Data Structures in Java

Lecture 5: Sequences and Series, Proofs

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Algorithms

• An algorithm is a clearly specified set of simple instructions to be followed to solve a problem.

• Algorithm Analysis — Questions:
  • Does the algorithm terminate?
  • Does the algorithm solve the problem? (correctness)
  • What resources does the algorithm use?
    • Time / Space
Contents

1. Sequences and Series

2. Proofs
Sequences

• What are these sequences?
  • 0, 2, 4, 6, 8, 10, ...
  • 2, 4, 8, 16, 32, 64, ...
  • 1, 1/2, 1/4, 1/8, 1/16, ...
Sequences

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  • 2, 4, 8, 16, 32, 64, ...
  • 1, 1/2, 1/4, 1/8, 1/16, ...

Arithmetic Sequence

\[ a_i = a + (i - 1)d \]
Sequences

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  • 0, 2, 4, 6, 8, 10, …  
  • 2, 4, 8, 16, 32, 64, …  
  • 1, 1/2, 1/4, 1/8, 1/16, …

Arithmetic Sequence

\[ a_i = a + (i - 1)d \]

Geometric Sequence

\[ a_i = a \cdot A^i \]
Arithmetic Series
Arithmetic Series

- Arithmetic Sequence of length $N$, with start term $a$ and common difference $d$.
  \[
  \{a, a + d, a + 2d, \ldots, a + (N - 1)d\}
  \]
Arithmetic Series

• Arithmetic Sequence of length $N$, with start term $a$ and common difference $d$.
  \[ \{a, a + d, a + 2d, \cdots, a + (N - 1)d\} \]

• Series: The sum of all elements of a sequence.

\[ \sum_{i=1}^{N} a + (i - 1)d \]

\[ = a + (a + d) + (a + 2d) + \cdots + (a + (N - 1)d) \]
Sum-Formulas for Arithmetic Series

\[ \sum_{i=1}^{N} a + (i - 1)d = N \cdot \frac{2a + (N - 1)d}{2} \]

• In particular (for \(a=1\) and \(d=1\)):

\[ \sum_{i=1}^{N} i = N \cdot \frac{N + 1}{2} \approx \frac{N^2}{2} \]
Geometric Series
Geometric Series

• Geometric Sequence with start term $s$ and common ratio $A$.

\[ \{s, s \cdot A, s \cdot A^2, \ldots, s \cdot A^N\} \]
Geometric Series

• Geometric Sequence with start term $s$ and common ratio $A$.
  \[ \{s, s \cdot A, s \cdot A^2, \cdots, s \cdot A^N\} \]

• Geometric Series:
  \[ \sum_{i=0}^{N} s \cdot A^i = s + s \cdot A + s \cdot A^2 + \cdots + s \cdot A^N \]
Geometric Series

• Geometric Sequence with start term \( s \) and common ratio \( A \).
  \[ \{ s, s \cdot A, s \cdot A^2, \ldots, s \cdot A^N \} \]

• Geometric Series:
  \[ \sum_{i=0}^{N} s \cdot A^i = s + s \cdot A + s \cdot A^2 + \cdots + s \cdot A^N \]

• Often \( 0 < A < 1 \) or \( A = 2 \)
Sum-Formulas for Finite Geometric Series

\[ \sum_{i=0}^{N} s \cdot A^i = \frac{s - s \cdot A^{N+1}}{1 - A} \]
Sum-Formulas for Finite Geometric Series

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• In particular, if \( s=1 \)

\[ \sum_{i=0}^{N} A^i = \frac{A^{N+1} - 1}{A - 1} \]
Sum-Formulas for Finite Geometric Series

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• In particular, if \( s = 1 \)

\[ \sum_{i=0}^{N} A^i = \frac{A^{N+1} - 1}{A - 1} \]

• In Computer Science we often have \( A = 2 \)

\[ \sum_{i=0}^{N} 2^i = 2^{N+1} - 1 \]
Sum-Formulas for Infinite Geometric Series

\[ \sum_{i=0}^{\infty} s \cdot A^i = \frac{s}{1 - A} \quad \text{only if } 0 < A < 1 \]
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\[ \sum_{i=0}^{\infty} s \cdot A^i = \frac{s}{1 - A} \quad \text{only if } 0 < A < 1 \]

• In particular, if \( s = 1 \)

\[ \sum_{i=0}^{\infty} A^i = \frac{1}{1 - A} \quad \text{and} \quad \sum_{i=0}^{N} A^i \leq \frac{1}{1 - A} \]
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• For instance,

\[ \sum_{i=0}^{\infty} \left( \frac{1}{2} \right)^i = \sum_{i=0}^{\infty} \frac{1}{2^i} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots \]

\[ = \frac{1}{1 - 1/2} = \frac{1}{1/2} = 2 \]
Analyzing the Towers of Hanoi Recurrence

\[ T(N) = 2 \cdot T(N - 1) + 1 \]

\[ = 2 \cdot (2 \cdot T(N - 2) + 1) + 1 = 2^2 \cdot T(N - 2) + 2 + 1 \]

\[ = 2 \cdot (2^2 \cdot T(N - 3) + 2 + 1) + 1 \]

\[ = 2^3 \cdot T(N - 3) + 2^2 + 2 + 1 = 2^3 \cdot T(N - 1) + 2^2 + 2^1 + 2^0 \]

\[ = 2^{N-1} \cdot T(1) + 2^{N-2} + 2^{N-3} \ldots + 2^0 \]

\[ = \sum_{k=0}^{N-1} 2^k = 2^N - 1 \]

geometric series

base case: \( T(1) = 1 \)
Legend says that, at the beginning of time, priests were given a puzzle with 64 golden disks. Once they finish moving all the disks according to the rules, the world is said to end.

If the priests move the disks at a rate of 1 disk/second, how long will it take to solve the puzzle?
The End of The World

Legend says that, at the beginning of time, priests were given a puzzle with 64 golden disks. Once they finish moving all the disks according to the rules, the world is said to end.

If the priests move the disks at a rate of 1 disk/second, how long will it take to solve the puzzle?

\[ T^N = 2^N - 1 = O(2^N) \]
Legend says that, at the beginning of time, priests were given a puzzle with 64 golden disks. Once they finish moving all the disks according to the rules, the world is said to end.

If the priests move the disks at a rate of 1 disk/second, how long will it take to solve the puzzle?

\[ T^N = 2^N - 1 = O(2^N) \]

\[ 2^{64} - 1 = 18,446,744,073,709,551,615 \text{ seconds} \]
\[ = 307,445,734,561,825,860 \text{ minutes} \]
\[ = 213,503,982,334,601 \text{ days} \]
\[ = 584,942,417,355 \text{ years} \]
Contents

1. Sequences and Series

2. Proofs
Types of Proofs

• Proof by Induction
• Proof by Contradiction
• Proof by Counterexample
Proofs by Induction

• We are proving a theorem $T$. (“this property holds for all cases.”).

• Step 1: Base case. We know that $T$ holds (trivially) for some small value.

• Step 2: Inductive step:
  • Inductive Hypothesis: Assume $T$ holds for all cases up to some limit $k$.
  • Show that $T$ also holds for $k+1$.

• This proves that $T$ holds for any $k$. 
Proof by Induction - Example

• For the Fibonacci numbers, we prove that

\[ F_i \leq (5/3)^i \text{ for any } i \geq 1 \]

• Base case:

\[ F_1 = F_2 = 1 < 5/3 \]

• Inductive step:

• Assume the theorem holds for \( i = 1, 2, \ldots, k \)
• We need to show that

\[ F_{k+1} < (5/3)^{k+1} \]
Proof by Induction - Inductive Step

- We know that $F_{k+1} = F_k + F_{k-1}$ and by the inductive hypothesis:

\[
F_{k+1} < (\frac{5}{3})^k + (\frac{5}{3})^{k-1}
\]

\[
< (\frac{3}{5})(\frac{5}{3})^{k+1} + (\frac{3}{5})^2(\frac{5}{3})^{k+1}
\]

\[
< (\frac{3}{5})(\frac{5}{3})^{k+1} + (\frac{9}{25})(\frac{5}{3})^{k+1}
\]

\[
< (\frac{3}{5} + \frac{9}{25})(\frac{5}{3})^{k+1}
\]

\[
< (\frac{24}{25})(\frac{5}{3})^{k+1}
\]

\[
< (\frac{5}{3})^{k+1}
\]

\[\square\]
Proof by Counter-Example

• We are proving that theorem $T$ is false. (“this property does not hold for all cases.”).

• It is sufficient to show that there is a case for which $T$ does not hold.

Example:

• Show that $F_i \leq i^2$ is false.

• There are $i$ for which $F_i > i^2$, e.g. $F_{13} = 233 > 13^2$
Proof by Contradiction

• We want to proof that $T$ is true.

• Step 1: Assume $T$ is false.

• Step 2: Show that this assumption leads to a contradiction.
Proof by Contradiction - Example

• We want to proof that there is an infinite number of primes.

• Assume the number of primes is finite and the largest prime it \( P_k \).

• Let the sequence of all primes be \( P_1, P_2, \ldots, P_k \)

\[
N = P_1 P_2 \cdots P_k + 1
\]

• Since \( N > P_k \), \( N \) cannot be prime, so it must have a factorization into primes. Such a factorization cannot exist: dividing \( N \) by any \( P_1, P_2, \ldots, P_k \) will always leave a remainder of 1.