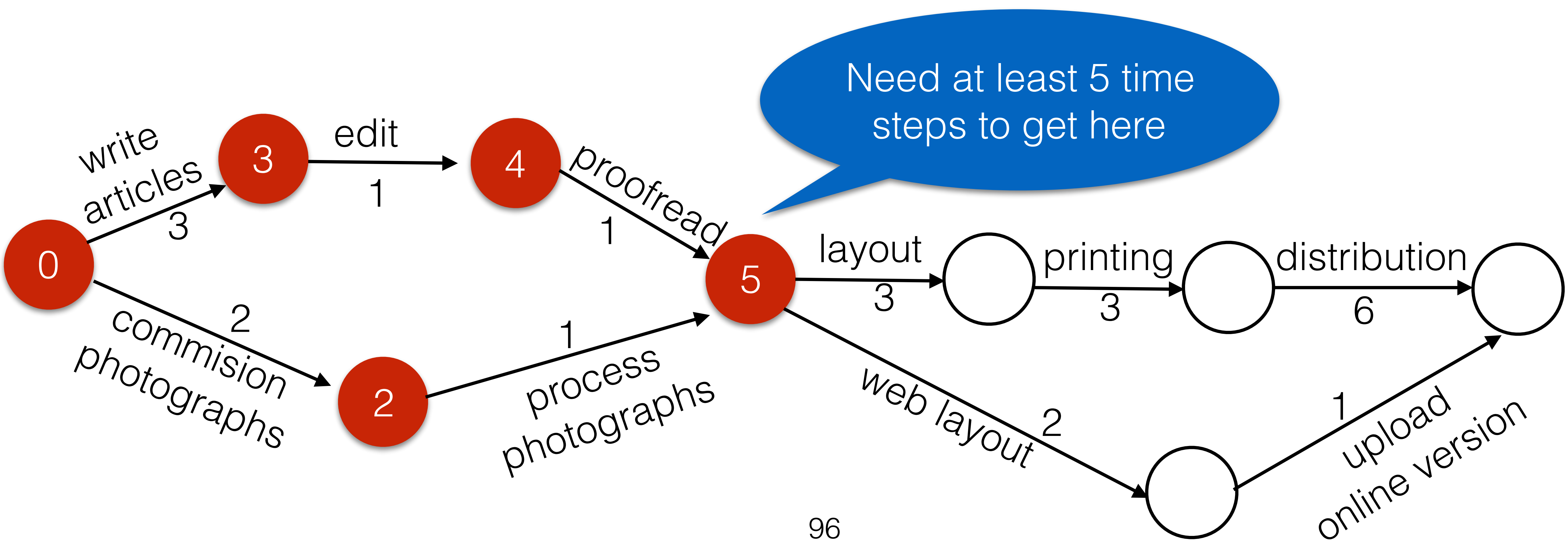
- For DAGs we can improve on Dijkstra's O(|E| log |V|) bound.
- Use topological sort.



Application: Critical Path Analysis

- Basic idea: Compute the earliest completion time for each event.

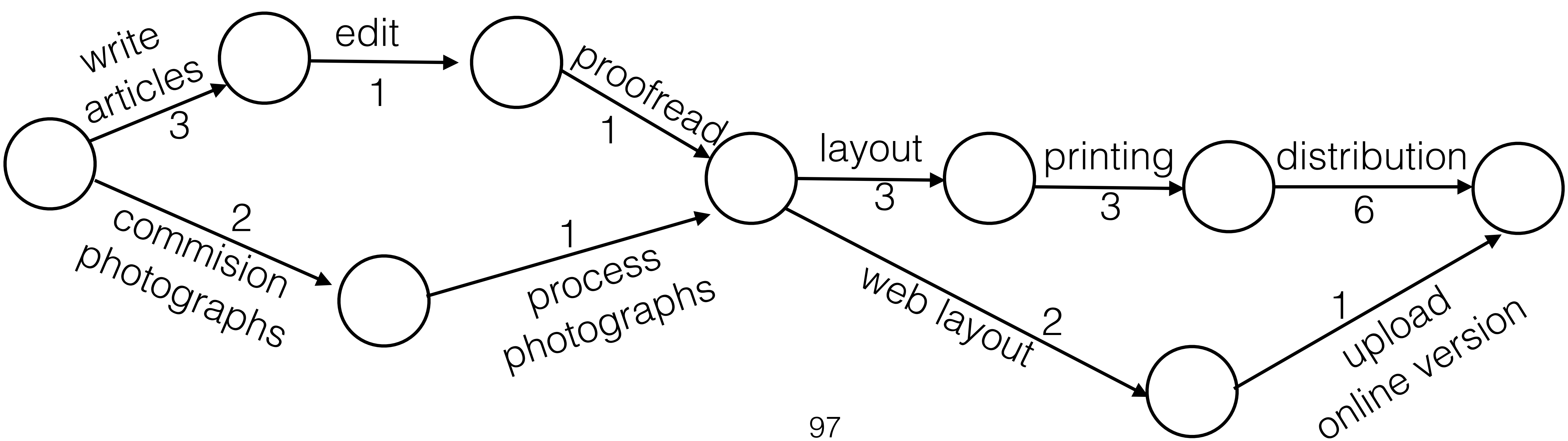
- Process events in topological order.



Computing Topological Order

- Basic idea: Use BFS!

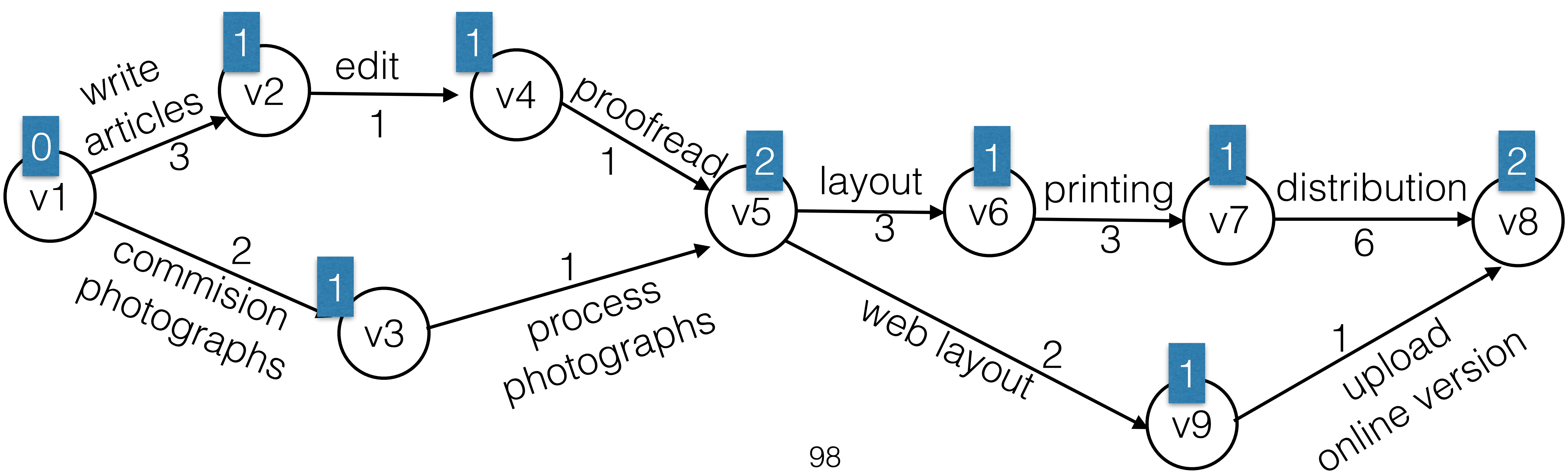
- To compute topological order, we need to find all incoming edges to a node first before visiting the node.



Computing Topological Order

- Example Application: Computing earliest completion time.

- First annotate each vertex with the number of incoming edges (the **indegree**).

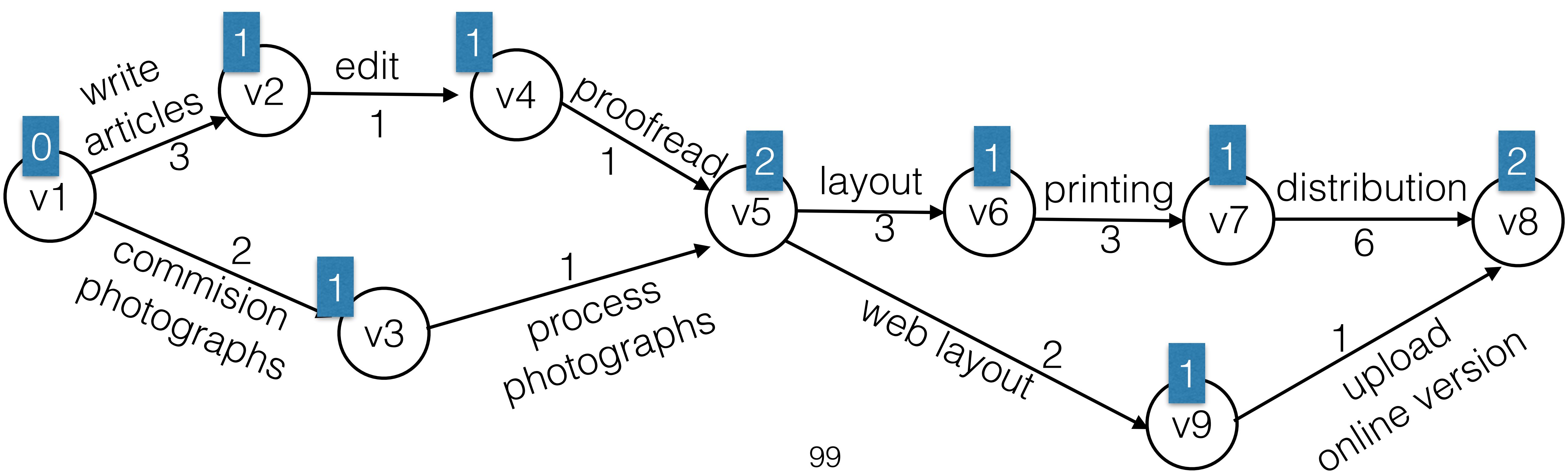


Computing Topological Order

- While the queue is not empty, dequeue a vertex, print it and decrement the indegree of its adjacent nodes.
- If the indegree of any new vertex becomes 0, enqueue it.

Queue: v1

Output:

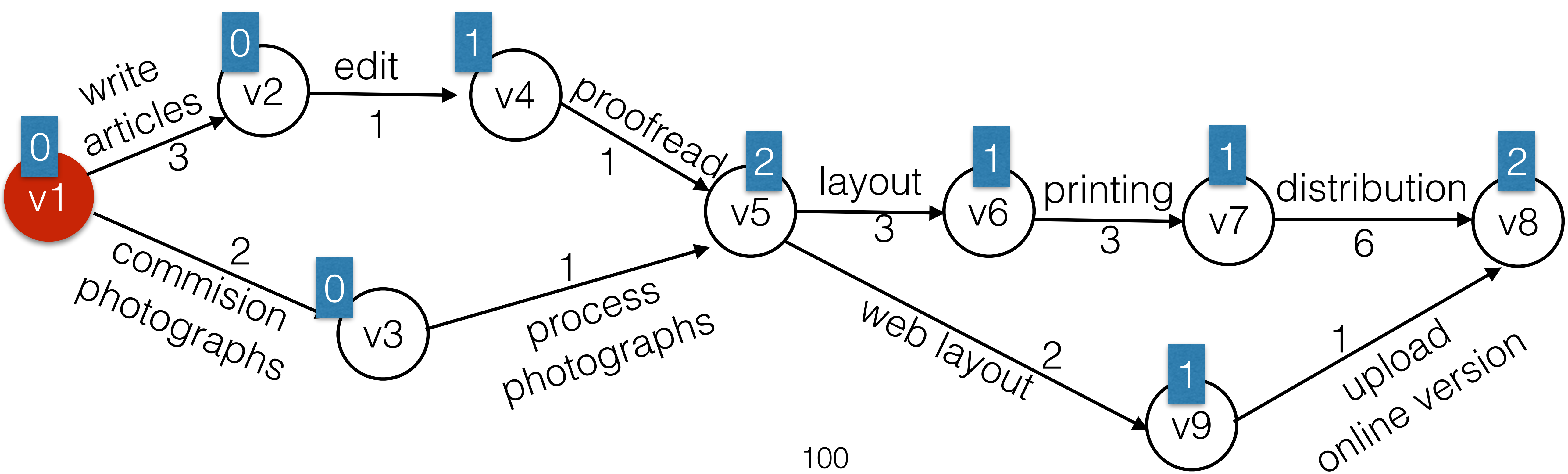


Computing Topological Order

- While the queue is not empty, dequeue a vertex, print it and decrement the indegree of its adjacent nodes.
- If the indegree of any new vertex becomes 0, enqueue it.

Queue: v2 v3

Output: v1

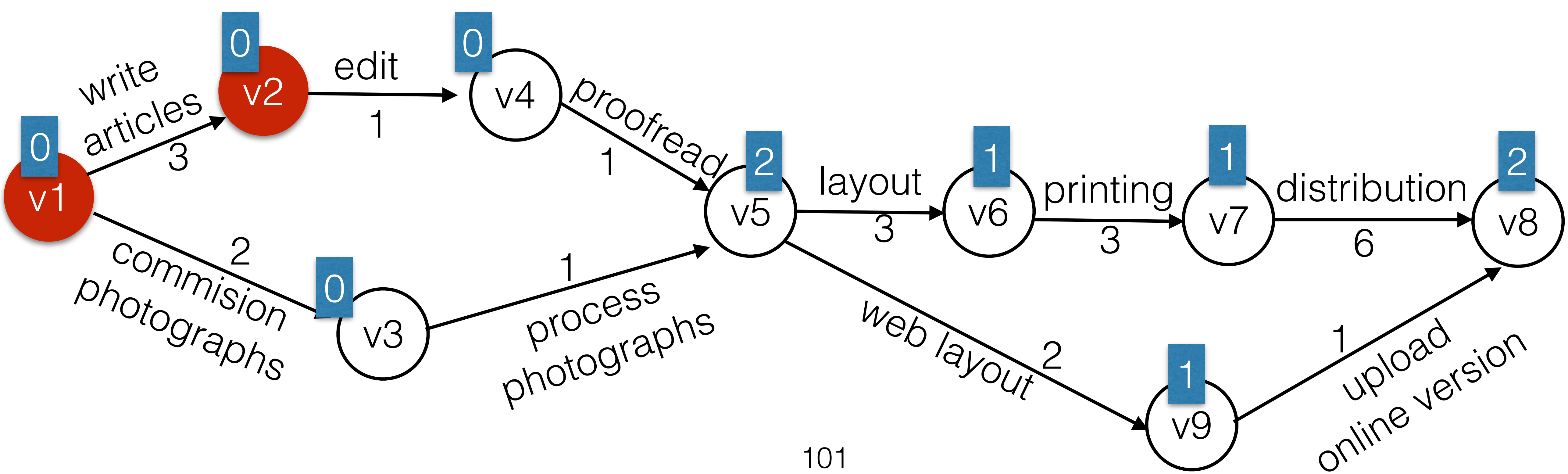


Computing Topological Order

- While the queue is not empty, dequeue a vertex, print it and decrement the indegree of its adjacent nodes.
- If the indegree of any new vertex becomes 0, enqueue it.

Queue: v3 v4

Output: v1 v2

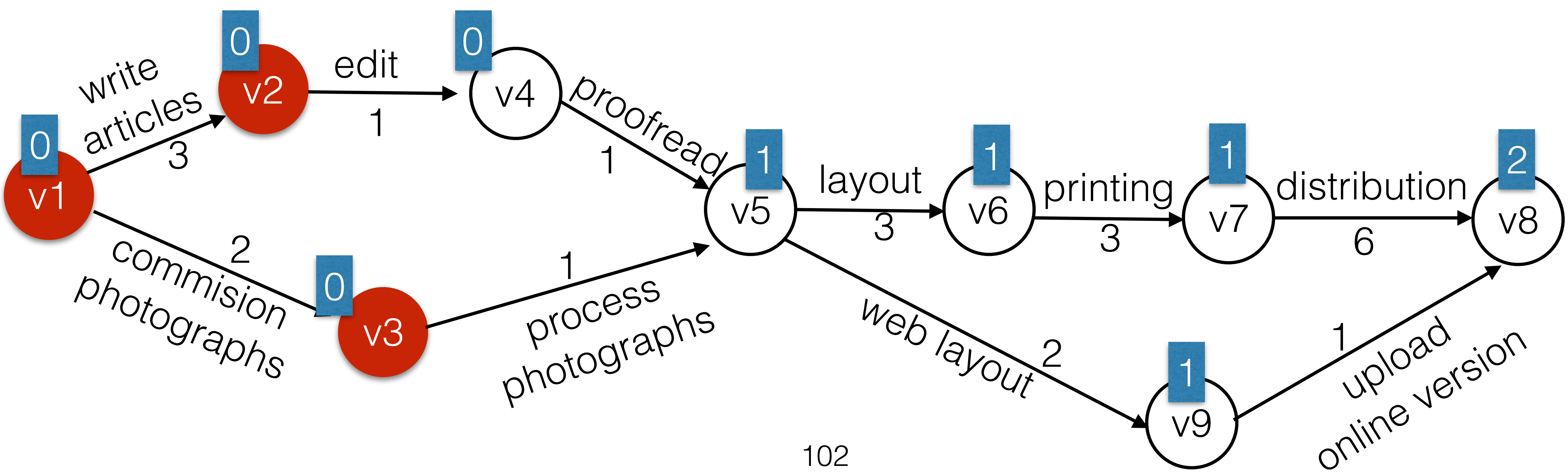


Computing Topological Order

- While the queue is not empty, dequeue a vertex, print it and decrement the indegree of its adjacent nodes.
- If the indegree of any new vertex becomes 0, enqueue it.

Queue: v4

Output: v1 v2 v3

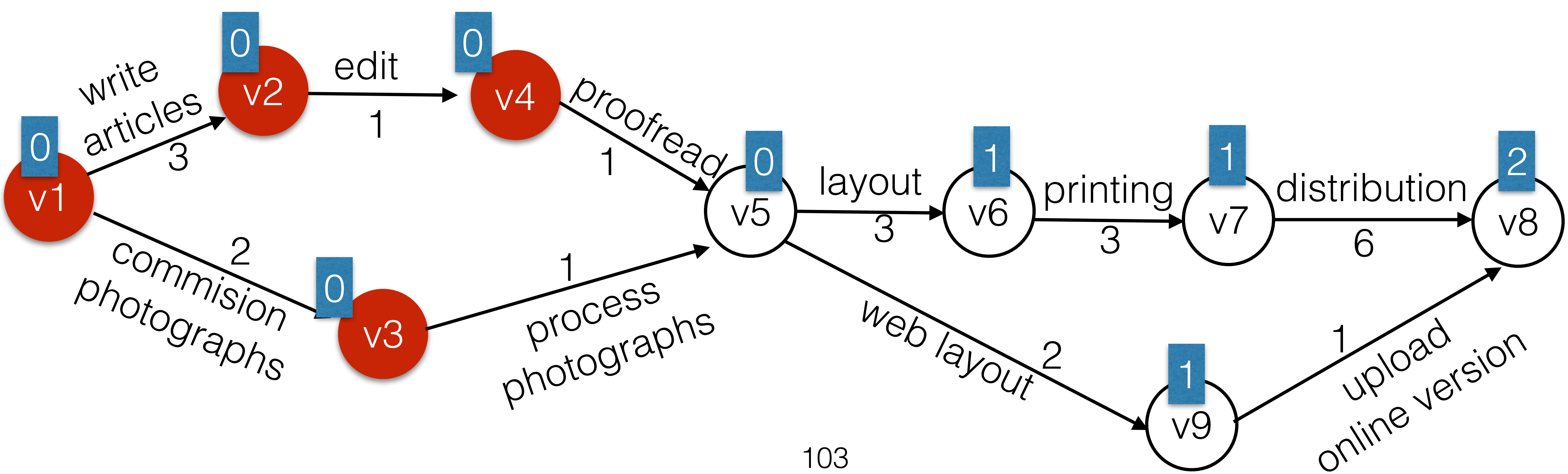


Computing Topological Order

- While the queue is not empty, dequeue a vertex, print it and decrement the indegree of its adjacent nodes.
- If the indegree of any new vertex becomes 0, enqueue it.

Queue: v5

Output: v1 v2 v3 v4

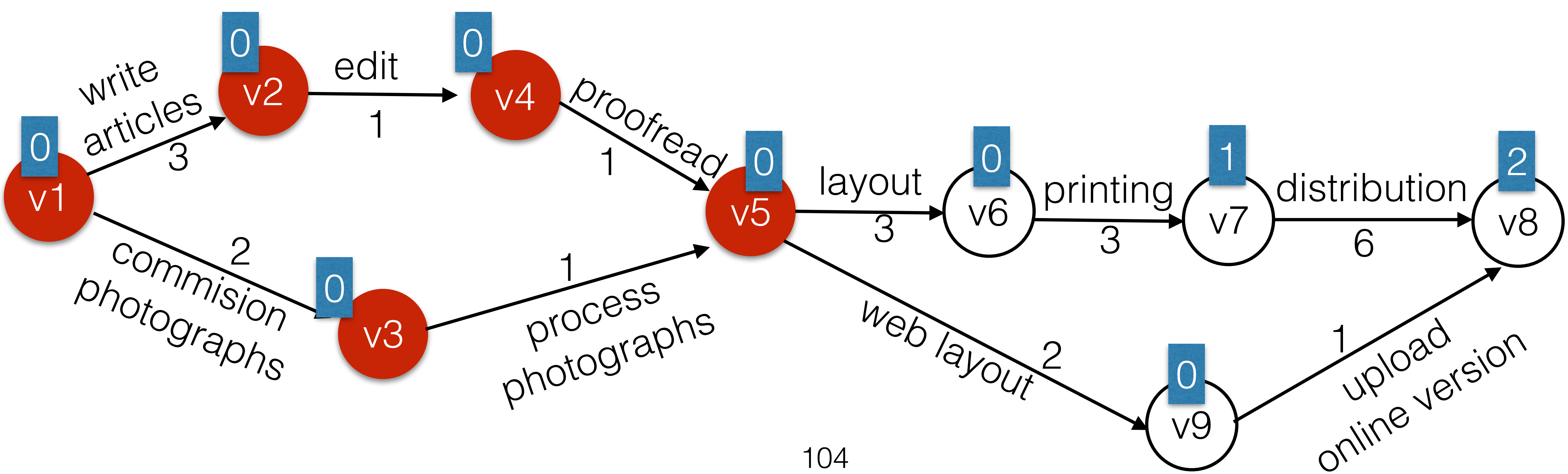


Computing Topological Order

- While the queue is not empty, dequeue a vertex, print it and decrement the indegree of its adjacent nodes.
- If the indegree of any new vertex becomes 0, enqueue it.

Queue: v6 v9

Output: v1 v2 v3 v4 v5

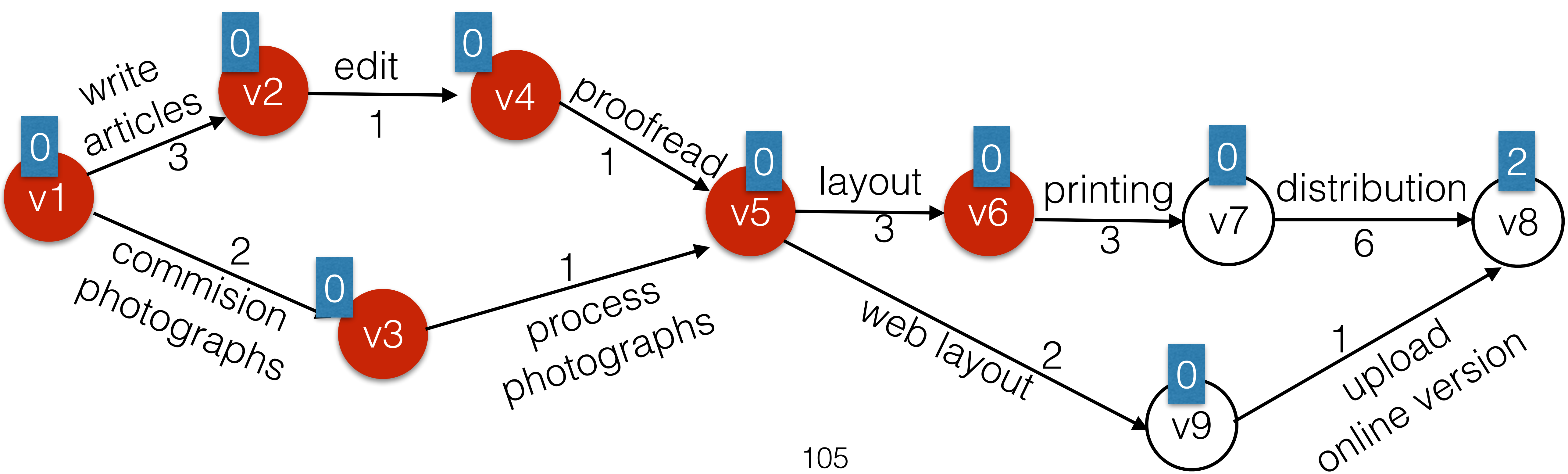


Computing Topological Order

- While the queue is not empty, dequeue a vertex, print it and decrement the indegree of its adjacent nodes.
- If the indegree of any new vertex becomes 0, enqueue it.

Queue: v9 v7

Output: v1 v2 v3 v4 v5 v6

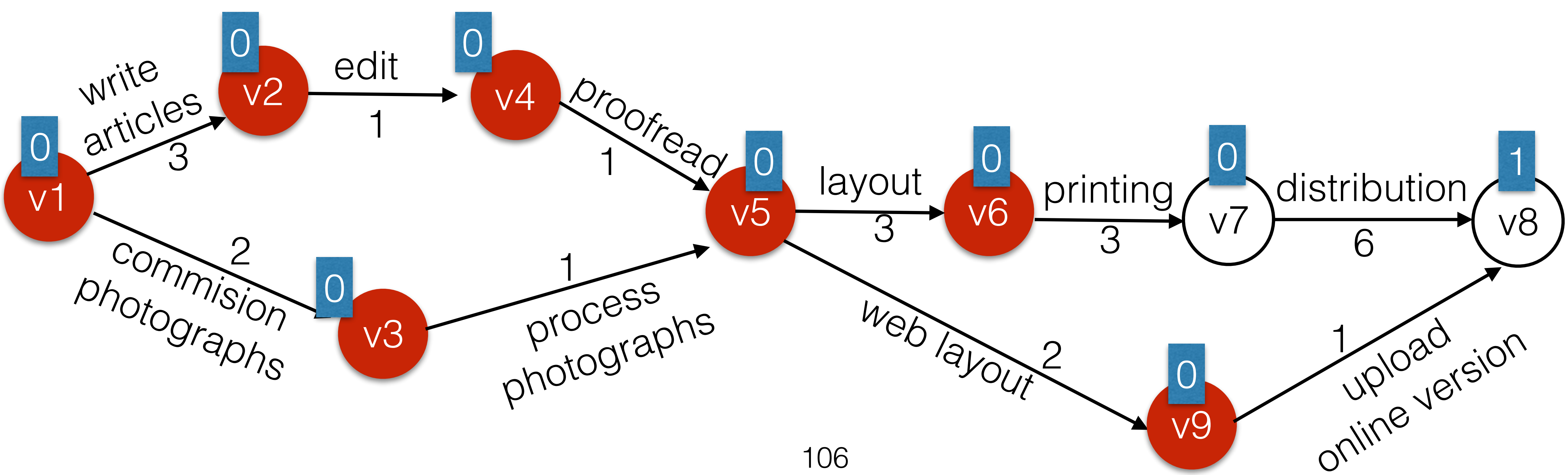


Computing Topological Order

- While the queue is not empty, dequeue a vertex, print it and decrement the indegree of its adjacent nodes.
- If the indegree of any new vertex becomes 0, enqueue it.

Queue: v7

Output: v1 v2 v3 v4 v5 v6 v9

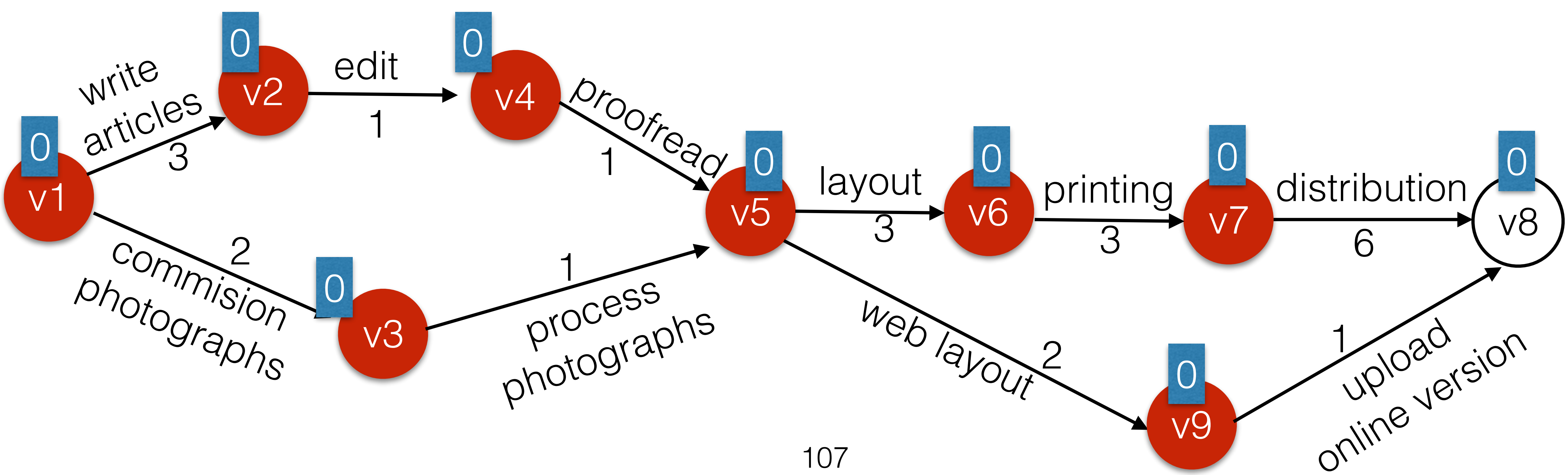


Computing Topological Order

- While the queue is not empty, dequeue a vertex, print it and decrement the indegree of its adjacent nodes.
- If the indegree of any new vertex becomes 0, enqueue it.

Queue: v8

Output: v1 v2 v3 v4 v5 v6 v9 v7

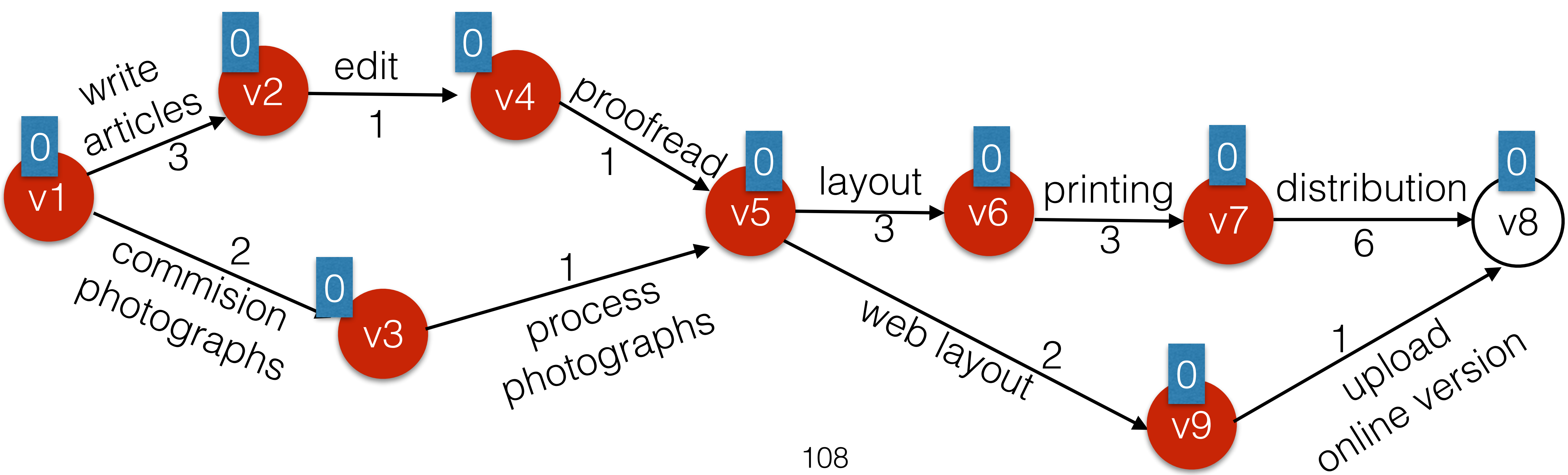


Computing Topological Order

- While the queue is not empty, dequeue a vertex, print it and decrement the indegree of its adjacent nodes.
- If the indegree of any new vertex becomes 0, enqueue it.

Queue:

Output: v1 v2 v3 v4 v5 v6 v9 v7 v8



Topological Sort - Running Time

- First annotate each vertex with its **indegree**.
- While the queue is not empty, dequeue a vertex, print it and decrement the indegree of its adjacent nodes.
- If the indegree of any new vertex becomes 0, enqueue it.

Topological Sort - Running Time

- First annotate each vertex with its **indegree**.
- While the queue is not empty, dequeue a vertex, print it and decrement the indegree of its adjacent nodes.
- If the indegree of any new vertex becomes 0, enqueue it.

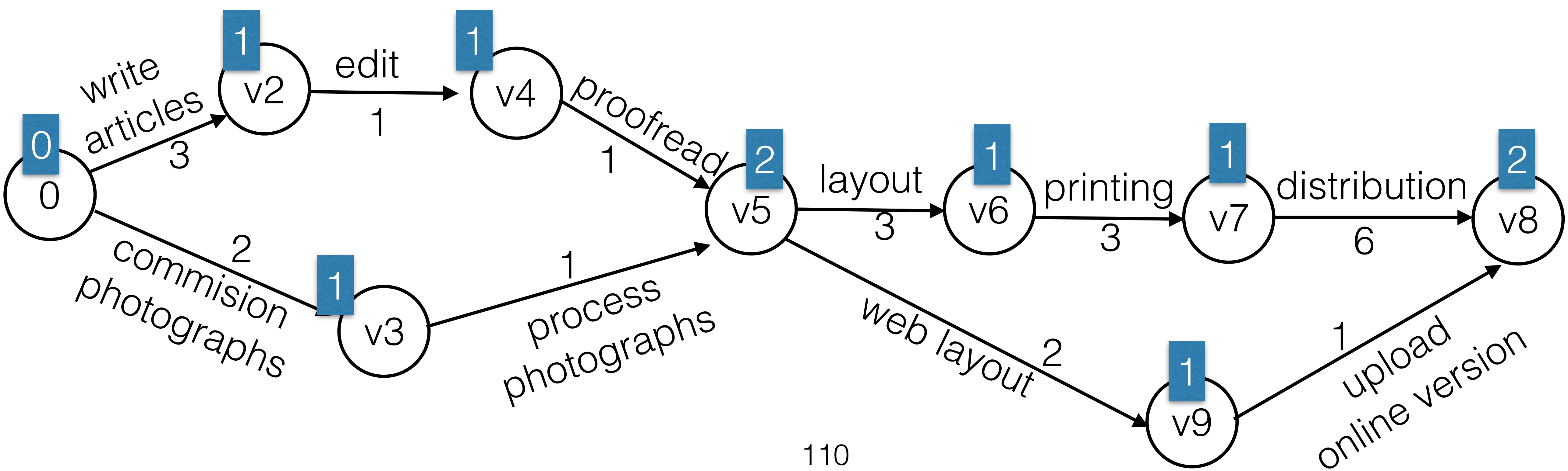
This is just BFS. Running time: O(|V|+|E|)

Earliest Completion Time

- While the queue is not empty, dequeue a vertex, print it and decrement the indegree of its adjacent nodes. **Update earliest completion time for each adjacent node.**
- If the indegree of any new vertex becomes 0, enqueue it.

Queue: v1

Output:



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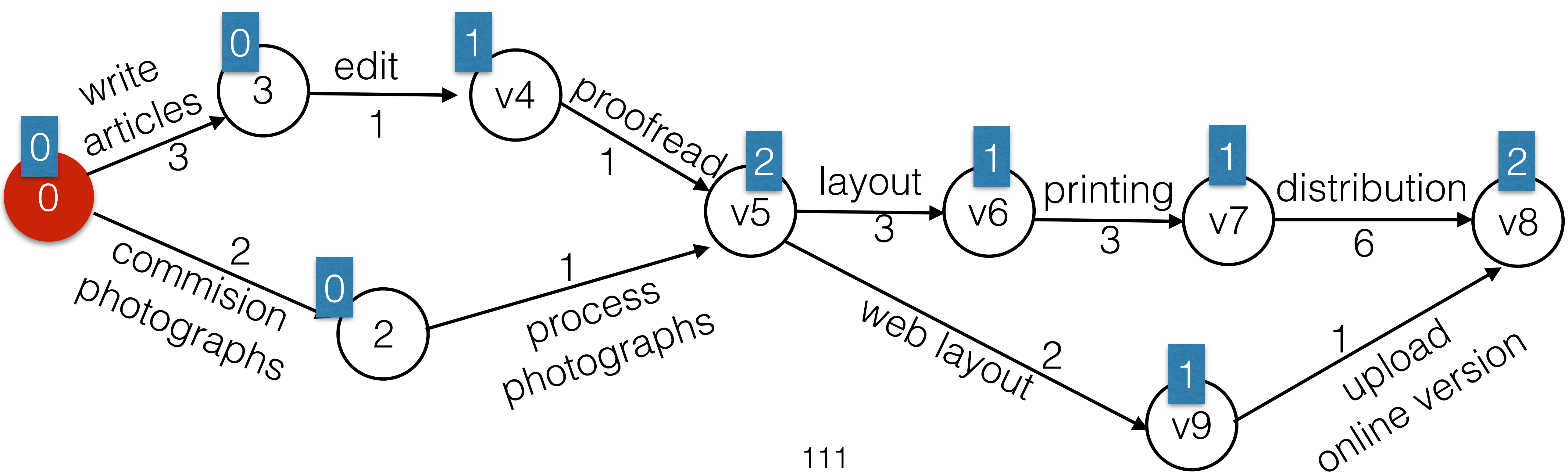
- While the queue is not empty, dequeue a vertex, print it and decrement the indegree of its adjacent nodes. **Update earliest completion time for each adjacent node.**
- If the indegree of any new vertex becomes 0, enqueue it.

Queue: v2 v3

Output: v1



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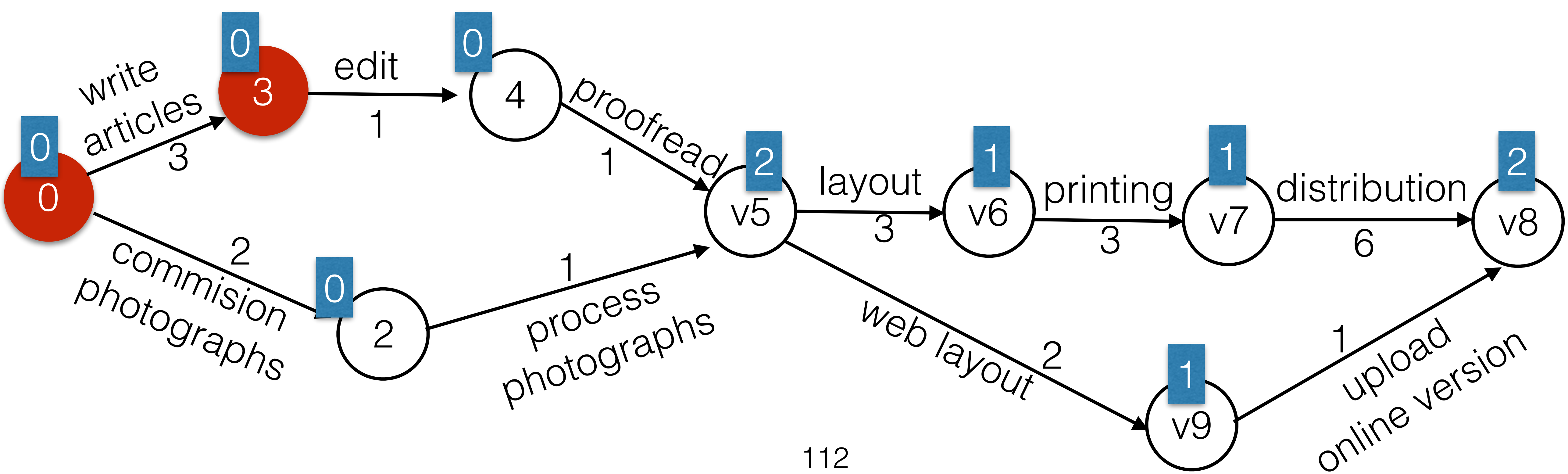
- While the queue is not empty, dequeue a vertex, print it and decrement the indegree of its adjacent nodes. **Update earliest completion time for each adjacent node.**
- If the indegree of any new vertex becomes 0, enqueue it.

Queue: v3 v4

Output: v1 v2

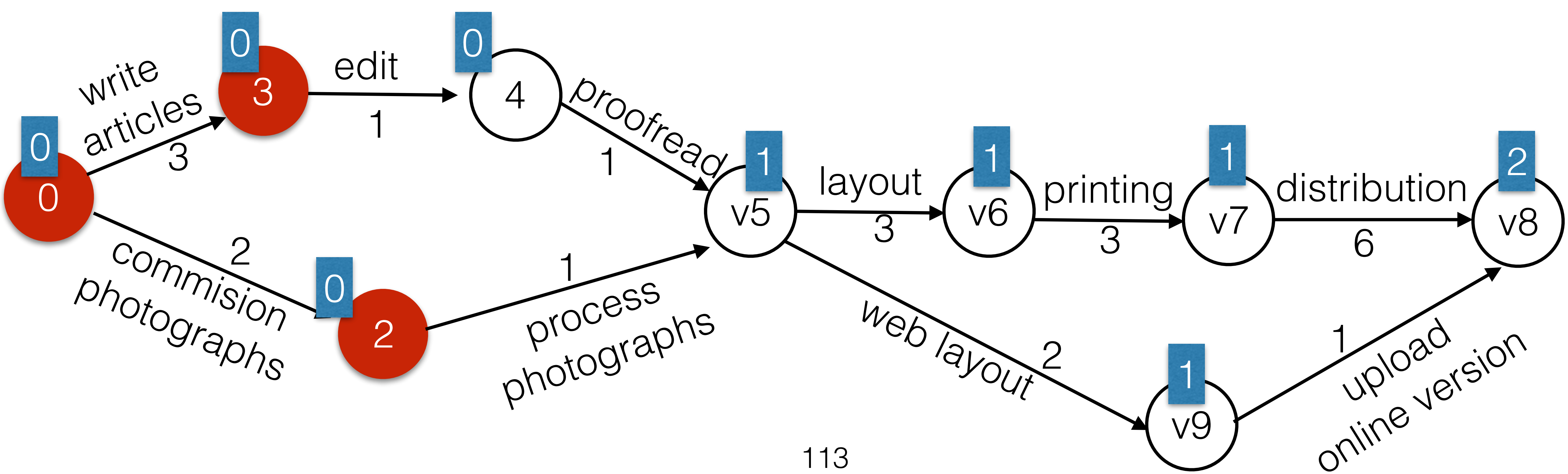


Earliest Completion Time

- While the queue is not empty, dequeue a vertex, print it and decrement the indegree of its adjacent nodes. **Update earliest completion time for each adjacent node.**
- If the indegree of any new vertex becomes 0, enqueue it.

Queue: v4

Output: v1 v2 v3

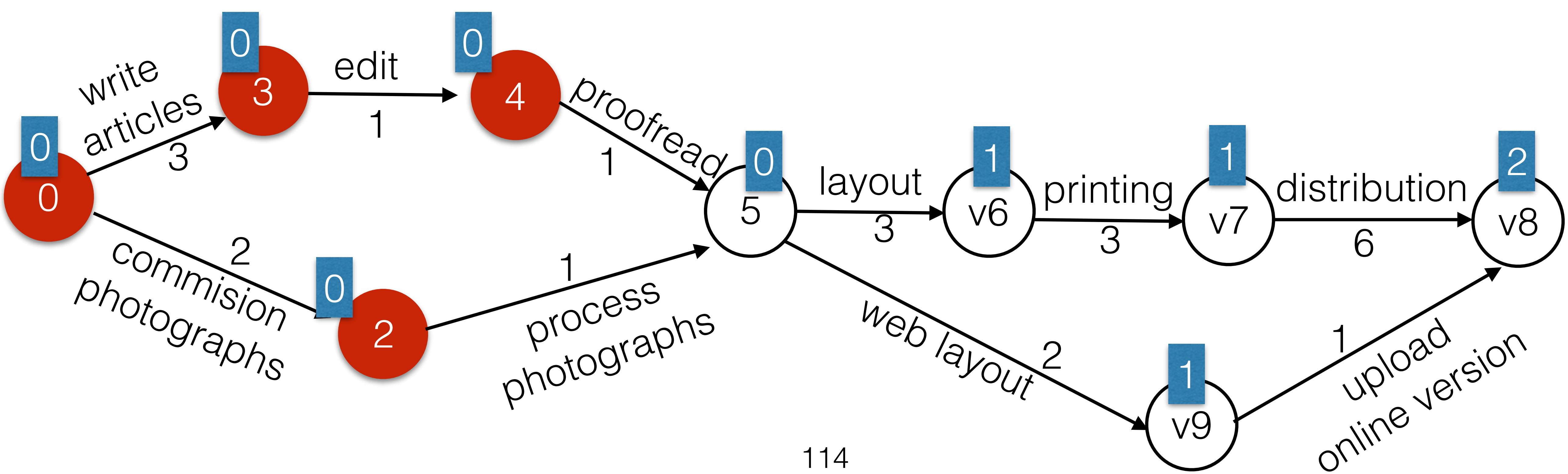


Earliest Completion Time

- While the queue is not empty, dequeue a vertex, print it and decrement the indegree of its adjacent nodes. **Update earliest completion time for each adjacent node.**
- If the indegree of any new vertex becomes 0, enqueue it.

Queue: v5

Output: v1 v2 v3 v4

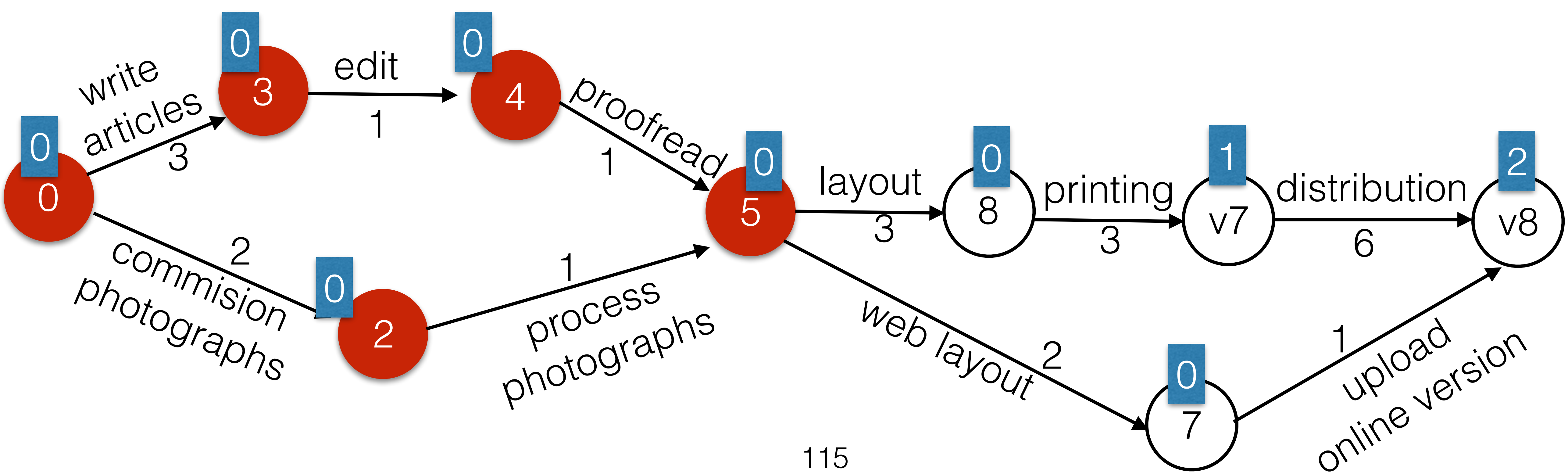


Earliest Completion Time

- While the queue is not empty, dequeue a vertex, print it and decrement the indegree of its adjacent nodes. **Update earliest completion time for each adjacent node.**
- If the indegree of any new vertex becomes 0, enqueue it.

Queue: v6 v9

Output: v1 v2 v3 v4 v5

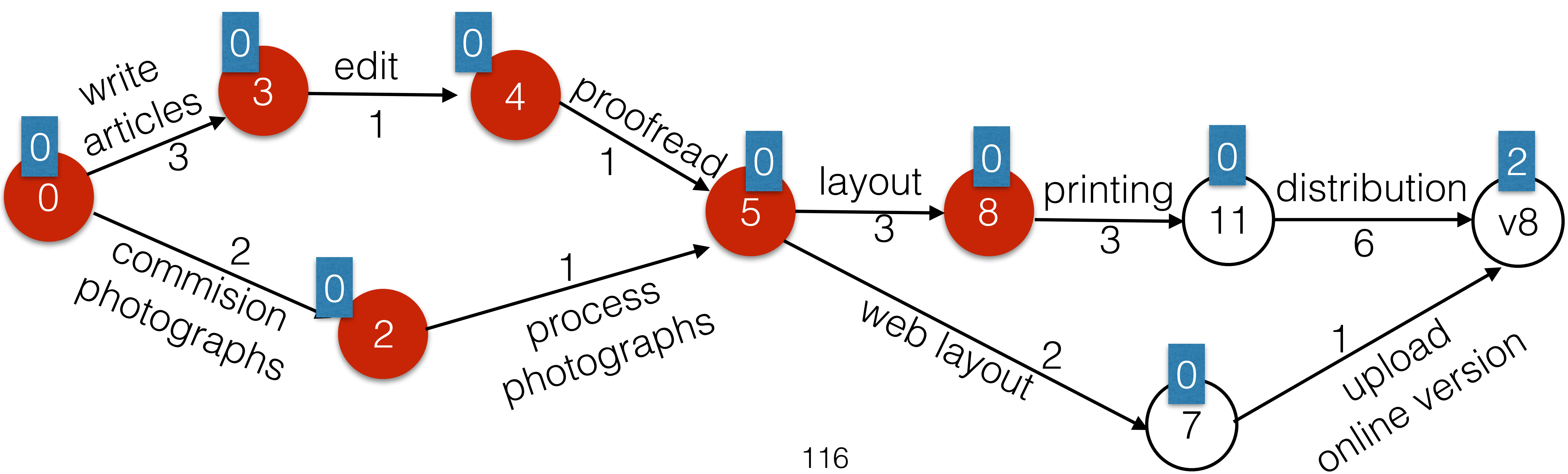


Earliest Completion Time

- While the queue is not empty, dequeue a vertex, print it and decrement the indegree of its adjacent nodes. **Update earliest completion time for each adjacent node.**
- If the indegree of any new vertex becomes 0, enqueue it.

Queue: v9 v7

Output: v1 v2 v3 v4 v5 v6

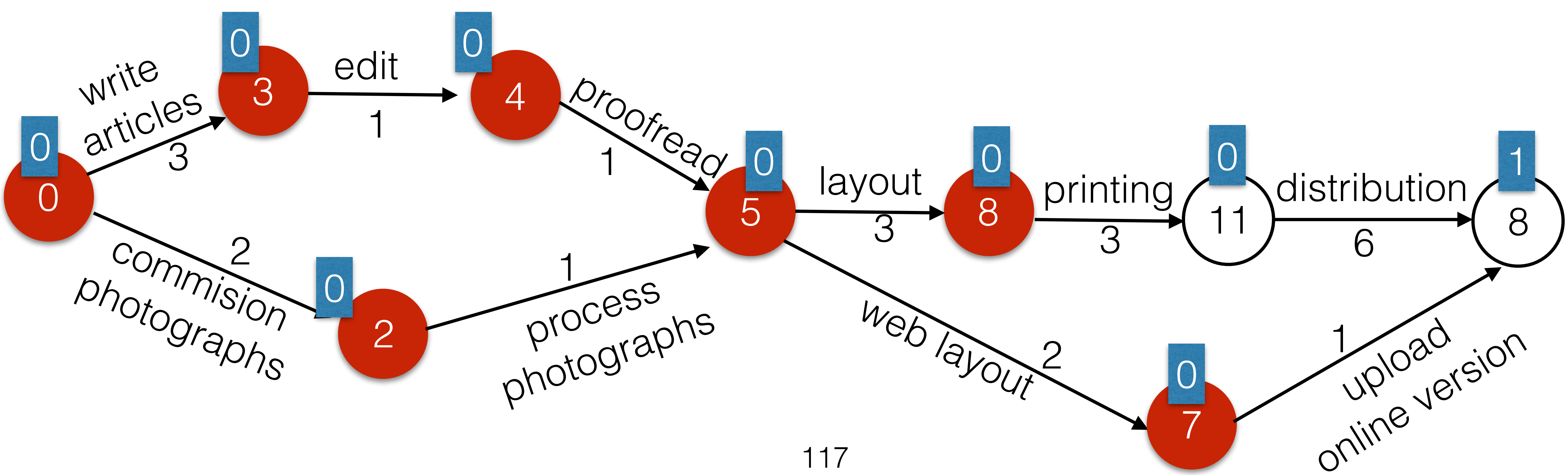


Earliest Completion Time

- While the queue is not empty, dequeue a vertex, print it and decrement the indegree of its adjacent nodes. **Update earliest completion time for each adjacent node.**
- If the indegree of any new vertex becomes 0, enqueue it.

Queue: v7

Output: v1 v2 v3 v4 v5 v6 v9



Earliest Completion Time

- While the queue is not empty, dequeue a vertex, print it and decrement the indegree of its adjacent nodes. **Update earliest completion time for each adjacent node.**
- If the indegree of any new vertex becomes 0, enqueue it.

Queue: v8

Output: v1 v2 v3 v4 v5 v6 v9 v7