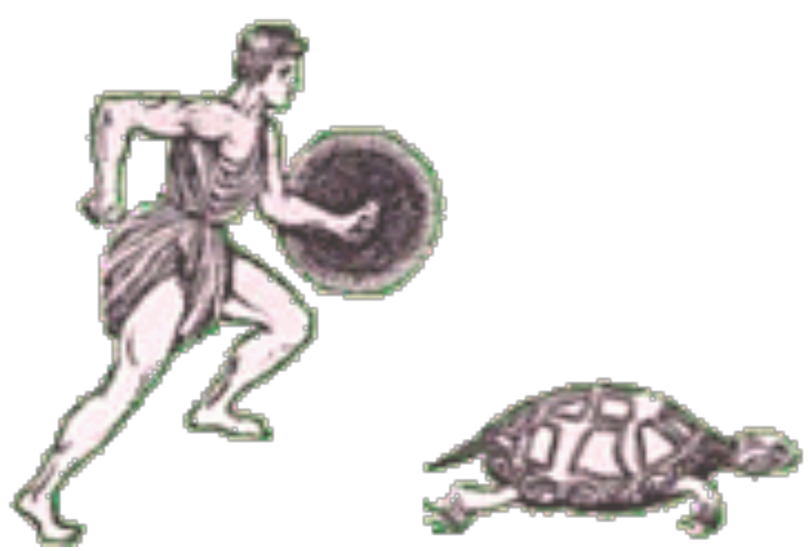
Data Structures in Java

Lecture 5: Sequences and Series, Proofs

9/23/2015



Daniel Bauer

Algorithms

- An algorithm is a clearly specified set of simple instructions to be followed to solve a problem.
- Algorithm Analysis — Questions:
    - Does the algorithm terminate?
    - Does the algorithm solve the problem? (correctness)
    - What resources does the algorithm use?
        - Time / Space

Contents

Sequences

- What are these sequences?
  - 0, 2, 4, 6, 8, 10, …
  - 2, 4, 8, 16, 32, 64, …
  - 1, 1/2, 1/4, 1/8, 1/16, …

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Arithmetic Sequence (0, 2, 4, 6, 8, 10, …)

$$a_i = a + (i-1)d$$

Geometric Sequence (2, 4, 8, 16, 32, 64, … and 1, 1/2, 1/4, 1/8, 1/16, …)

$$a_i = a \cdot A^i$$

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- Arithmetic Sequence of length *N*, with start term *a* and common difference *d.*

$$\{a, a+d, a+2d, \cdots, a+(N-1)d\}$$

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- Series: The sum of all elements of a sequence.

$$\sum_{i=1}^{N} a + (i - 1)d$$

$$= a + (a + d) + (a + 2d) + \cdots + (a + (N - 1)d)$$

Sum-Formulas for Arithmetic Series

$$\sum_{i=1}^{N} a + (i-1)d = N \cdot \frac{2a + (N-1)d}{2}$$

- In particular (for *a*=1 and *d*=1):

$$\sum_{i=1}^{N} i = N\,\frac{N+1}{2} \approx \frac{N^2}{2}$$

Geometric Series

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- Geometric Sequence with start term *s* and common ratio *A.*

$$\{s, s \cdot A, s \cdot A^2, \cdots, s \cdot A^N\}$$

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- Often $0 < A < 1$ or $A = 2$

Sum-Formulas for Finite Geometric Series

$$\sum_{i=0}^{N} s \cdot A^i = \frac{s - s \cdot A^{N+1}}{1 - A}$$

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- In Computer Science we often have *A = 2*

$$\sum_{i=0}^{N} 2^i = 2^{N+1} - 1$$

Sum-Formulas for Infinite Geometric Series

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- For instance,

$$\sum_{i=0}^{\infty} (\frac{1}{2})^i = \sum_{i=0}^{\infty} \frac{1}{2^i} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$$

$$= \frac{1}{1-1/2} = \frac{1}{1/2} = 2$$

Analyzing the Towers of Hanoi Recurrence

$$T(N) = 2 \cdot T(N-1) + 1$$

$$= 2 \cdot (2 \cdot T(N-2) + 1) + 1 = 2^2 \cdot T(N-2) + 2 + 1$$

$$= 2 \cdot (2^2 \cdot T(N-3) + 2 + 1) + 1$$

$$= 2^3 \cdot T(N-3) + 2^2 + 2 + 1 = 2^3 \cdot T(N-1) + 2^2 + 2^1 + 2^0$$

$$= 2^{N-1} \cdot T(1) + 2^{N-2} + 2^{N-3} \cdots + 2^0$$

$$= \sum_{k=0}^{N-1} 2^k = 2^N - 1$$

base case: $T(1) = 1$

geometric series

The End of The World

Legend says that, at the beginning of time, priests were given a puzzle with 64 golden disks. Once they finish moving all the disks according to the rules, the wold is said to end.

If the priests move the disks at a rate of 1 disk/second, how long will it take to solve the puzzle?

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$$T^N = 2^N - 1 = O(2^N)$$

$$\begin{aligned} 2^{64} - 1 &= 18,446,744,073,709,551,615 \quad \text{seconds} \\ &= 307,445,734,561,825,860 \quad \text{minutes} \\ &= 213,503,982,334,601 \quad \text{days} \\ &= 584,942,417,355 \quad \text{years} \end{aligned}$$

Contents

Types of Proofs

- Proof by Induction
- Proof by Contradiction
- Proof by Counterexample

Proofs by Induction

- We are proving a theorem **T.** (“this property holds for all cases.”).
- Step 1: Base case. We know that **T** holds (trivially) for some small value.
- Step 2: Inductive step:
    - Inductive Hypothesis: Assume **T** holds for all cases up to some limit *k.*
    - Show that **T** also holds for *k+1.*
- This proves that **T** holds for any *k.*

Proof by Induction - Example

- For the Fibonacci numbers, we prove that

$$F_i \leq (5/3)^i \text{ for any } i \geq 1$$

- Base case:

$$F_1 = F_2 = 1 < 5/3$$

- Inductive step:
  - Assume the theorem holds for i = 1,2,…,*k*
  - We need to show that

$$F_{k+1} < (5/3)^{k+1}$$

Proof by Induction - Inductive Step

- We know that $F_{k+1} = F_k + F_{k-1}$ and by the inductive hypothesis:

$$
\begin{aligned}
F_{k+1} &< (5/3)^k + (5/3)^{k-1} \\
&< (3/5)(5/3)^{k+1} + (3/5)^2(5/3)^{k+1} \\
&< (3/5)(5/3)^{k+1} + (9/25)(5/3)^{k+1} \\
&< (3/5 + 9/25)(5/3)^{k+1} \\
&< (24/25)(5/3)^{k+1} \\
&< (5/3)^{k+1} \qquad \square
\end{aligned}
$$

Proof by Counter-Example

- We are proving that theorem **T** is false**.** ("this property does not hold for all cases.").

- It is sufficient to show that there is a case for which **T** does not hold.

- Example:
  - Show that $F_i \leq i^2$ is false.
  - There are *i* for which $F_i > i^2$,e.g. $F_{13} = 233 > 13^2$

$\square$

Proof by Contradiction

- We want to proof that **T** is true.
- Step 1: Assume **T** is false.
- Step 2: Show that this assumption leads to a contradiction.

Proof by Contradiction - Example

- We want to proof that there is an infinite number of primes.
- Assume the number of primes is finite and the largest prime it P_k.
- Let the sequence of all primes be P_1, P_2,,…,P_k

$$N = P_1 P_2 \cdots P_k + 1$$

- Since N>P_k, N cannot be prime, so it must have a factorization into primes. Such a factorization cannot exist: dividing N by any P_1, P_2,,…,P_k will always leave a remainder of 1. $\square$