Data Structures in Java

Lecture 13: Priority Queues (Heaps)

11/4/2015

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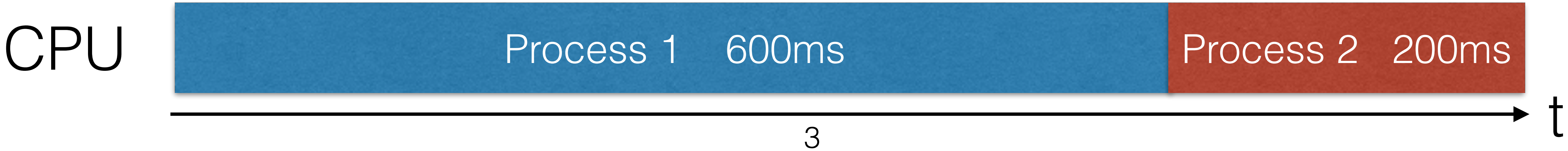
The Selection Problem

- Given an **unordered** sequence of N numbers $S = (a_1, a_2, \ldots a_N)$, select the k-th largest number.

Process Scheduling



CPU

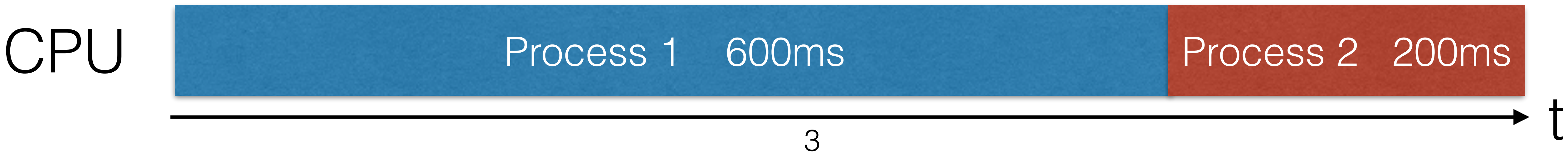
Process 1 600ms

Process 2 200ms

t

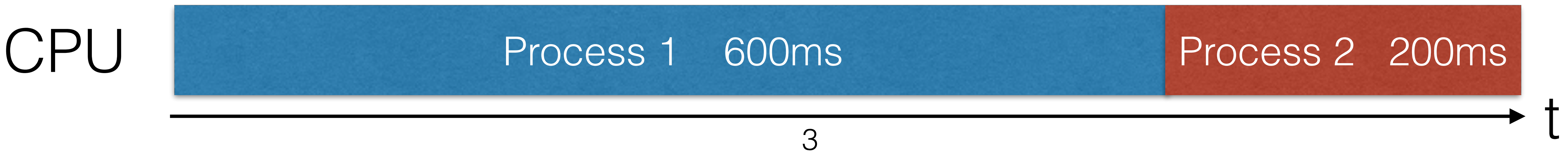
Process Scheduling

- Assume a system with a single CPU core.
  - Only one process can run at a time.
  - Simple approach: Keep new processes on a Queue, schedule them in FIFO oder. (Why is a Stack a terrible idea?)



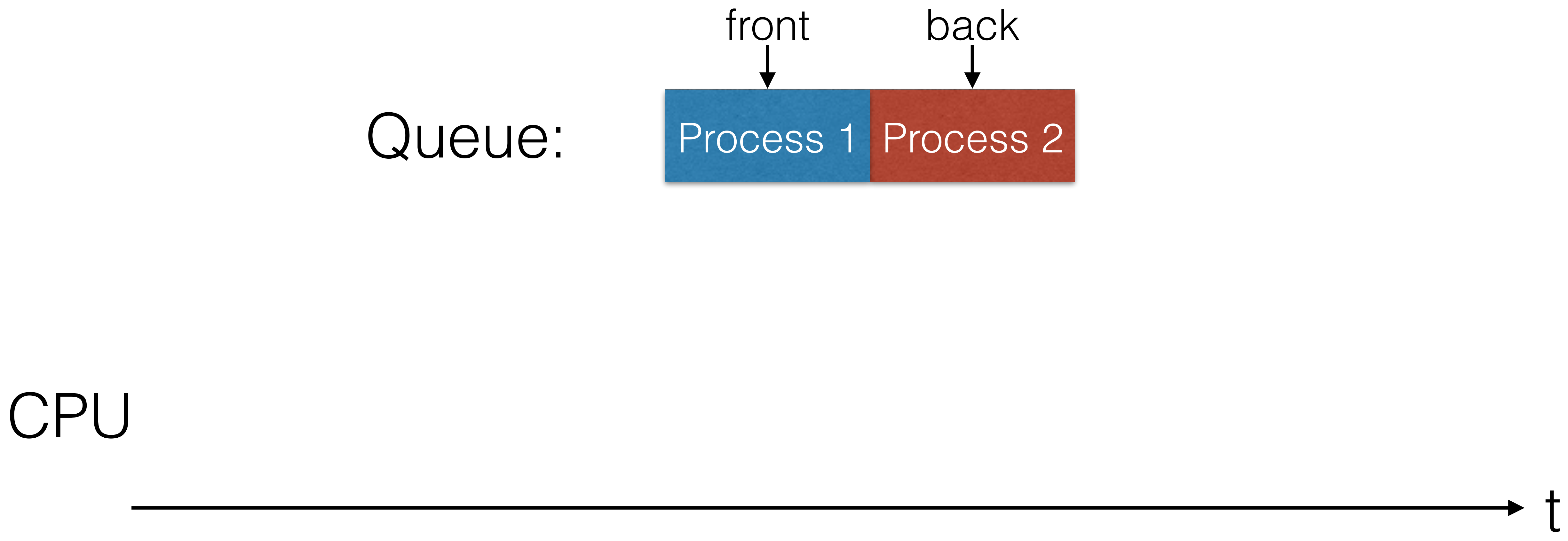
Process Scheduling

- Assume a system with a single CPU core.
  - Only one process can run at a time.
  - Simple approach: Keep new processes on a Queue, schedule them in FIFO oder. (Why is a Stack a terrible idea?)
  - Problem: Long processes may block CPU (usually we do not even know how long).
  - Observation: Processes may have different priority (CPU vs. I/O bound, critical real time systems)



Round Robin Scheduling

- Idea: processes take turn running for a certain time interval in round robin fashion.



front

back

Queue:

Process 1

Process 2

CPU

t

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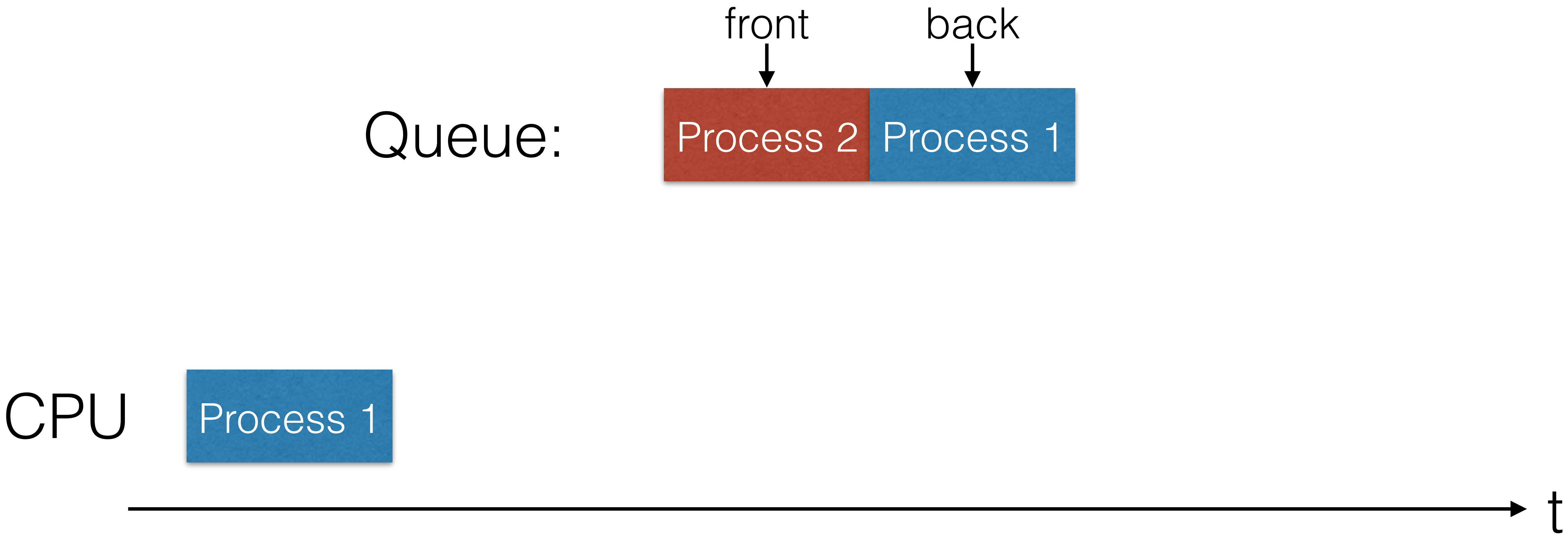

front

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Queue:

Process 2

Process 1

CPU

Process 1

t

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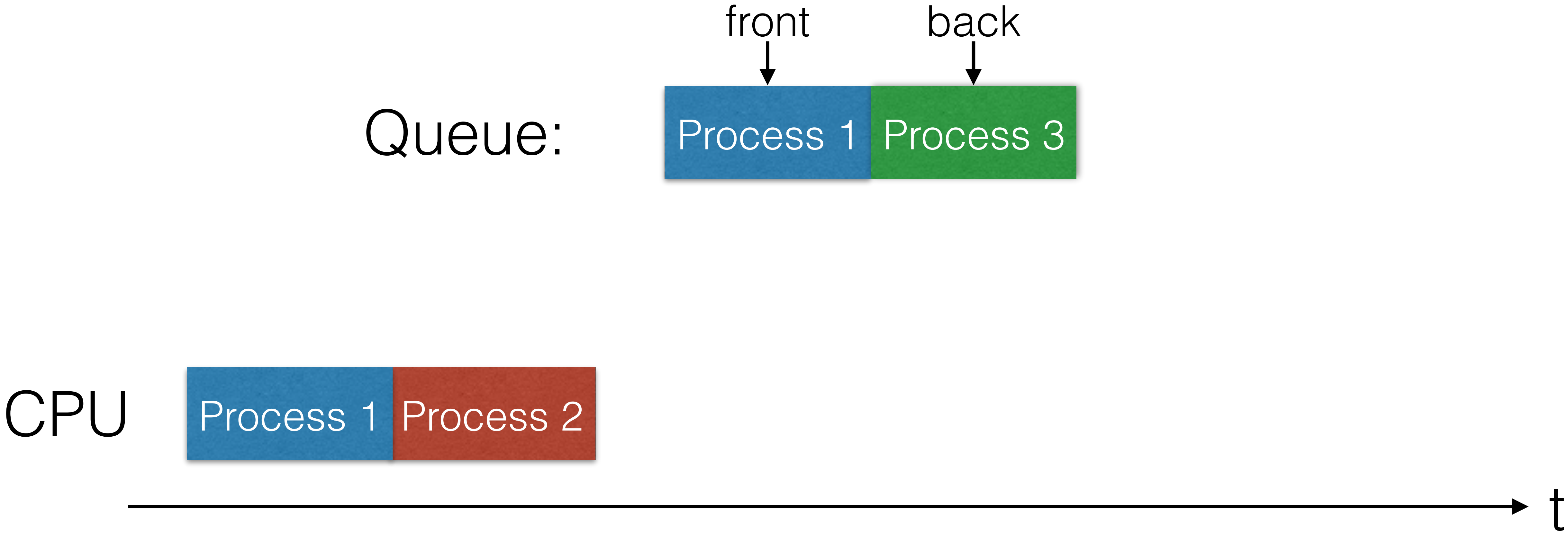

front

back

Queue:

Process 1

Process 3

CPU

Process 1

Process 2

t

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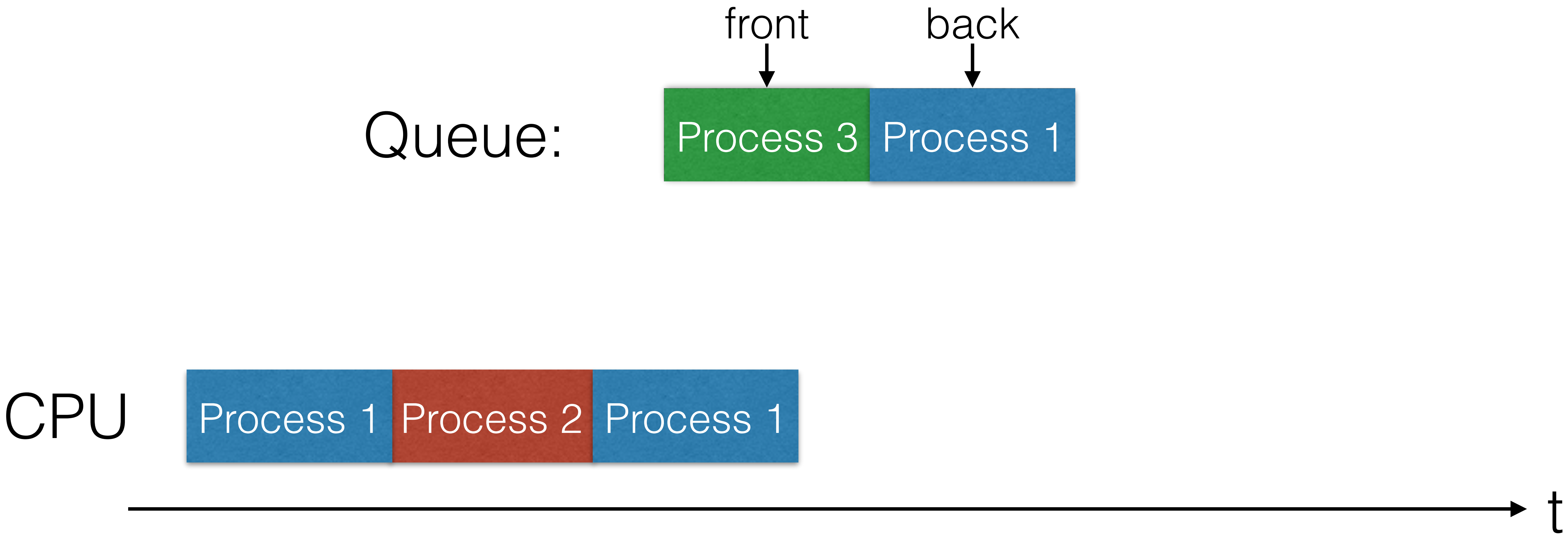

front
back

Queue:
Process 3
Process 1

CPU
Process 1
Process 2
Process 1

t

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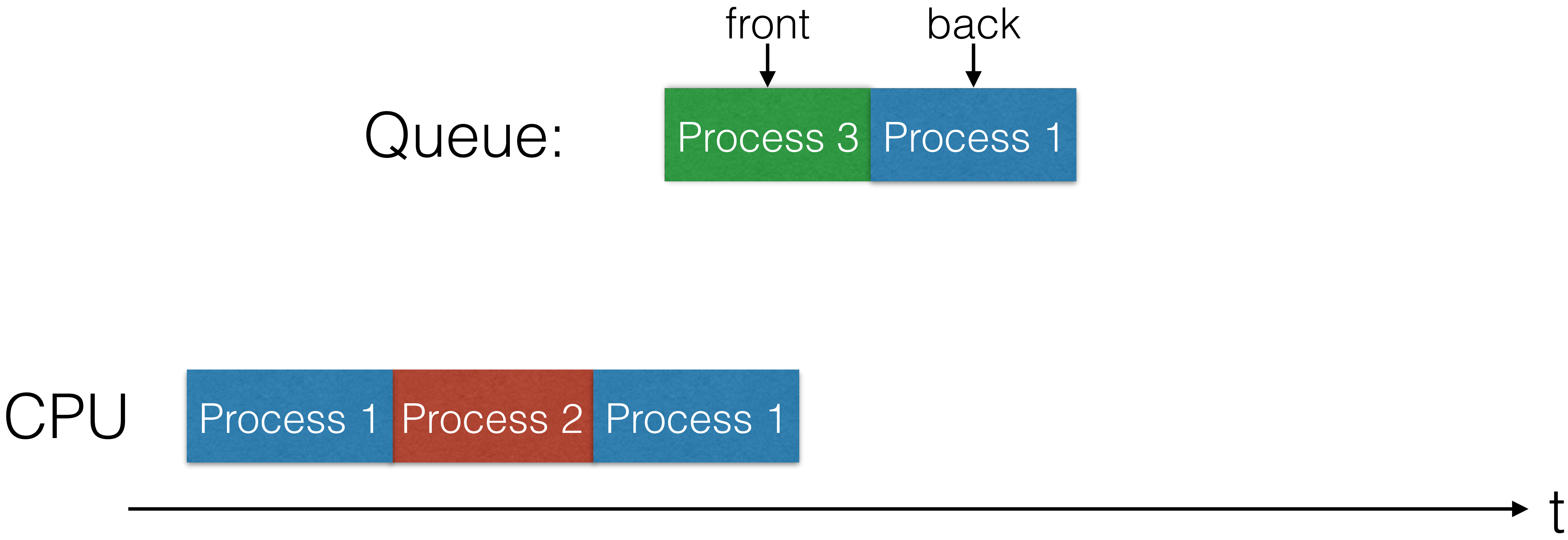


front back

Queue: Process 3 Process 1

CPU Process 1 Process 2 Process 1

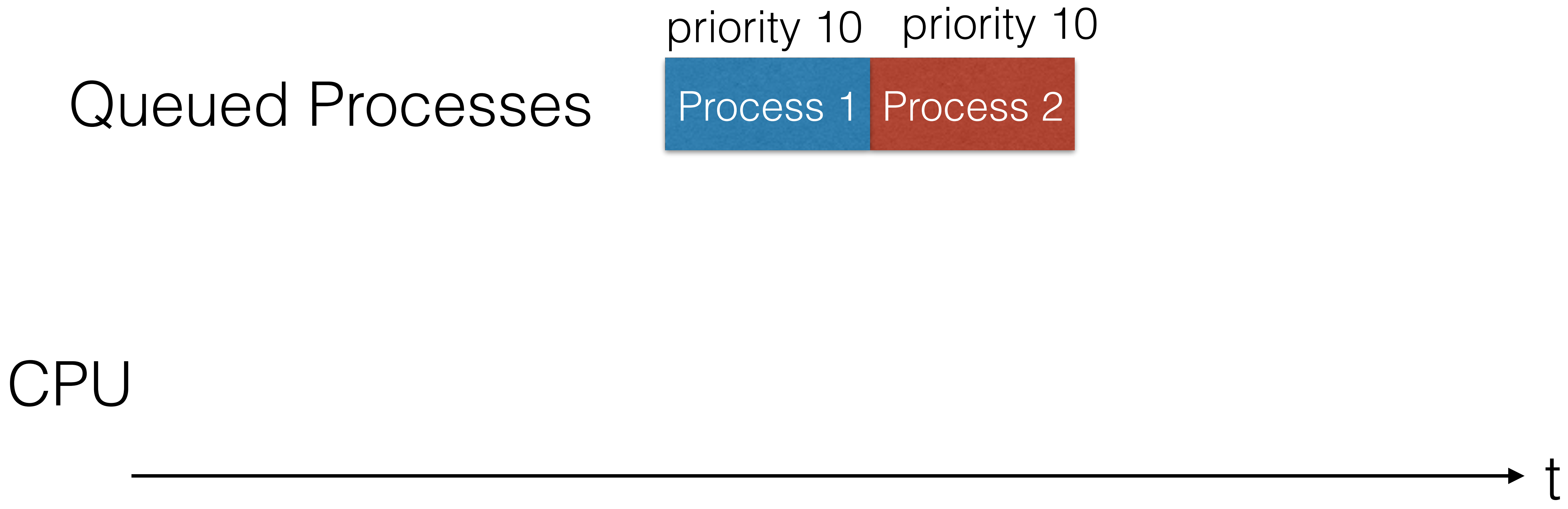
t

Sometimes Process 3 is so crucial that we want to run it immediately when the CPU becomes available!

Priority Scheduling

- Idea: Keep processes ordered by priority. Run the process with the highest priority first.

- Usually lower number = higher priority.



priority 10 priority 10

Queued Processes Process 1 Process 2

CPU

t

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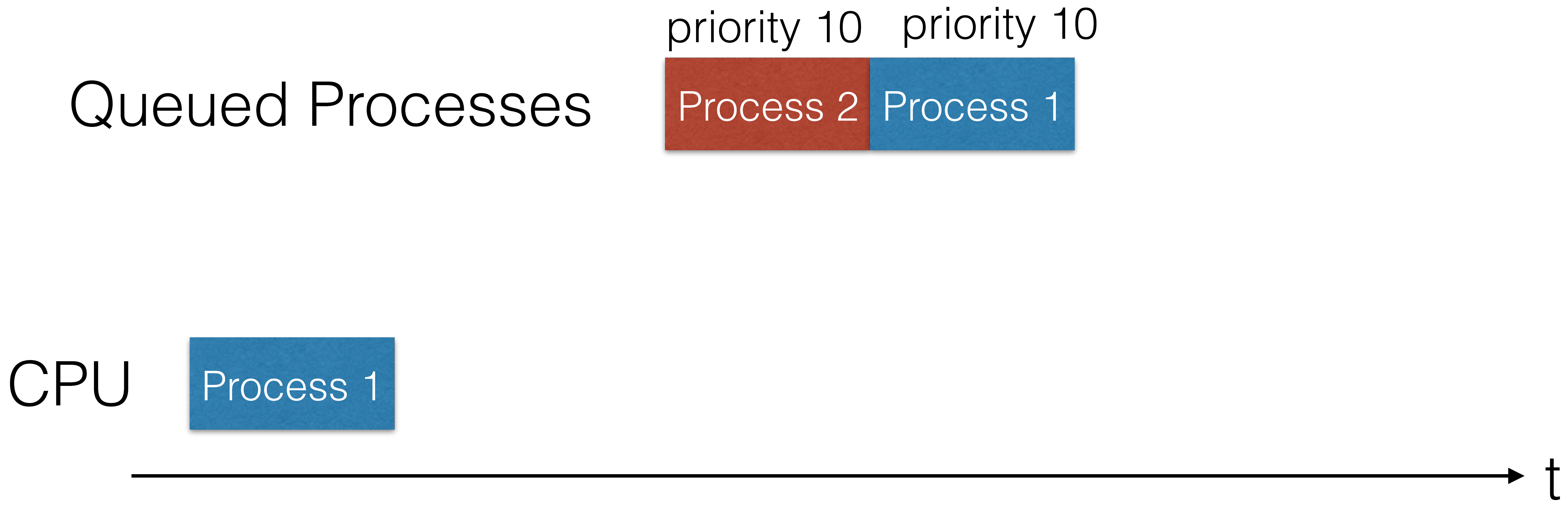

priority 10

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Queued Processes

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Process 1

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Process 1

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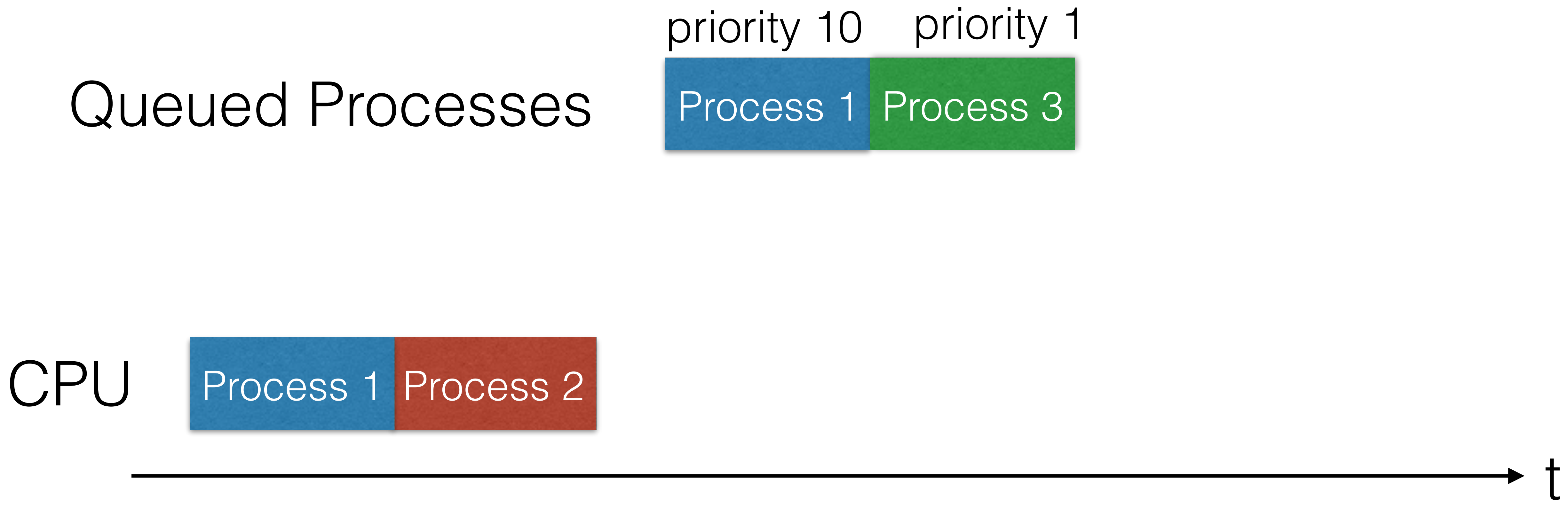

priority 10

priority 1

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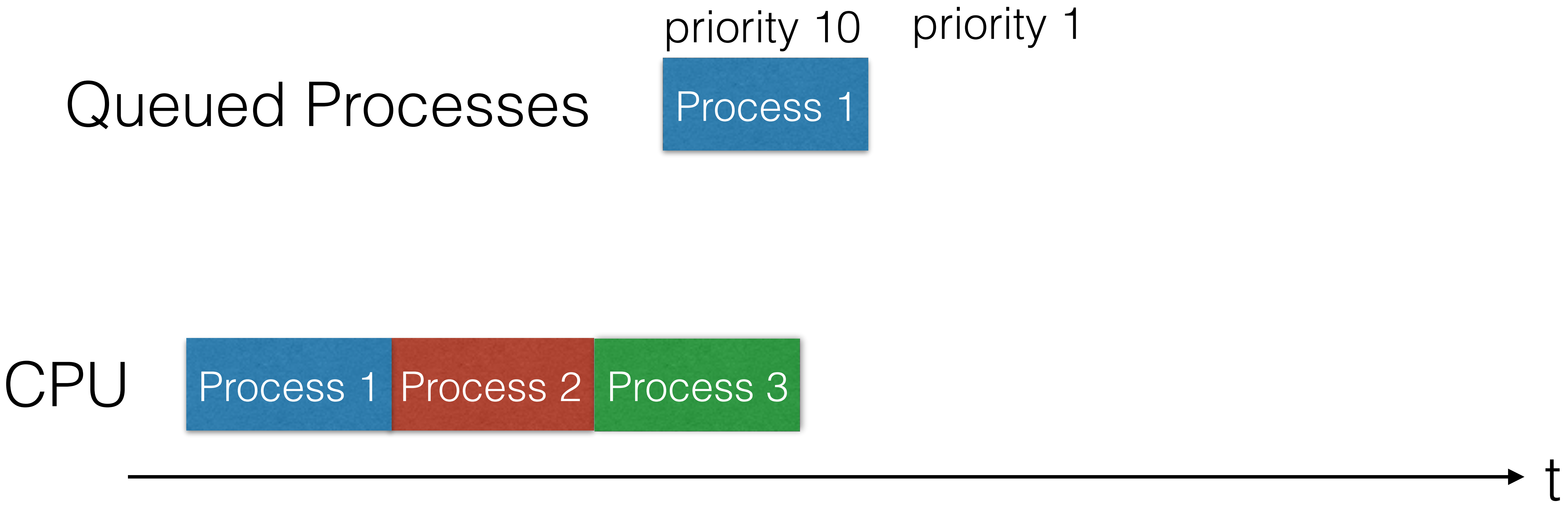

priority 10

priority 1

Queued Processes

Process 1

CPU

Process 1

Process 2

Process 3

t

The Priority Queue ADT

- A collection Q of comparable elements, that supports the following operations:
    - `insert(x)` - add an element to Q (compare to enqueue).
    - `deleteMin()` - return the minimum element in Q and delete it from Q (compare to dequeue).

Other Applications for Priority Queues

- Selection problem.
- Implementing sorting efficiently.
- Keep track of the *k*-best solutions of some dynamic programing algorithm.
- Implementing greedy algorithms (e.g. graph search).

Implementing Priority Queues

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- Idea 1: Use a Linked List.
  `insert(x)`:O(1), `deleteMin()`: O(N)

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- Idea 2: Use a Binary Search Tree.
  `insert(x):`O(log N), `deleteMin()`: O(log N)

Implementing Priority Queues

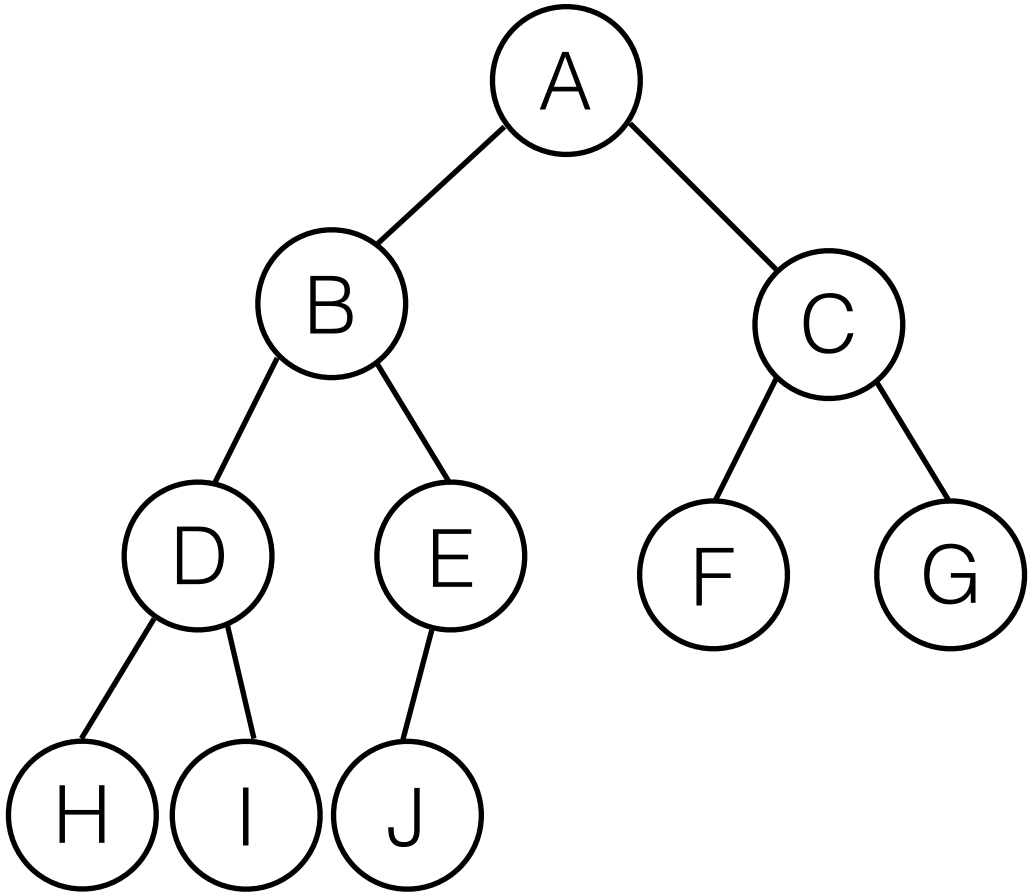
- Idea 1: Use a Linked List.
  `insert(x):`O(1), `deleteMin()`: O(N)

- Idea 2: Use a Binary Search Tree.
  `insert(x):`O(log N), `deleteMin()`: O(log N)

- Can do even better with a **Heap** data structure**:**
  - Inserting N items in O(N).
  - This gives a sorting algorithm in O(N log N).

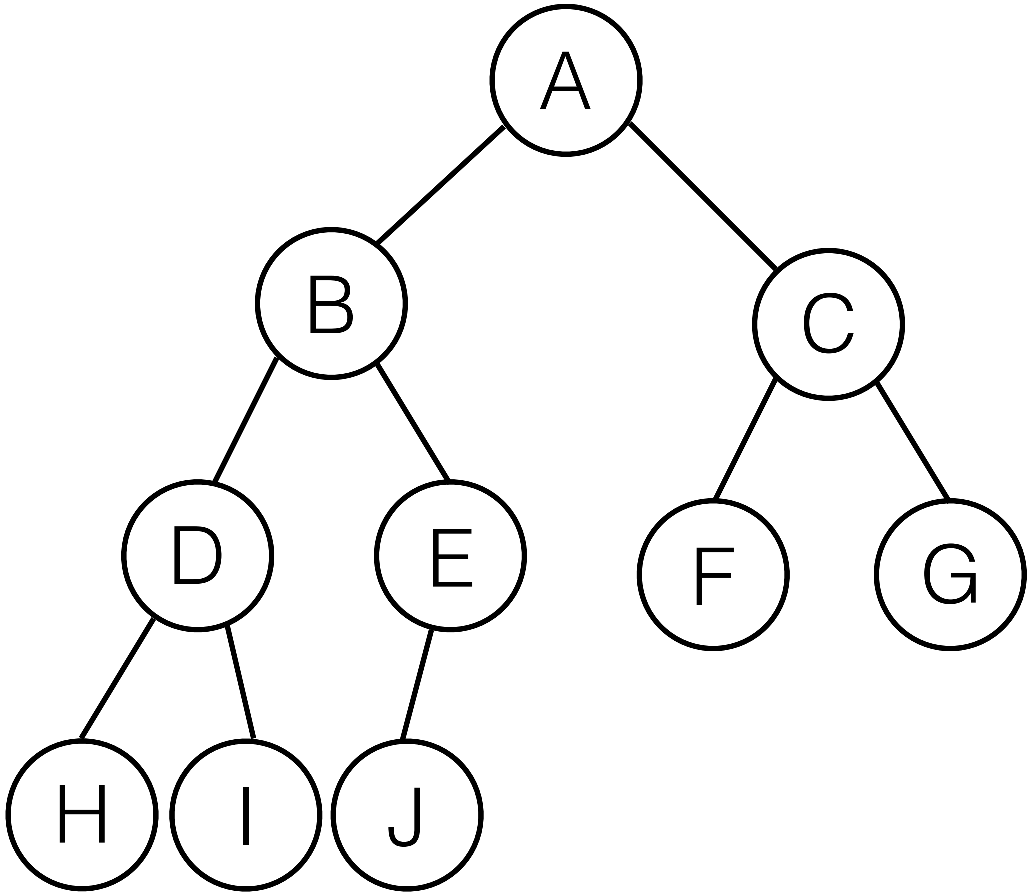
Review: Complete Binary Trees

- All non-leaf nodes have exactly 2 children (full binary tree)
- All levels are completely full (except possibly the last)



Storing Complete Binary Trees in Arrays

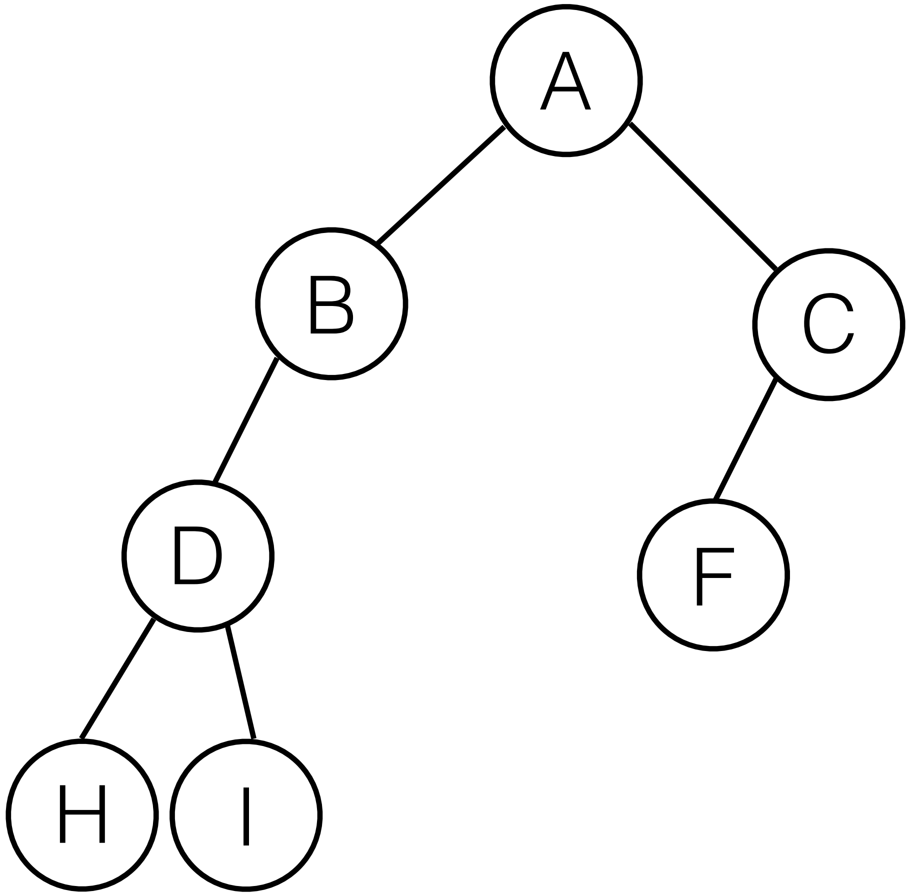
- The shape of a complete binary tree with N nodes is unique.
- We can store such trees in an array in level-order.
- Traversal is easy:
  - leftChild(i) = 2i
  - rightChild(i) = 2i +1
  - parent(i) = $\lfloor i/2 \rfloor$



	A	B	C	D	E	F	G	H	I	J			

Storing Incomplete Binary Trees in Arrays

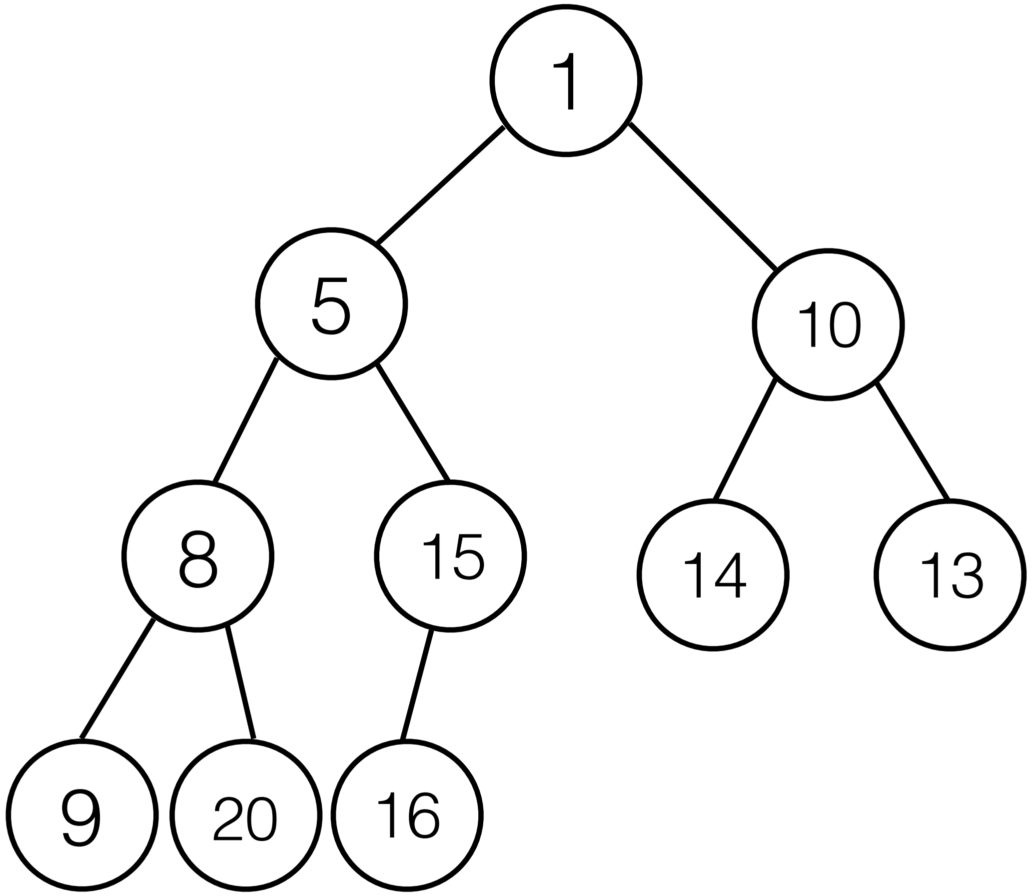
- Assume the tree takes as much space as a complete binary tree, but only store the nodes that actually exist.



	A	B	C	D		F			I				

Heap

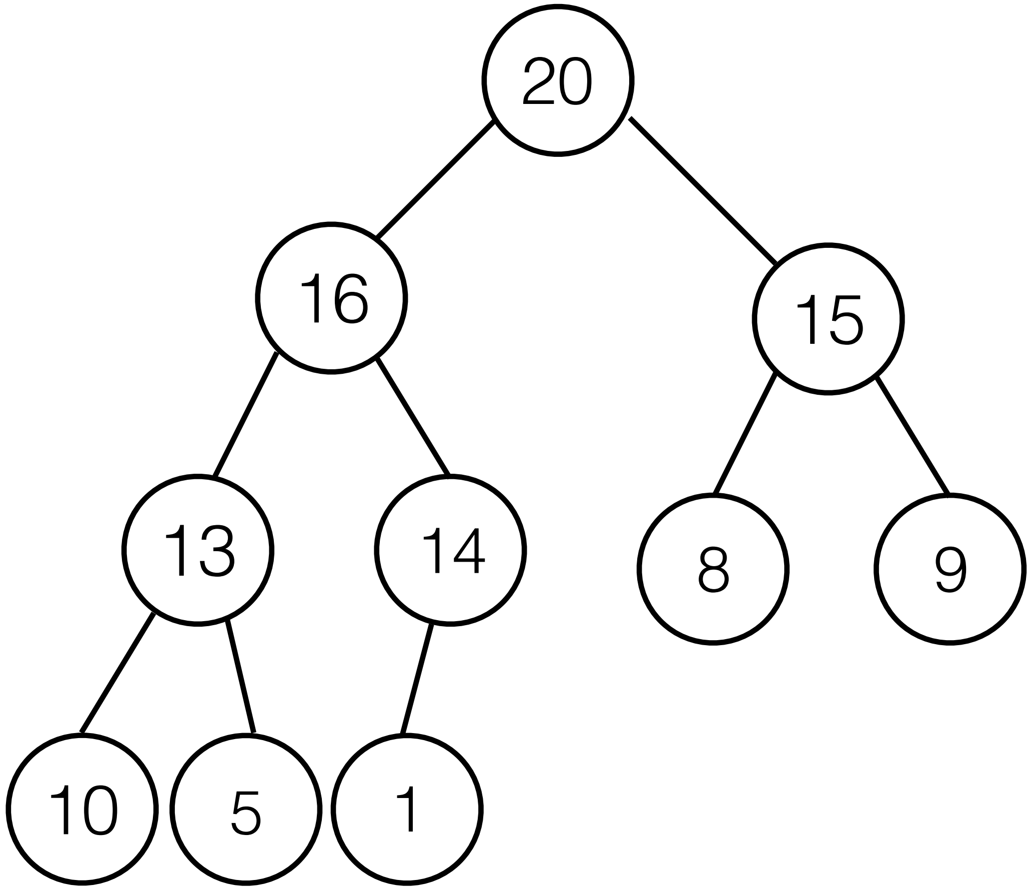
- A heap is a complete binary tree stored in an array, with the following **heap order property**:
  - For every node *n* with value x:
    - the values of all nodes in the subtree rooted in *n* are greater or equal than x.



	1	5	10	8	15	14	13	9	20	16			

Max Heap

- A heap is a complete binary tree stored in an array, with the following **heap order property**:
  - For every node *n* with value x:
    - the values of all nodes in the subtree rooted in *n* are **less or equal** than x.

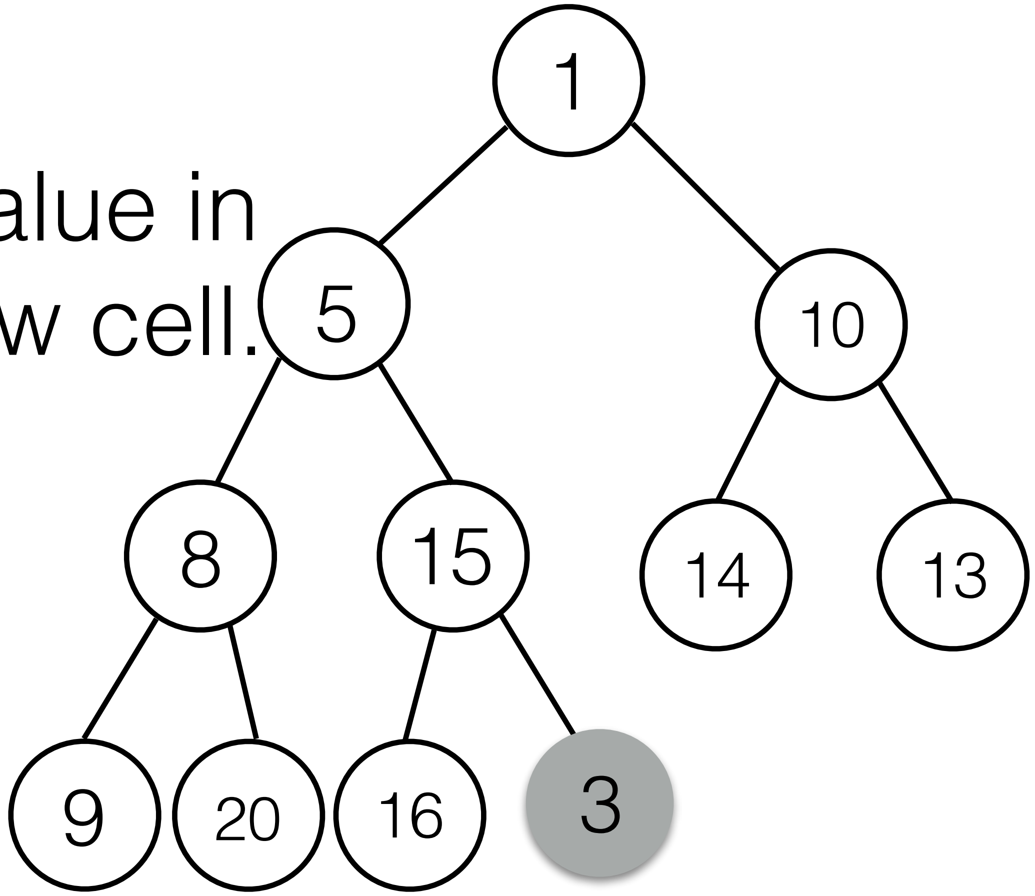


	20	16	15	13	14	8	9	10	5	1			

Min Heap - `insert(x)`

- Attempt to insert at last array position (next possible leaf in the last layer).
- If heap order property is violated, ***percolate*** the value ***up***.
  - Swap that value ('hole') and value in the parent cell, then try the new cell.
  - If heap order is still violated, continue until correct position is found.

insert(3)



	1	5	10	8	15	14	13	9	20	16	3		

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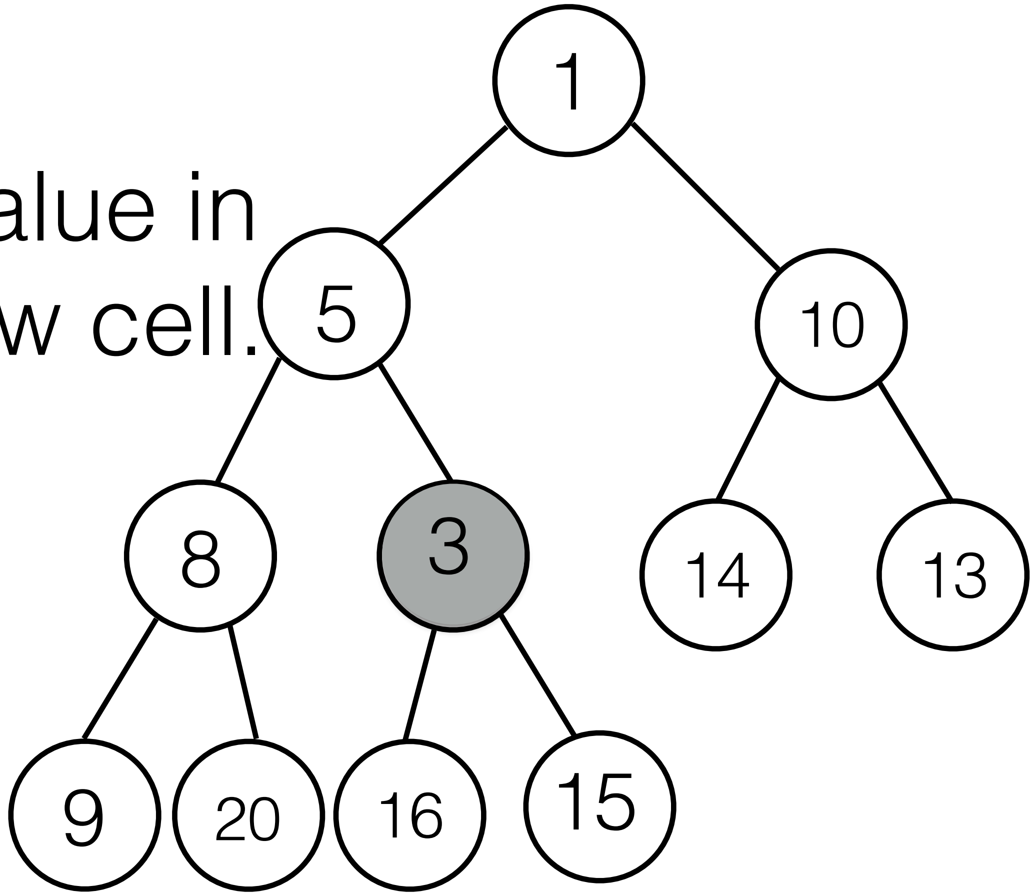

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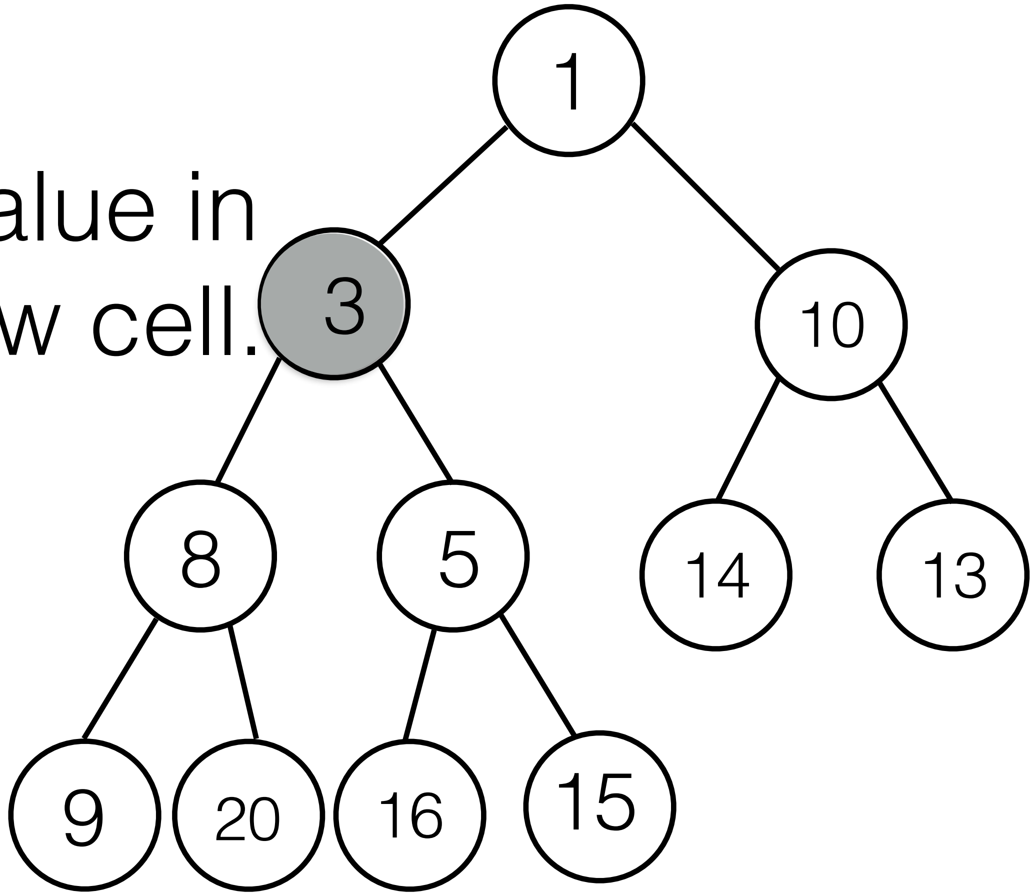
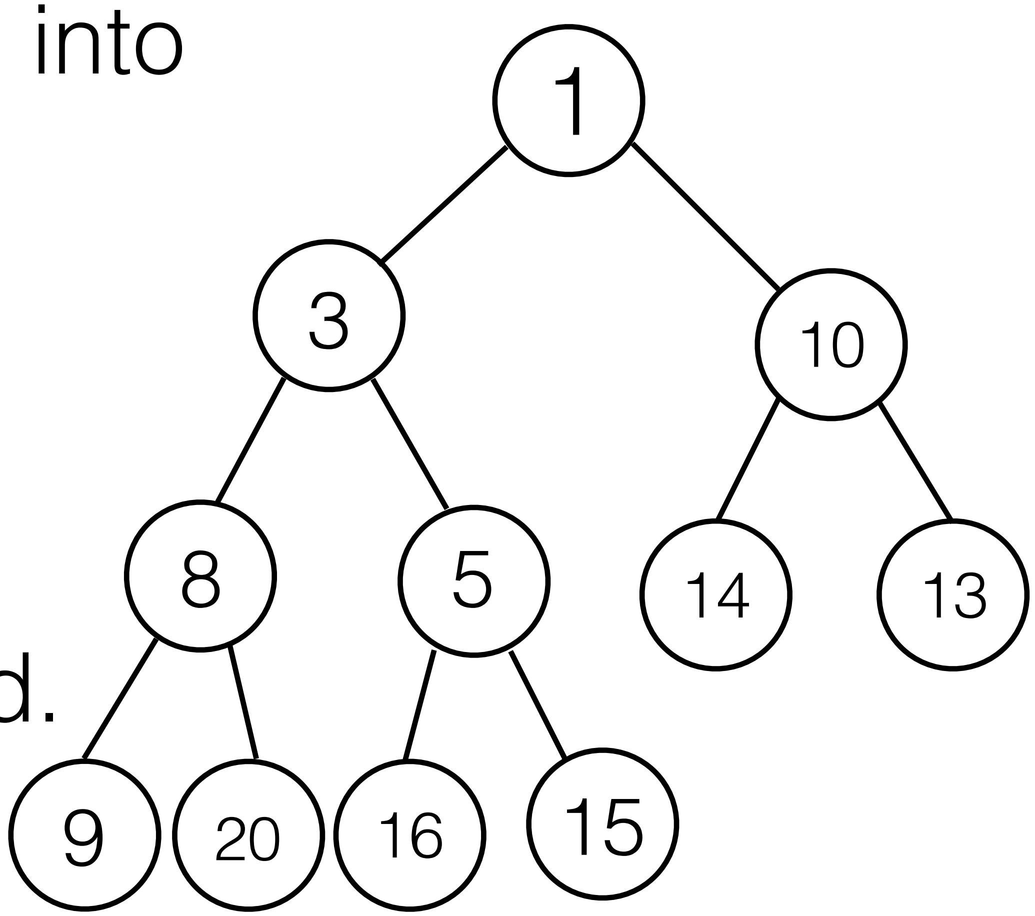


	1	3	10	8	5	14	13	9	20	16	15		

Min Heap - `deleteMin()`

- The minimum is always at the root of the tree.
- Remove lowest item, creating an empty cell in the root.
- Try to place last item in the heap into the root.
  - If heap order is violated, ***percolate*** the value ***down:***
    - Swap with the smaller child until correct position is found.



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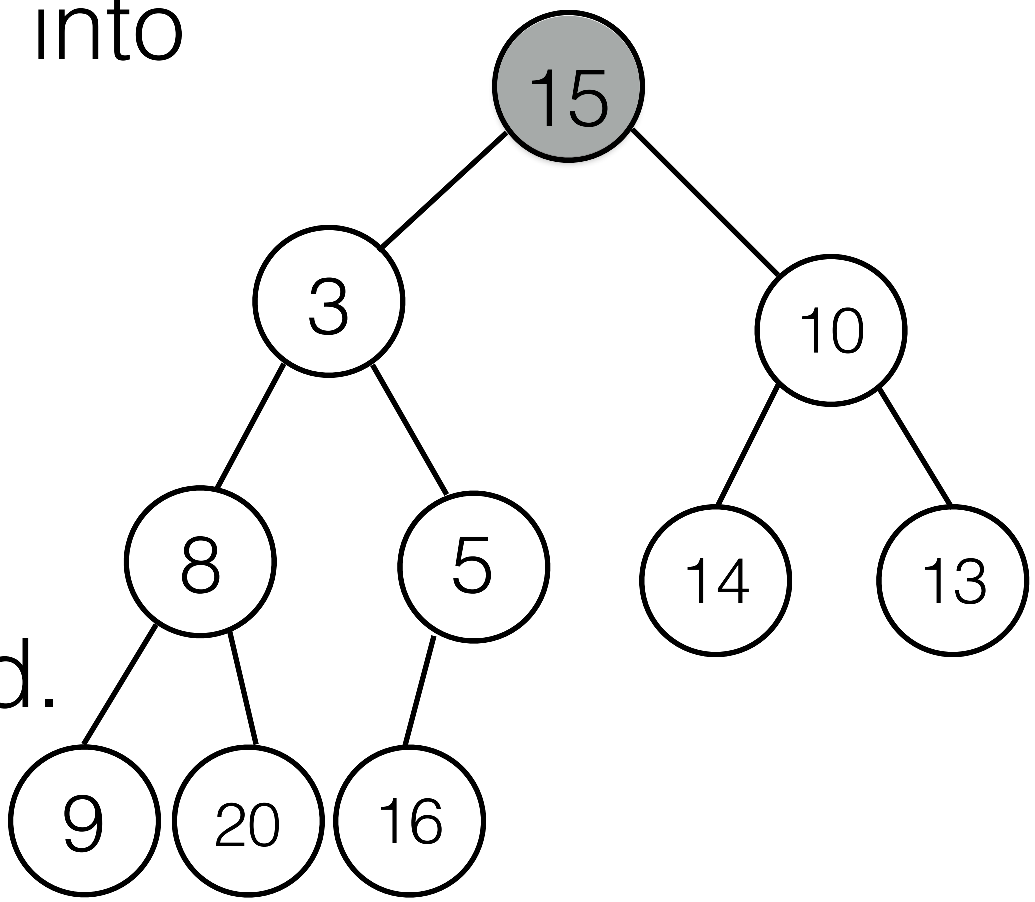

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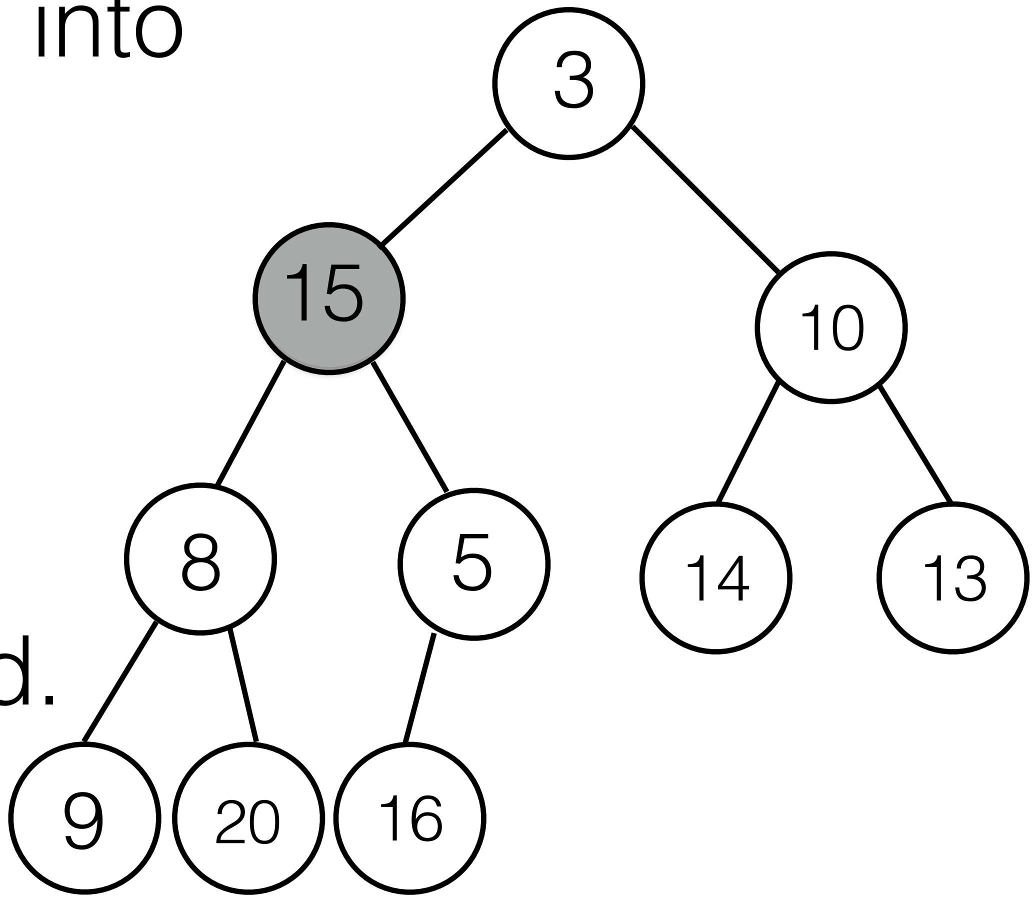

	3	15	10	8	5	14	13	9	20	16			

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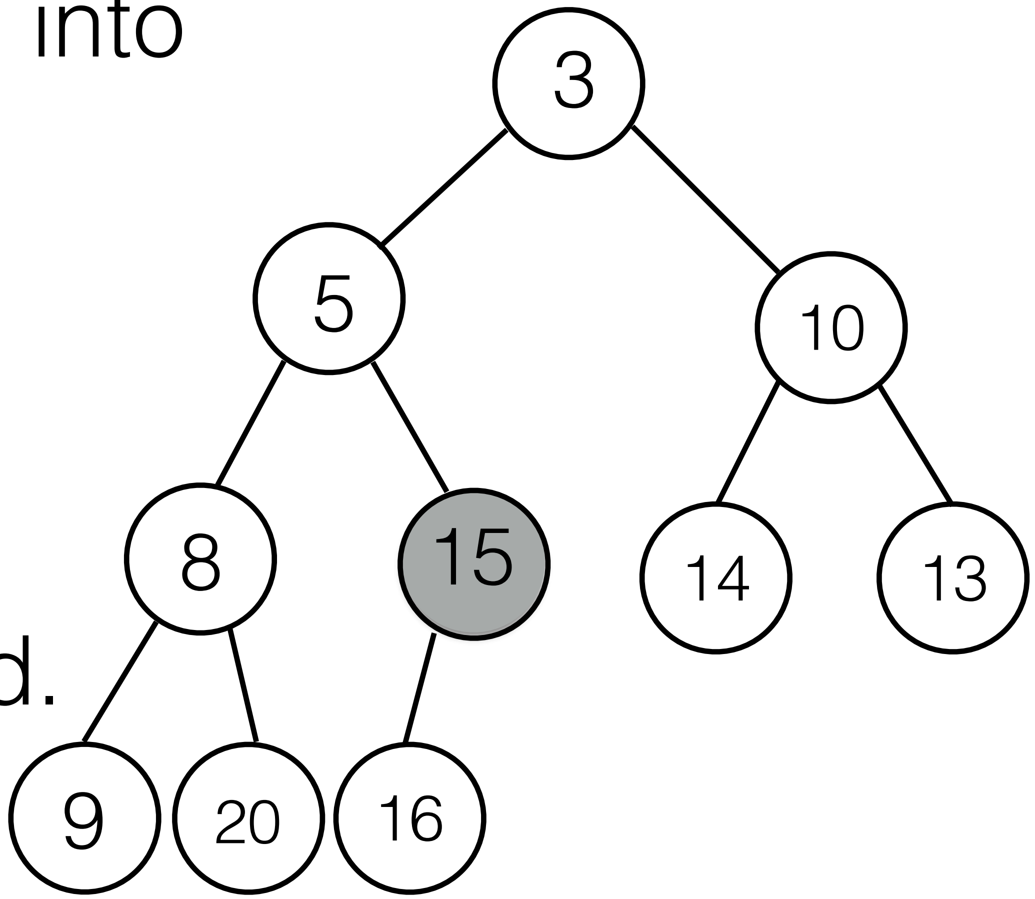


	3	5	10	8	15	14	13	9	20	16			

Running Time for Heap Operations

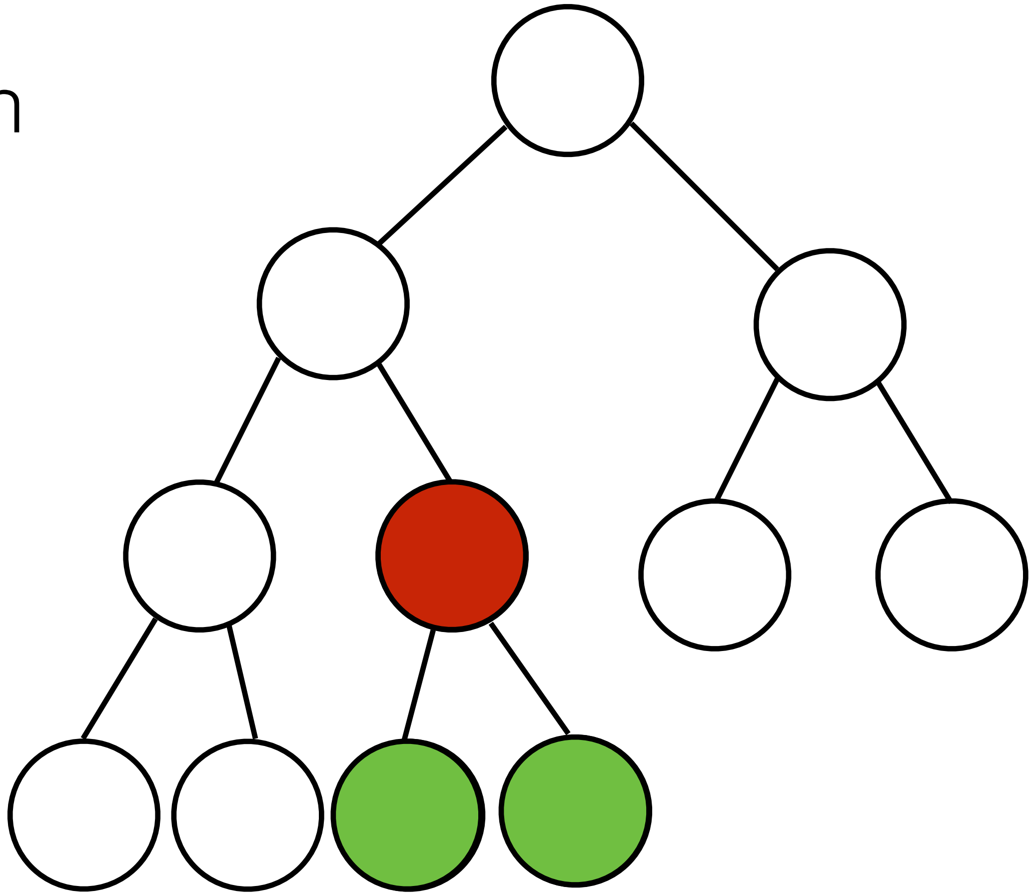
- Because a Heap is a complete binary tree, it's height is about log N.
- Worst-case running time for `insert(x)` and delete`Min()` is therefore O(log N).
- getMin() is O(1).

Building a Heap

- Want to convert an collection of N items into a heap.
- Each `insert(x)` takes O(log N) in the worst case, so the total time is O(N log N).
- Can show a better bound O(N) for building a heap.

Building a Heap Bottom-Up

- Start with an unordered array.
- `percolateDown(i)` assumes that both subtrees under *i* are already heaps.
- Idea: restore heap property bottom-up.
  - Make sure all subtrees in the two last layers are heaps.
  - Then move up layer-by-layer.



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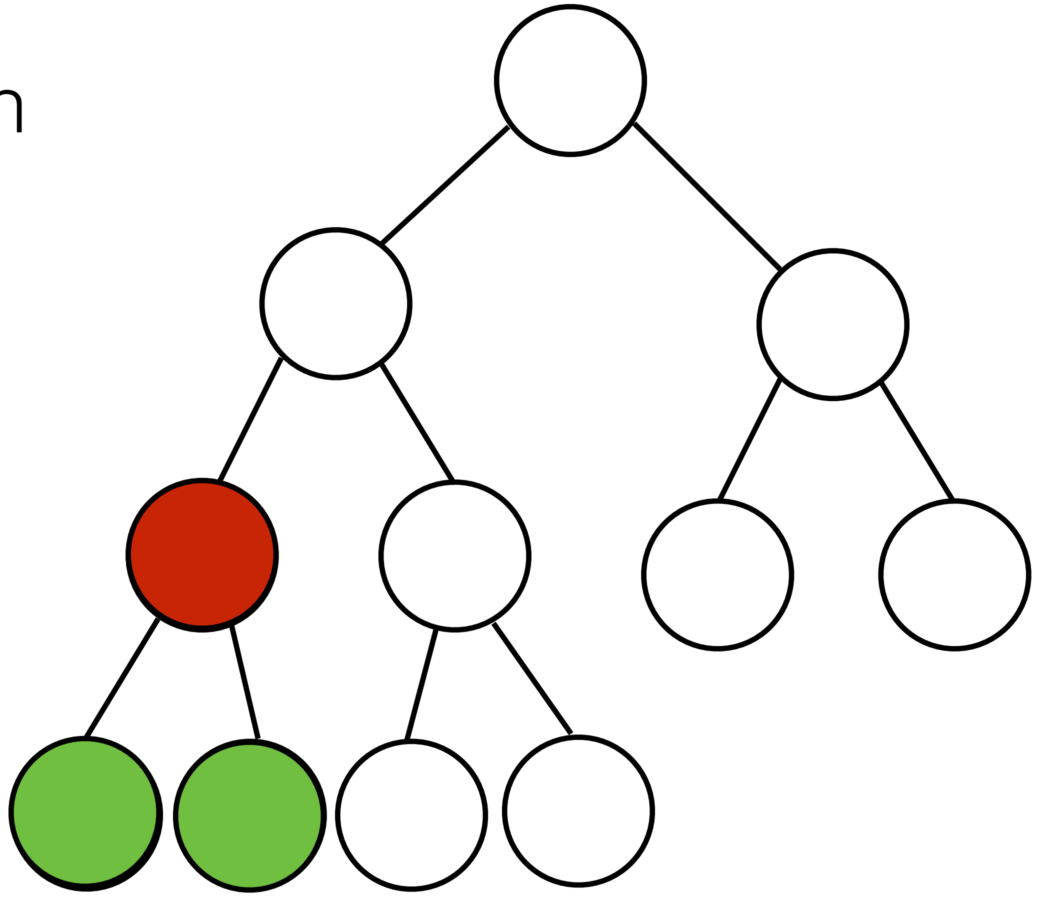

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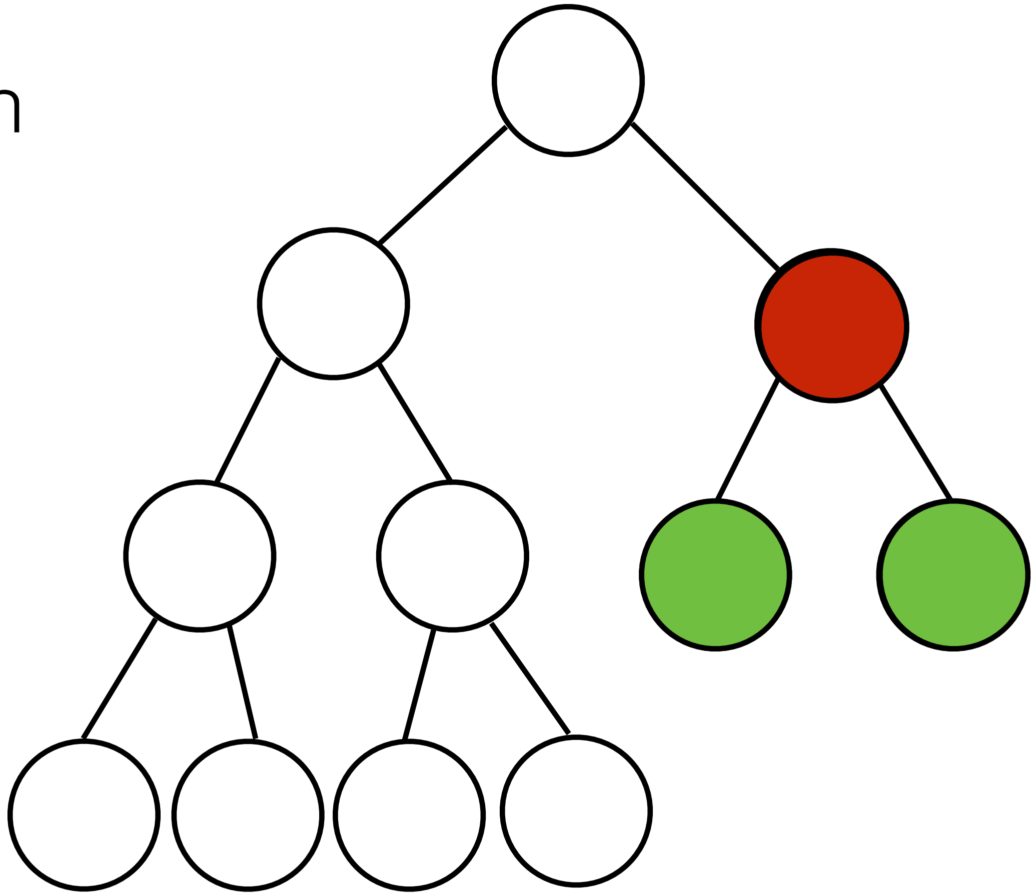

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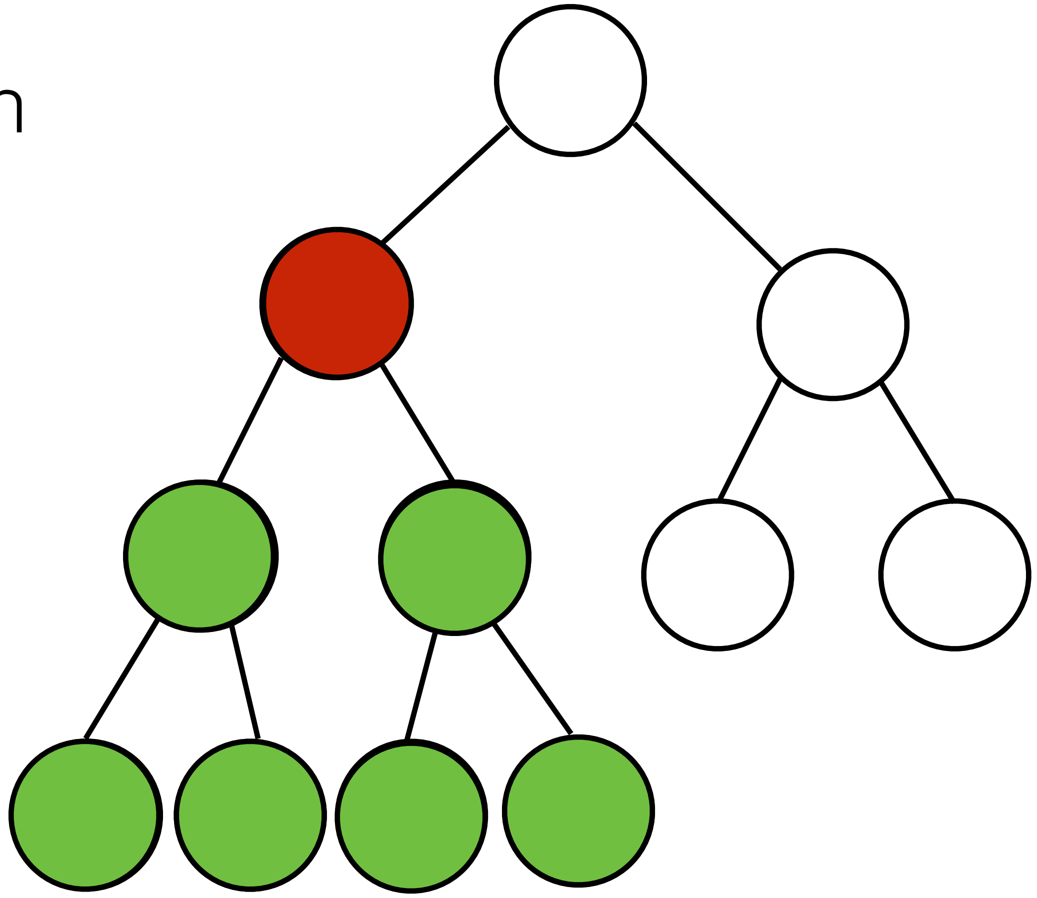

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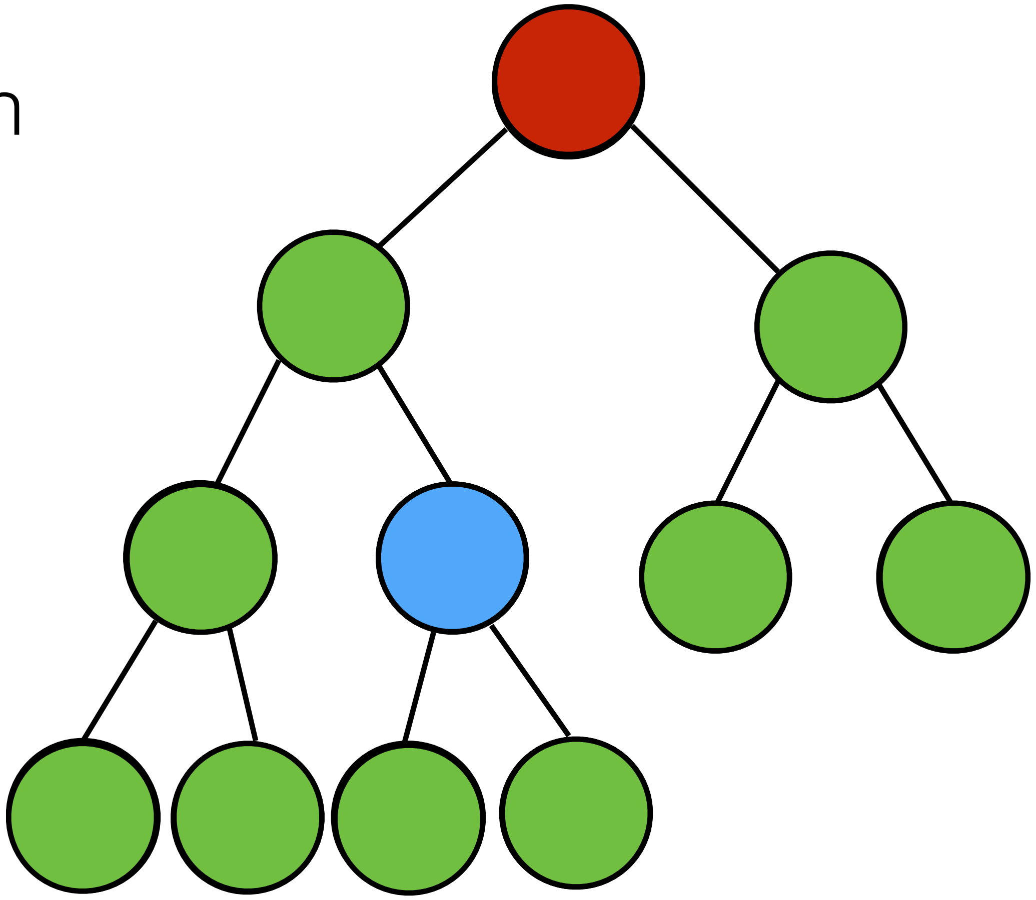

```
For i = $\lfloor N/2 \rfloor$ … 1
    percolateDown(i)
```

Building a Heap - Example

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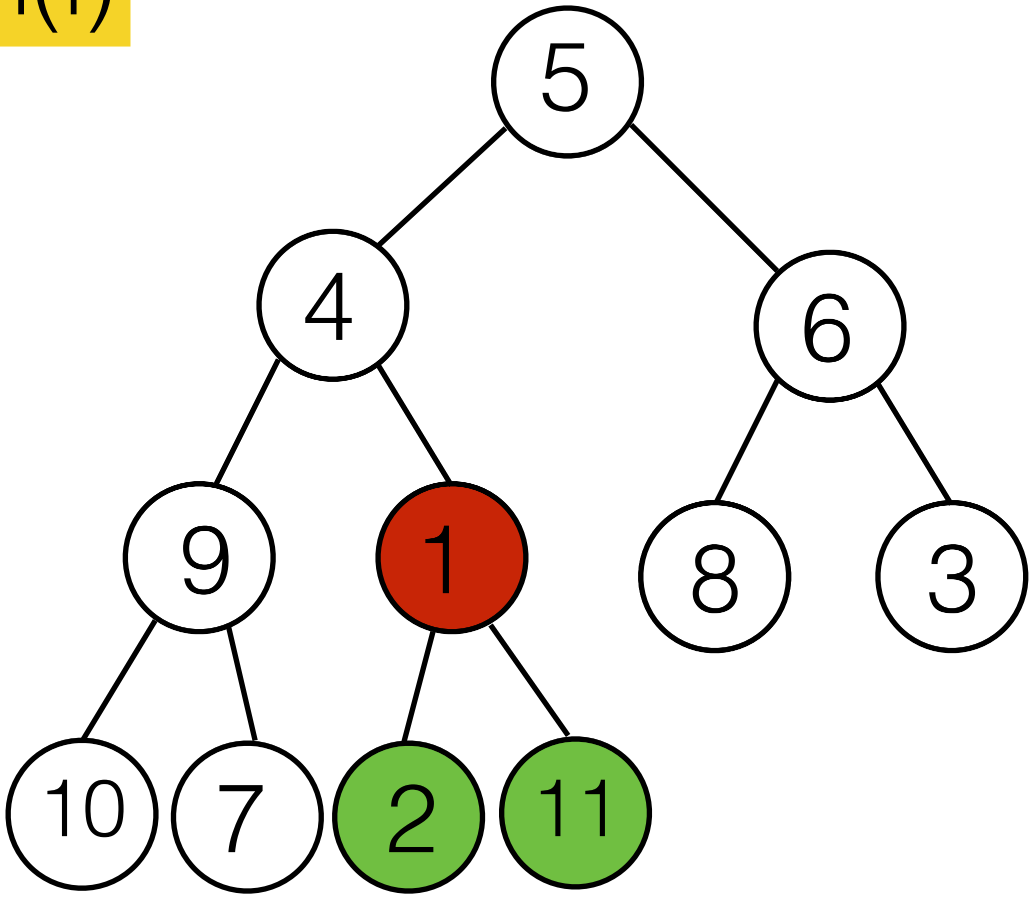

i=11/2 = 5

	5	4	6	9	1	8	3	10	7	2	11		

Building a Heap - Example

```
For i = ⌊N/2⌋ … 1
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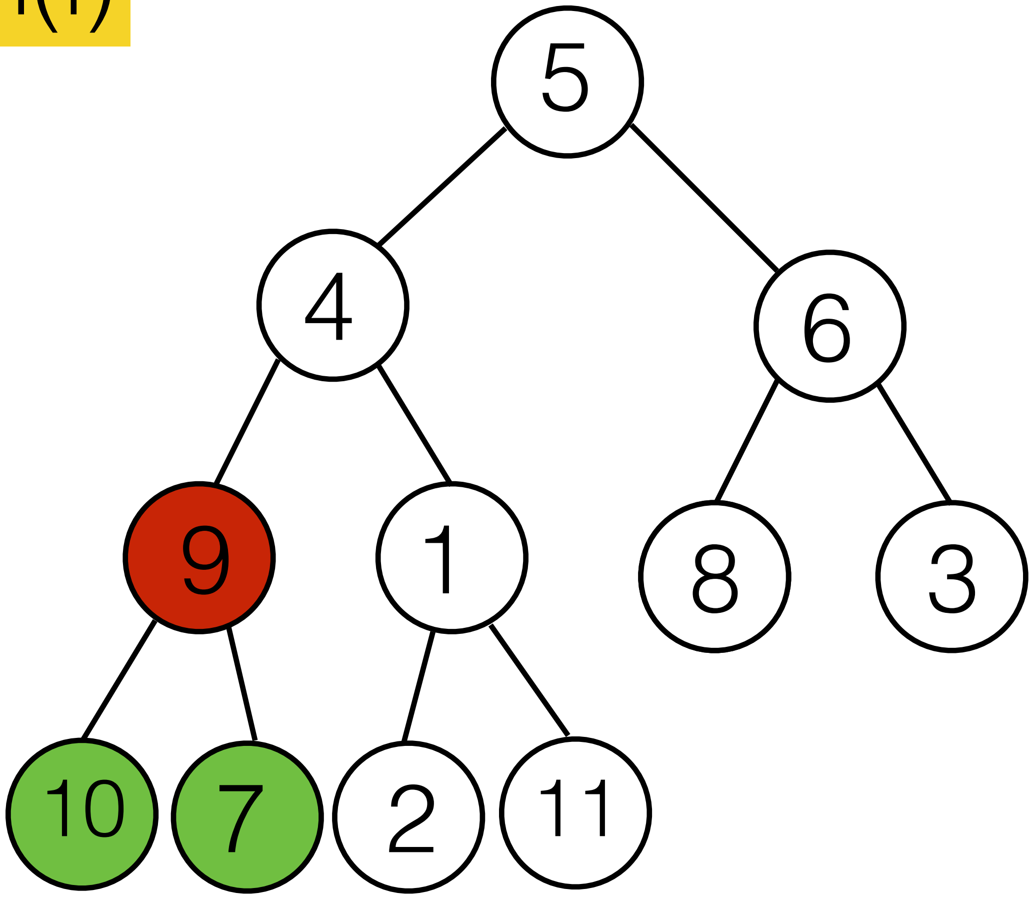
i=4



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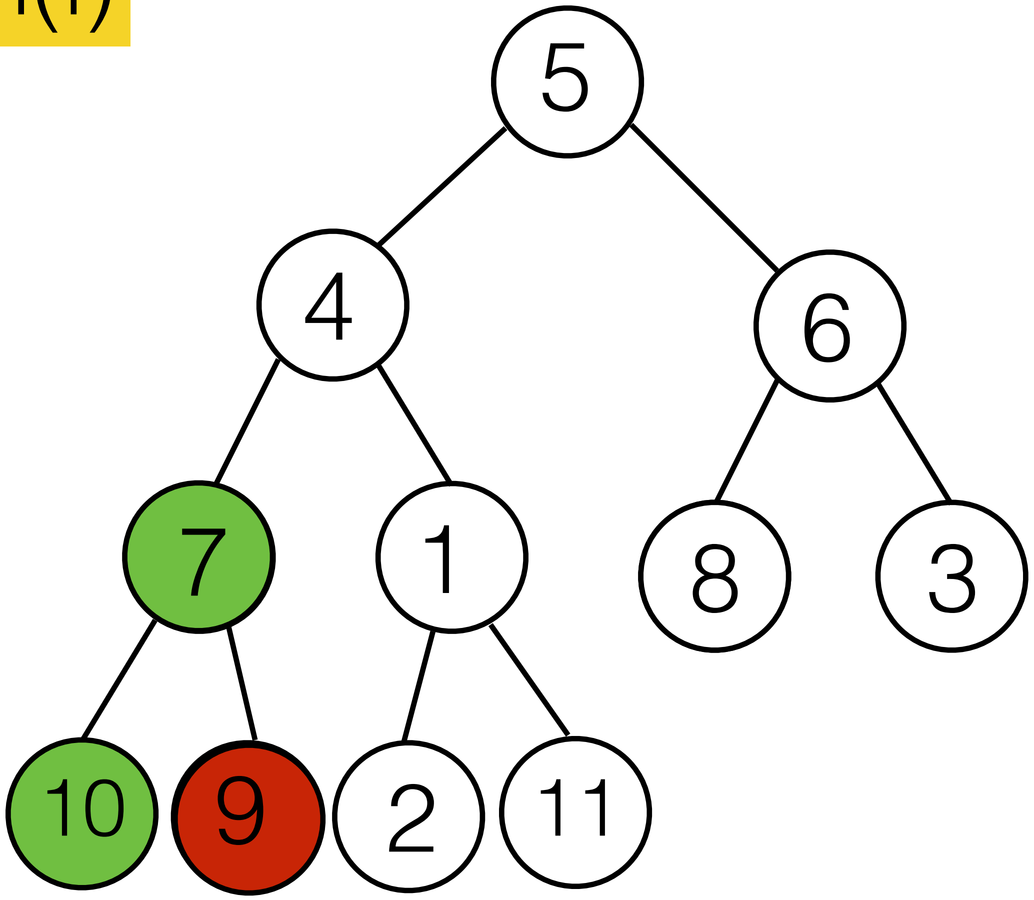

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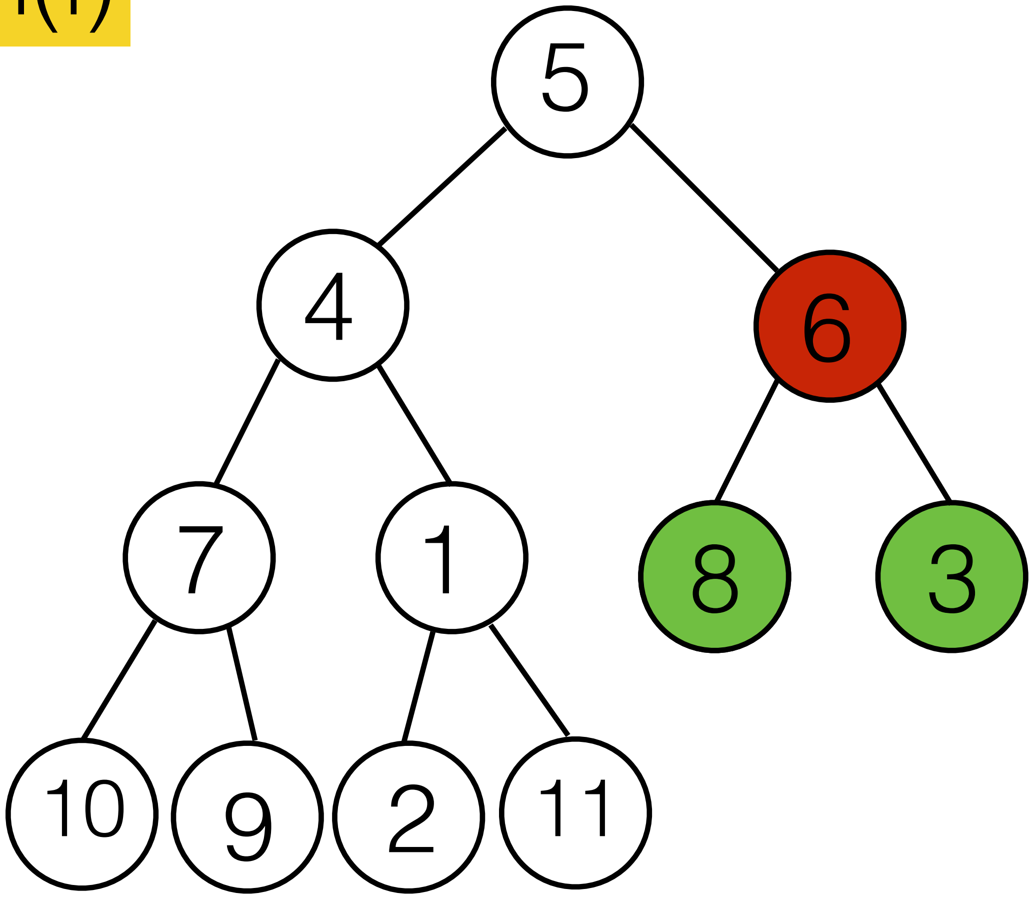

i=3

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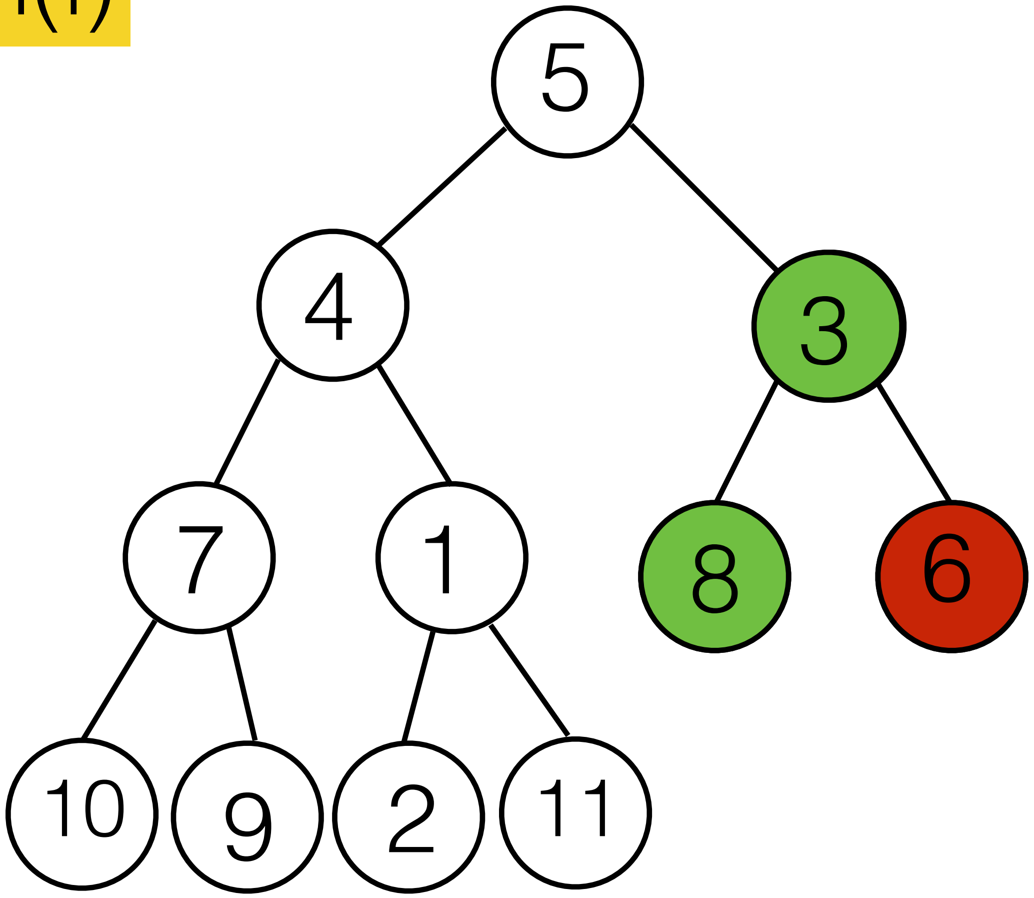

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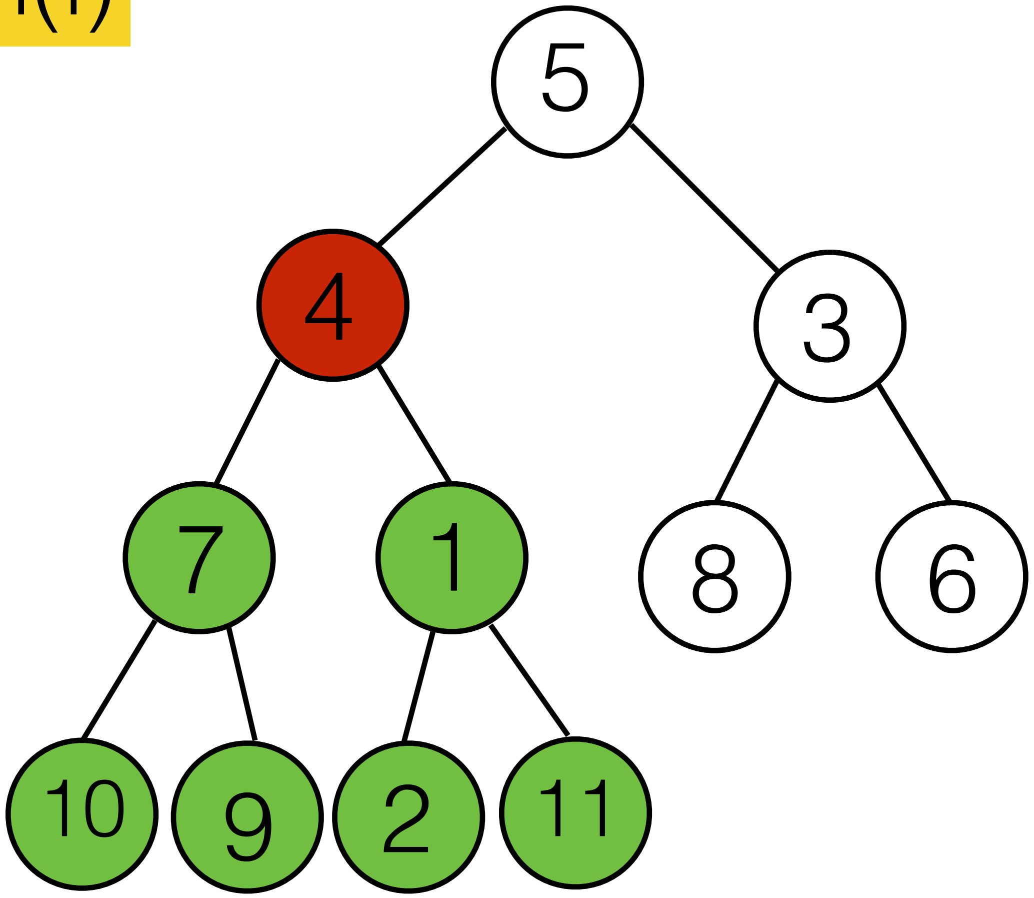

i=2

	5	4	3	7	1	8	6	10	9	2	11		

Building a Heap - Example

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For i = $\lfloor N/2 \rfloor$ … 1
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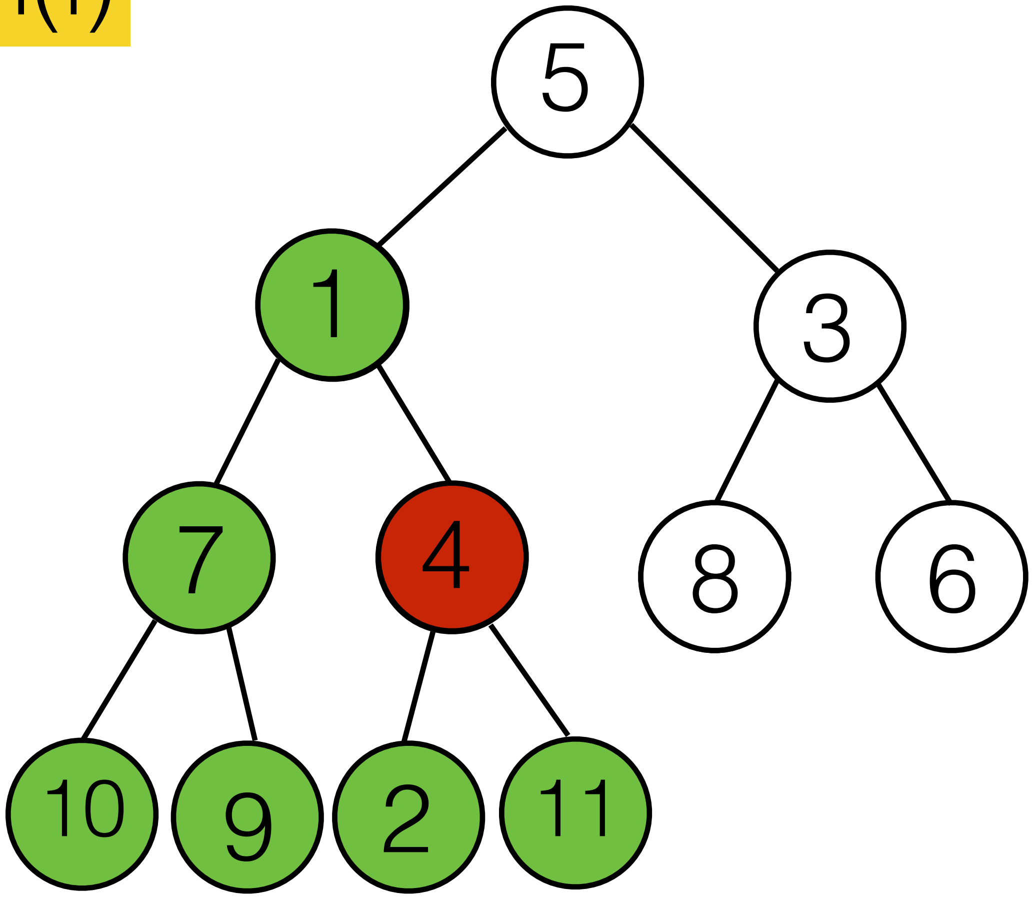
i=2



	5	1	3	7	4	8	6	10	9	2	11		

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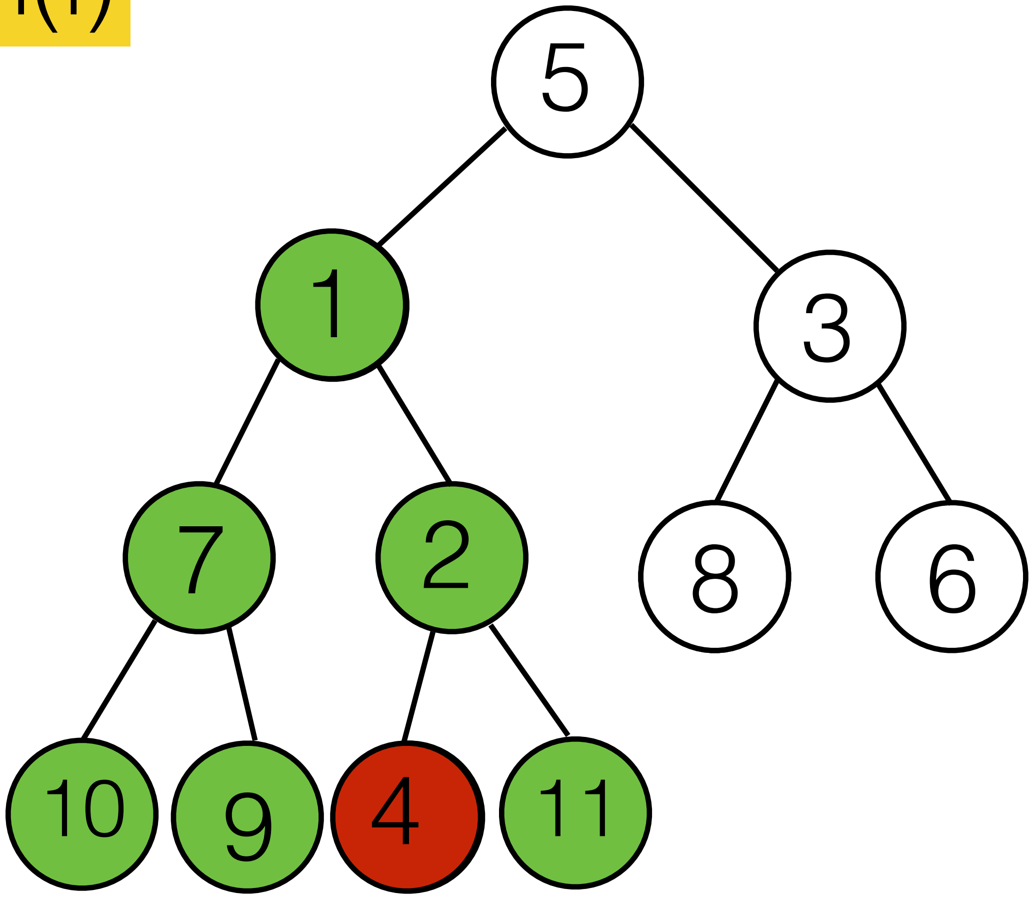

i=2

	5	1	3	7	2	8	6	10	9	4	11		

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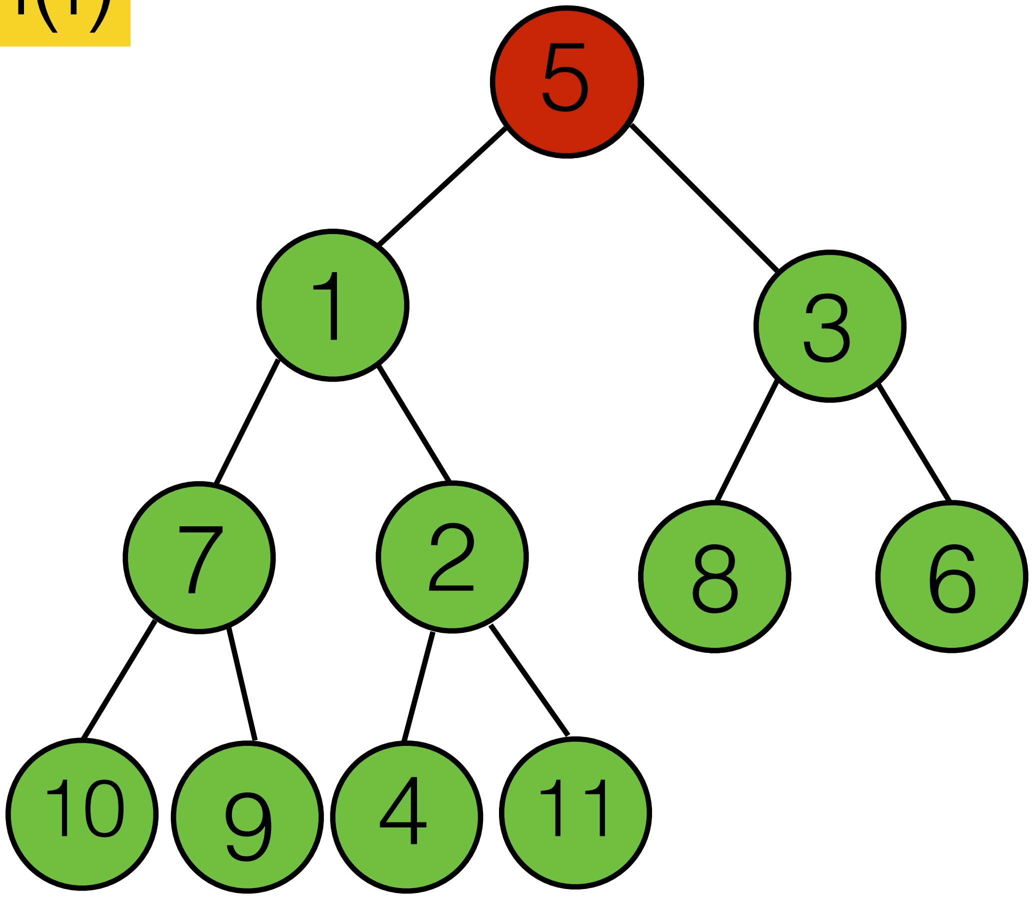

i=1

	5	1	3	7	2	8	6	10	9	4	11		

Building a Heap - Example

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    percolateDown(i)
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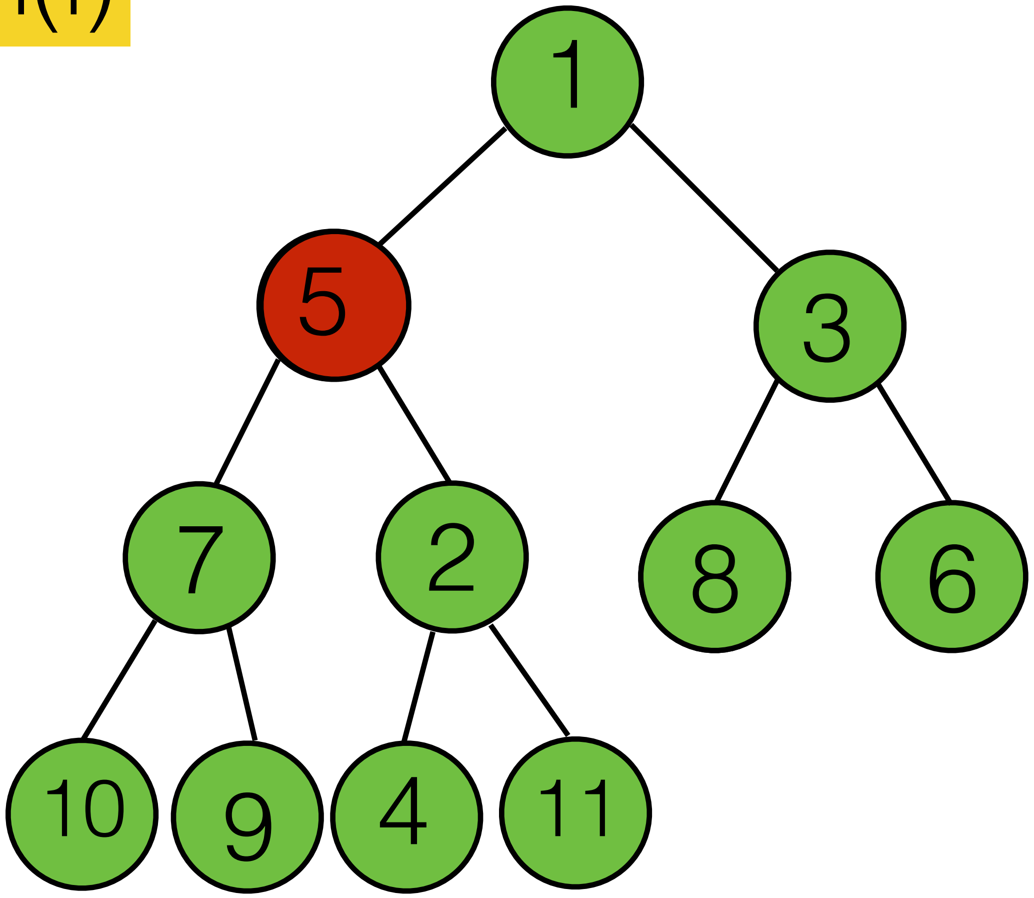

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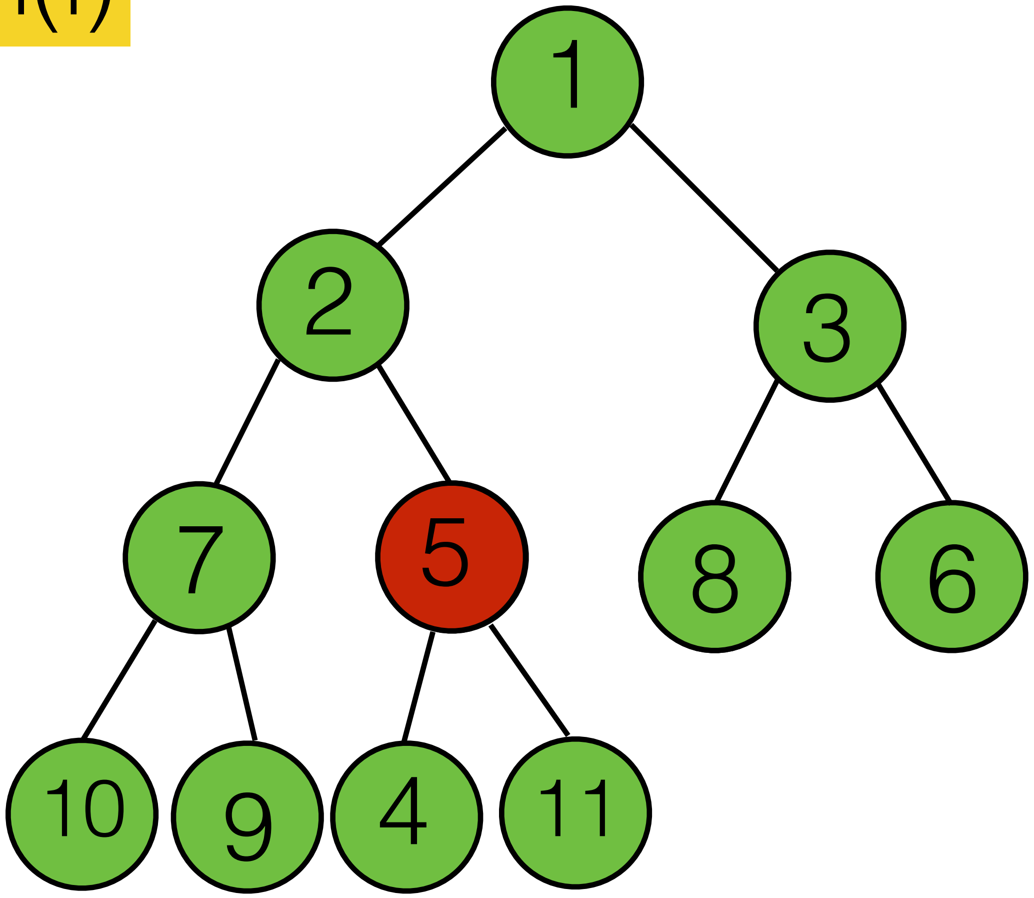

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```
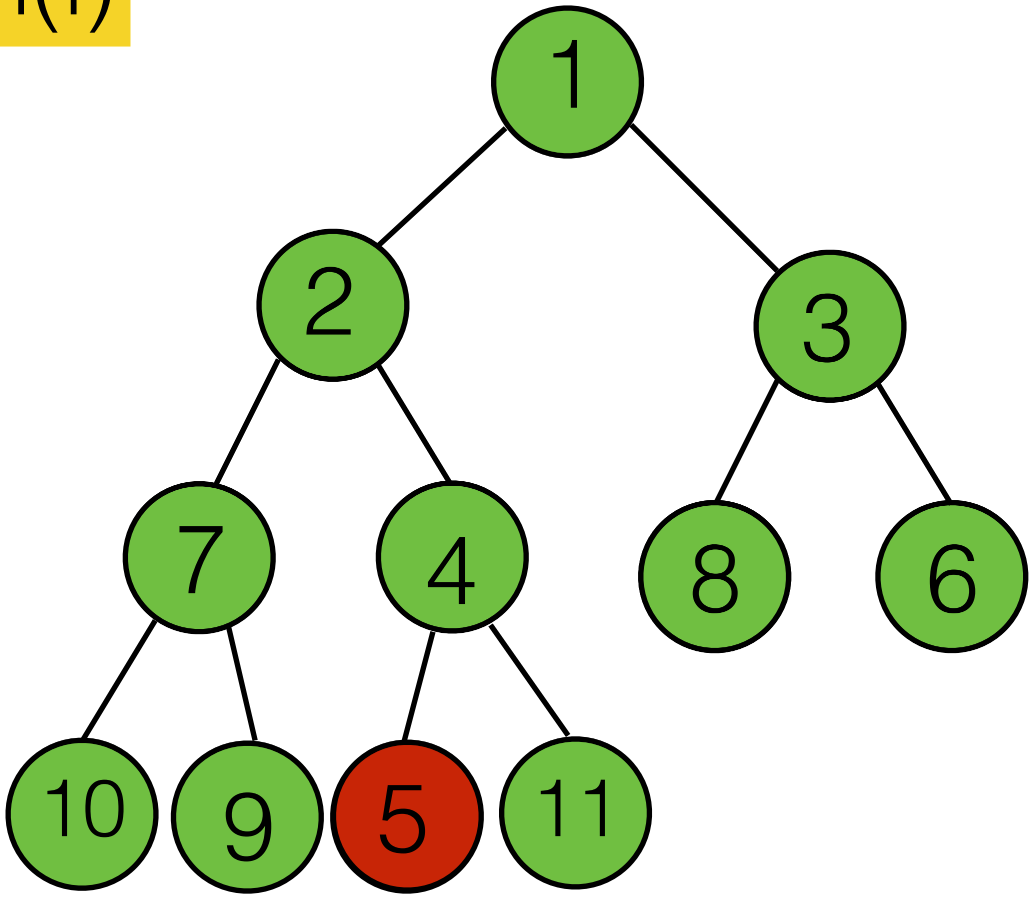

i=1



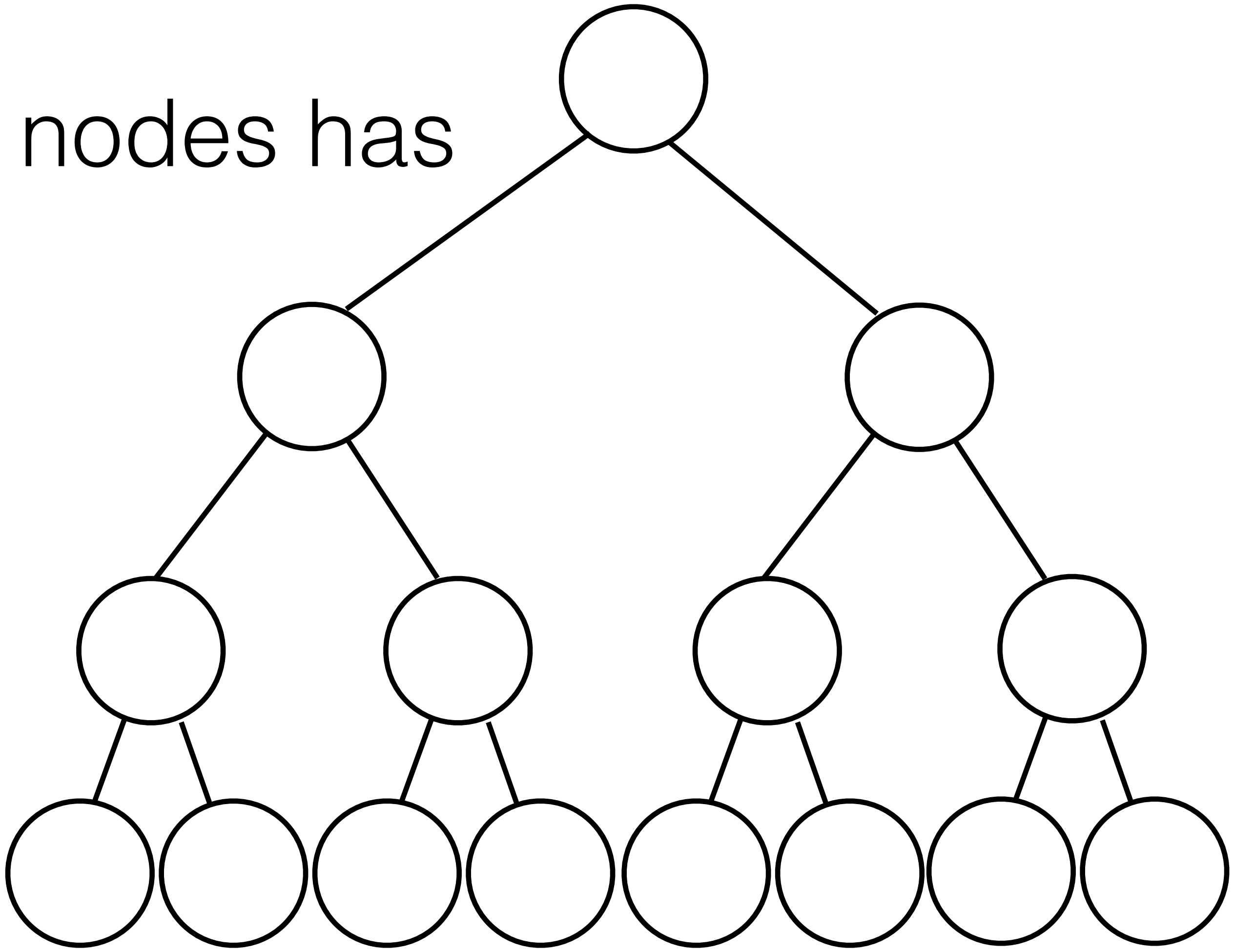
	1	2	3	7	4	8	6	10	9	5	11		

`BuildHeap` - Running Time

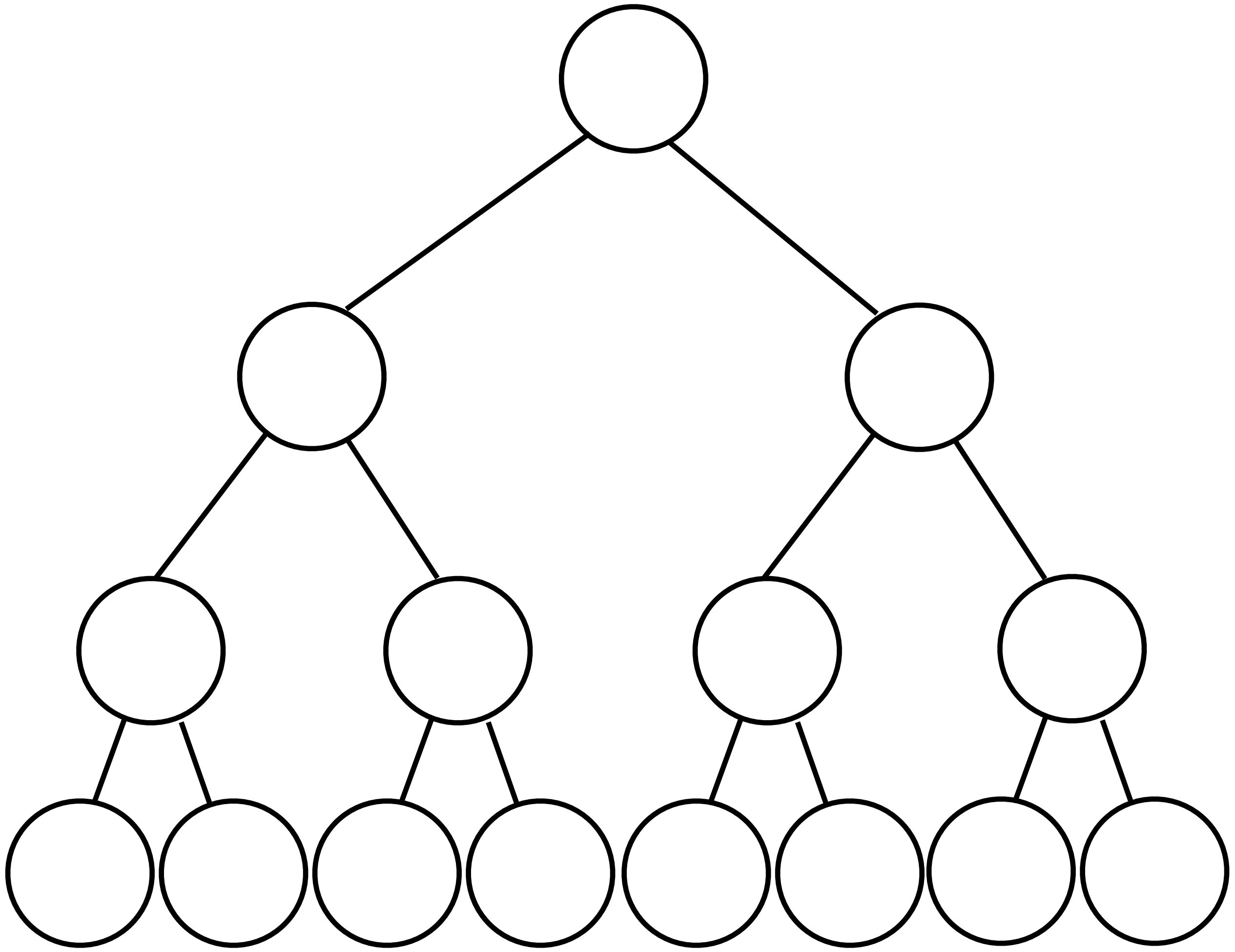
- How many comparisons do we need in each of the N/2 `percolateDown` calls?
    - In the worst case, each call to `percolateDown` needs to move the value all the way down to the leaf level.
    - We need to sum the possible steps for each level of the tree.

`BuildHeap` - Running Time

- Upper bound for nodes in a complete binary tree (if all levels are full) : $2^{h+1} - 1$
- A complete binary tree with N nodes has height: $h = \lfloor \log(N+1) \rfloor$



`BuildHeap` - Running Time



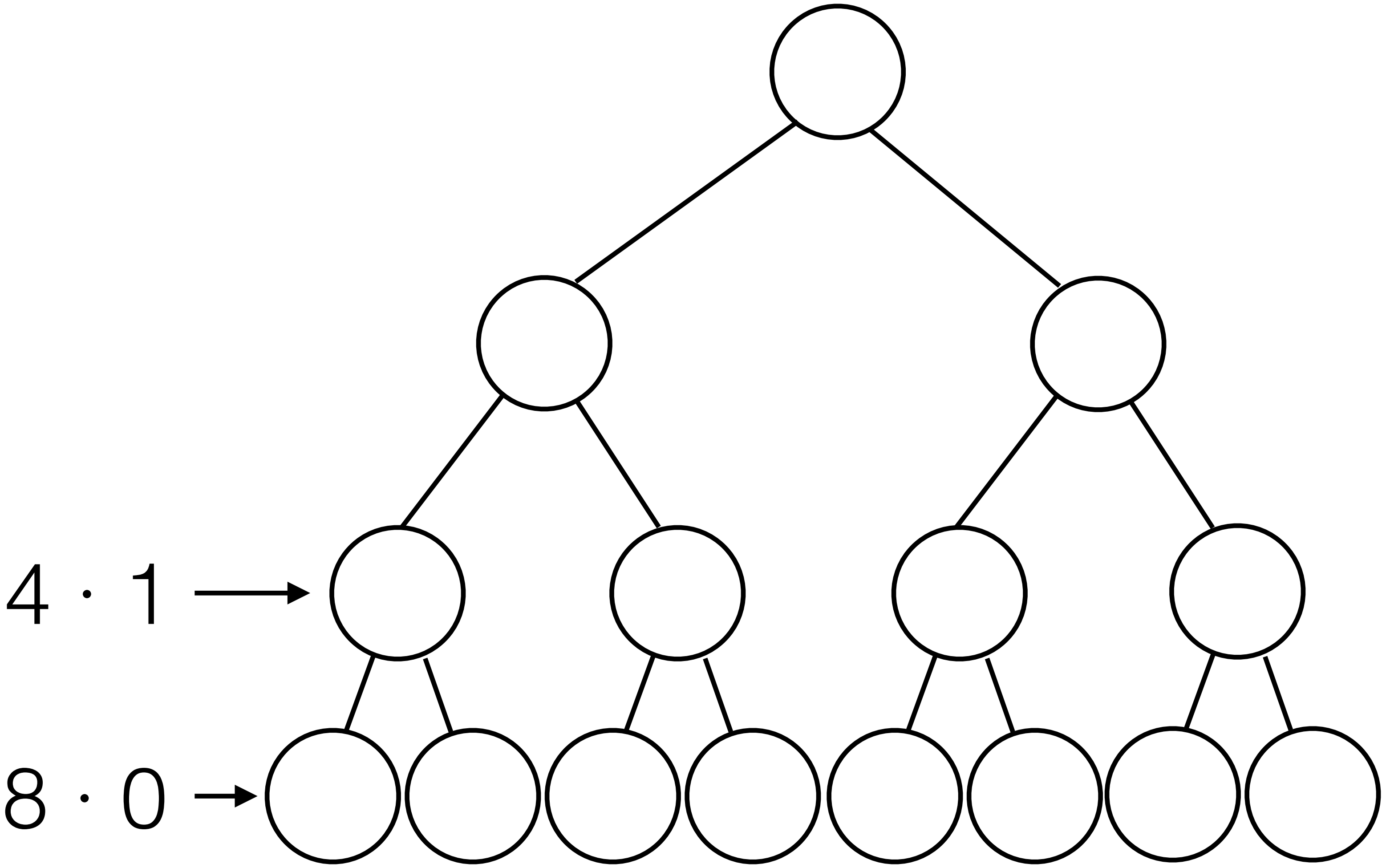
2^h nodes · 0 steps

8 · 0 →

`BuildHeap` - Running Time

2^{h-1} nodes · 1 steps

2^{h} nodes · 0 steps

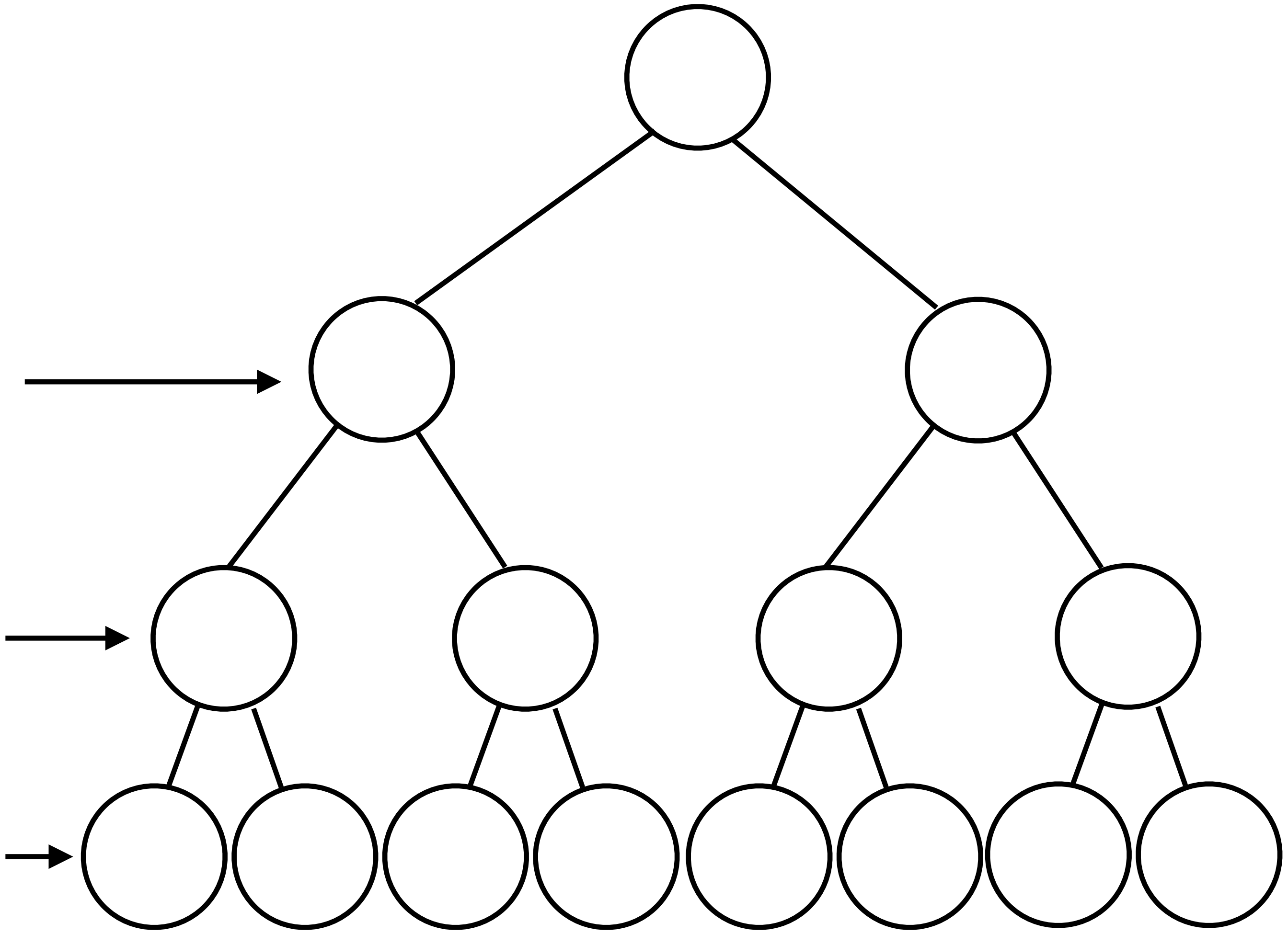


`BuildHeap` - Running Time

2^{h-2} nodes $\cdot$ 2 steps

2^{h-1} nodes $\cdot$ 1 steps

2^{h} nodes $\cdot$ 0 steps



$2 \cdot 2$

$4 \cdot 1$

$8 \cdot 0$

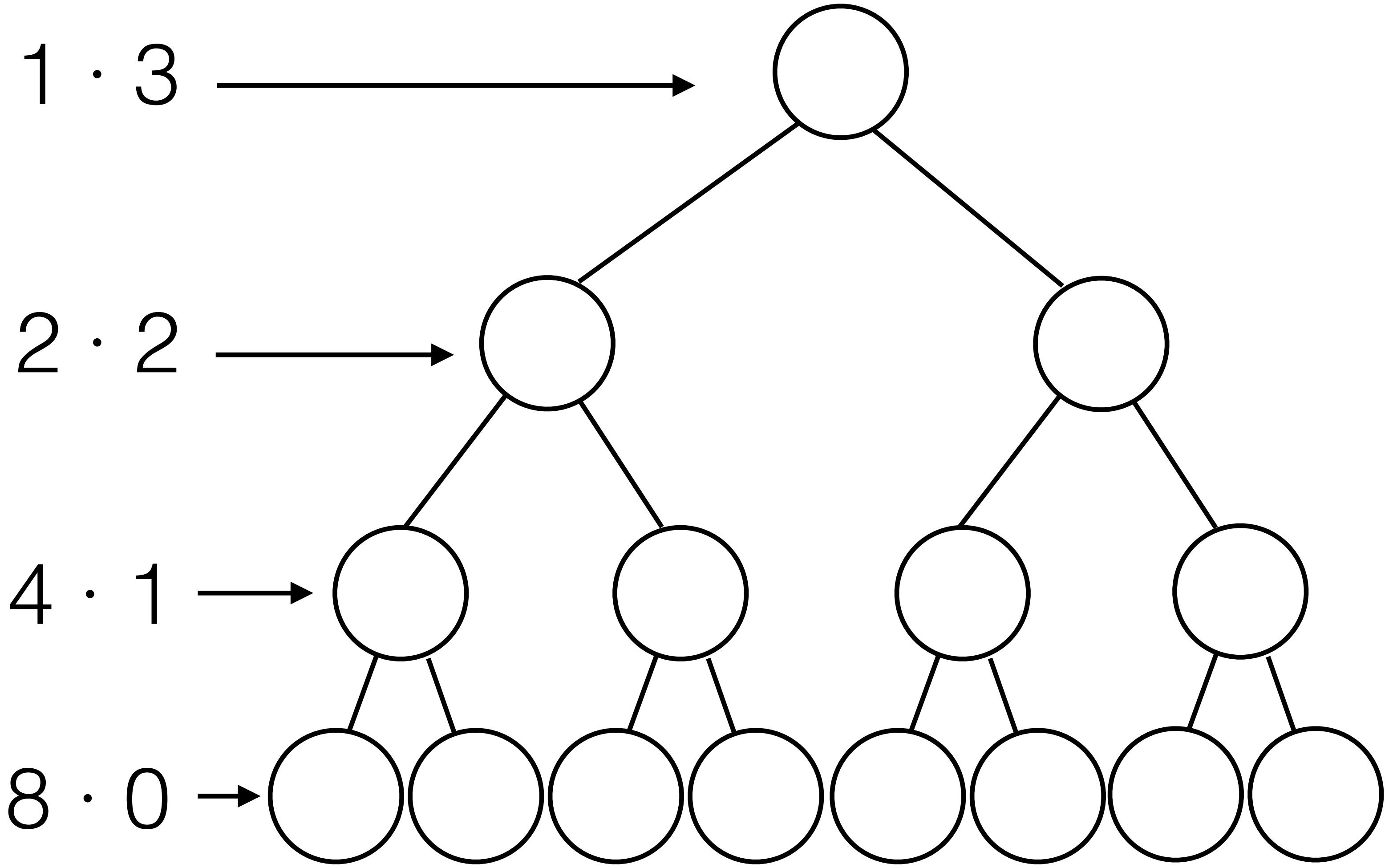
`BuildHeap` - Running Time

2^{h-3} nodes · 3 steps

2^{h-2} nodes · 2 steps

2^{h-1} nodes · 1 steps

2^{h} nodes · 0 steps



$1 \cdot 3$

$2 \cdot 2$

$4 \cdot 1$

$8 \cdot 0$

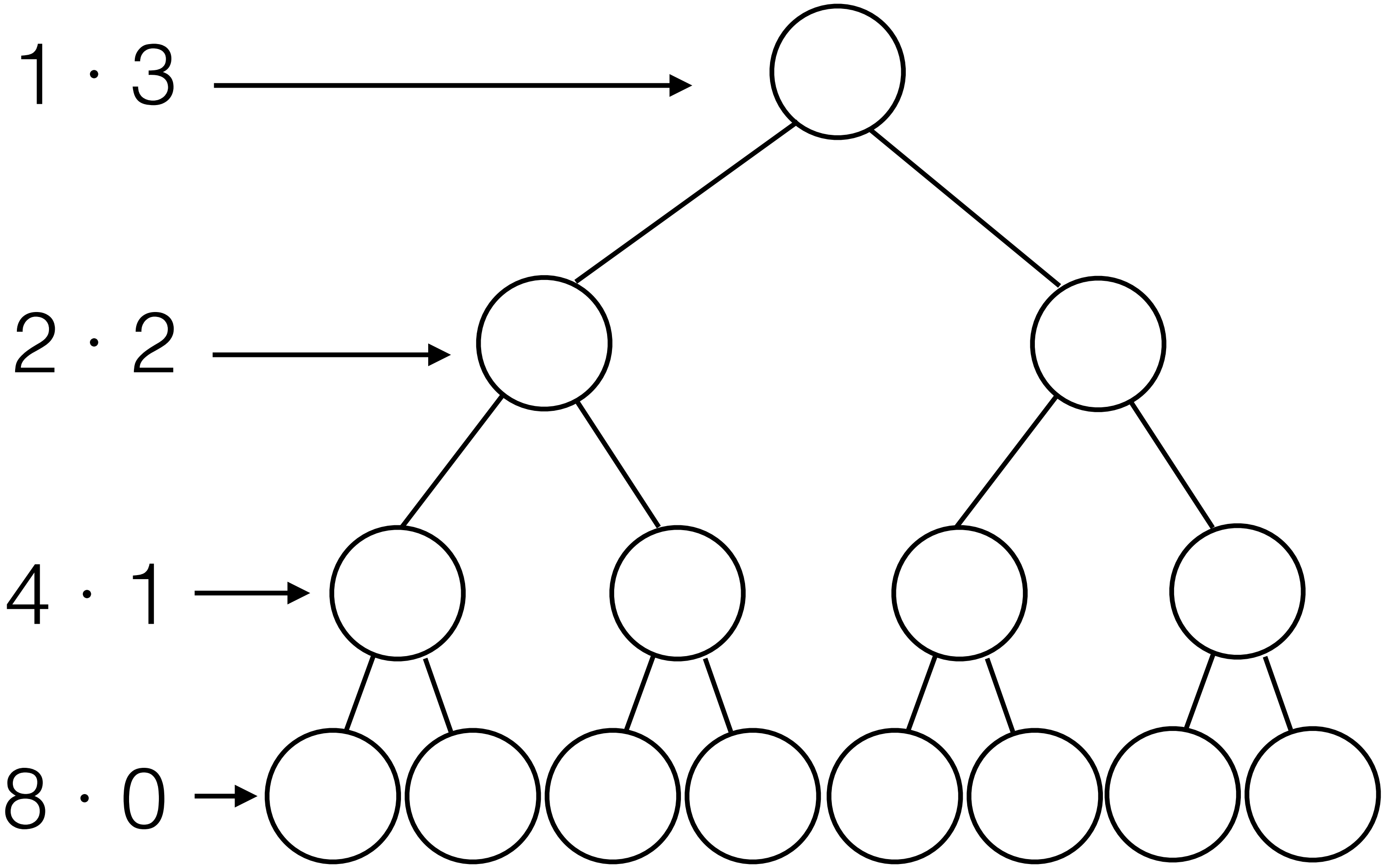
`BuildHeap` - Running Time

2^{h-3} nodes · 3 steps

2^{h-2} nodes · 2 steps

2^{h-1} nodes · 1 steps

2^{h} nodes · 0 steps



1 · 3

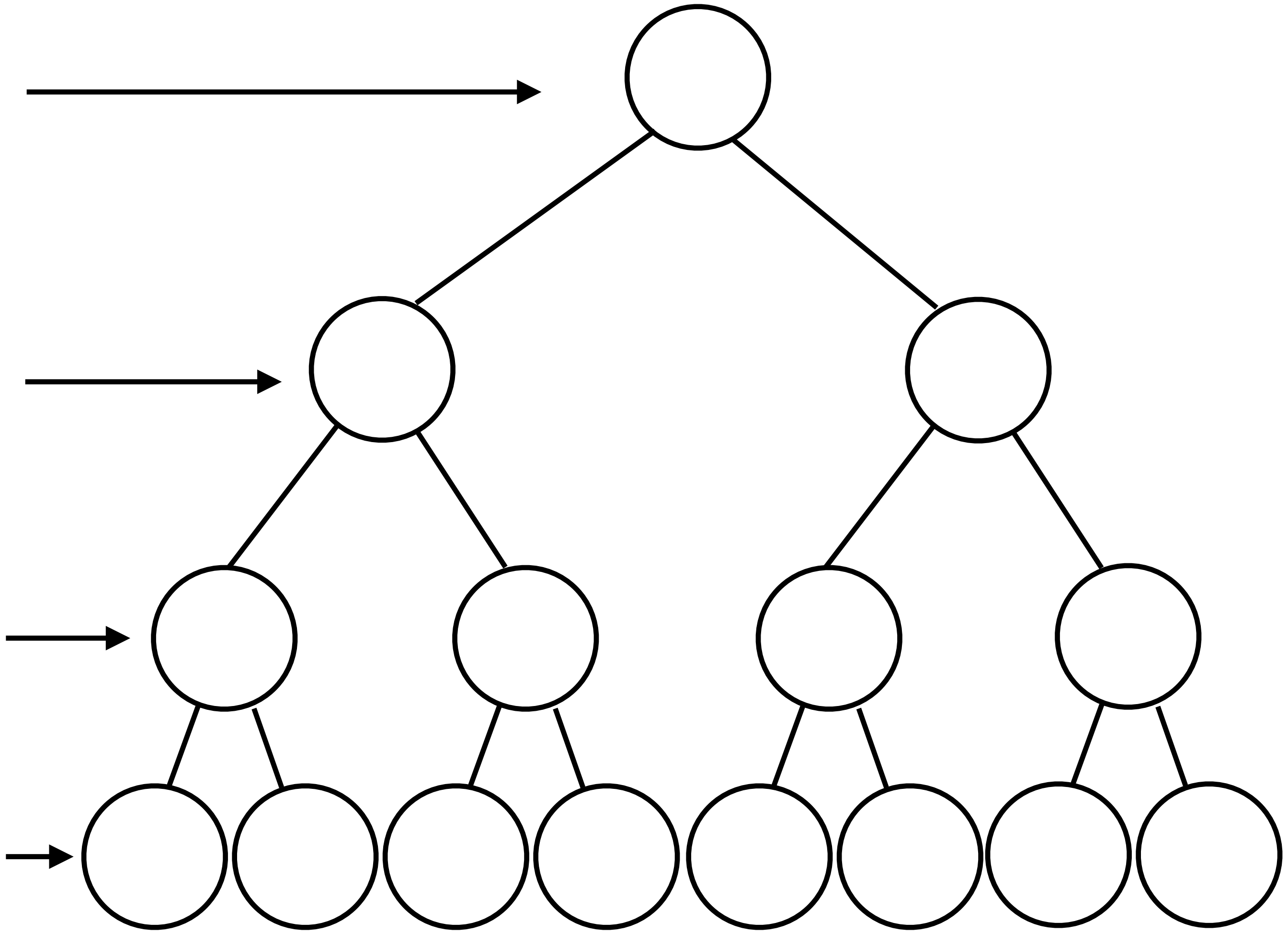
2 · 2

4 · 1

8 · 0

$$T(N) = 2^{h-1} \cdot 1 + \cdots + 4 \cdot (h-2) + 2 \cdot (h-1) + h \cdot 1$$

`BuildHeap` - Running Time



2^{h-3} nodes · 3 steps 1 · 3

2^{h-2} nodes · 2 steps 2 · 2

2^{h-1} nodes · 1 steps 4 · 1

2^{h} nodes · 0 steps 8 · 0

$$T(N) = 2^{h-1} \cdot 1 + \cdots + 4 \cdot (h-2) + 2 \cdot (h-1) + h \cdot 1 = \sum_{j=0}^{h} j \cdot 2^{h-j}$$

`BuildHeap` - Running Time

$$2T(N) = 2^h \cdot 1 + \cdots + 8 \cdot (h-2) + 4 \cdot (h-1) + h \cdot 2$$

$$T(N) = 2^{h-1} \cdot 1 + \cdots + 4 \cdot (h-2) + 2 \cdot (h-1) + h \cdot 1$$

$$2T(N) - T(N) = \underbrace{2^h + 2^{h+1} + \cdots + 8 + 4 + 2}_{\left(\sum_{i=0}^{h} 2^i\right) - 1 \quad = (2^{h+1} - 1) - 1} + h$$

$$T(N) = (2^{h+1} - 1) - (h + 1)$$

$$T(N) = (2^{h+1} - 1) - (\log(N+1) + 1) \quad = O(N)$$

=N

The Selection Problem

- Given an **unordered** sequence of *N* numbers
  $S = (a_1, a_2, \dots a_N)$, select the *k*-th largest number.

- Approach 1: Sort the numbers in decreasing order. Then pick the number at *k*-th position. => O(N log N + k)

- Approach 2: Initialize array of size *k* with the first *k* numbers. Sort the array in decreasing order. For every element in the sequence, if it is larger than the k-th entry in the array, replace the appropriate entry in the array with the new number.
  => O(k log k) + O(N · k)

The Selection Problem

- Given an unordered sequence of *N* numbers
  $S = (a_1, a_2, \ldots a_N)$, select the *k*-th largest number.
- Using a Heap (Option 1):
  - First build a Max-Heap in O(N).
  - Then call deleteMax() k times O(k log N).
  - Total: O(N + k log N)
  - If k has a linear dependence on N (e.g. k=N/2), then the total is O(N log N).

The Selection Problem

- Given an unordered sequence of N numbers
  $S = (a_1, a_2, \dots a_N)$, select the k-th largest number.

- Using a Heap (Option 2):

  - Build a Min-Heap S from the first k unordered elements in O(k).

  - The root of S now contains the k-th largest element.

  - Iterate through the remaining N-k = O(N) numbers:

    - If a number is larger than the root of S, remove the root of S and insert the new number into S. This takes O(log k) time.

  - Total: $O(k + N \cdot \log k) = O(N \log k)$