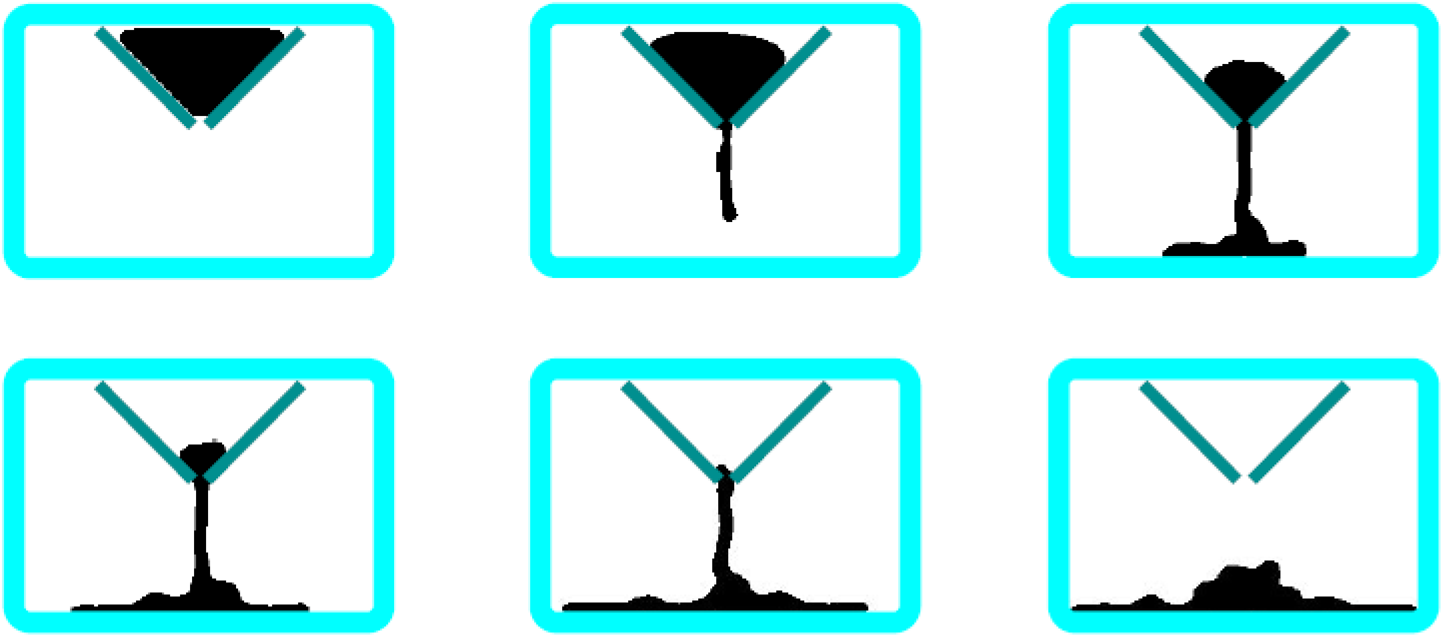
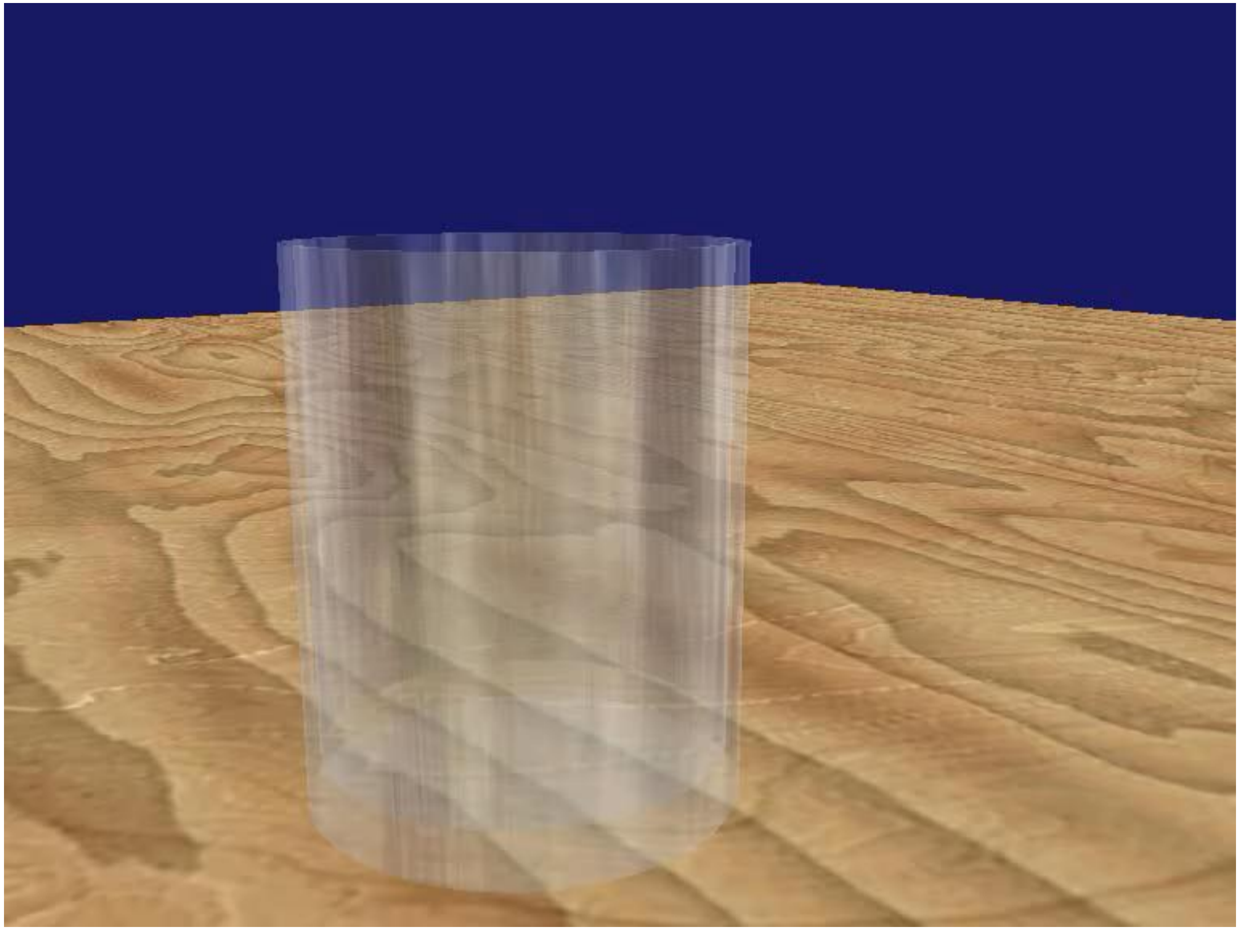
Particle-Based Fluids

COMS6998 – Problem Solving for Physical Simulation

SPH - 1996



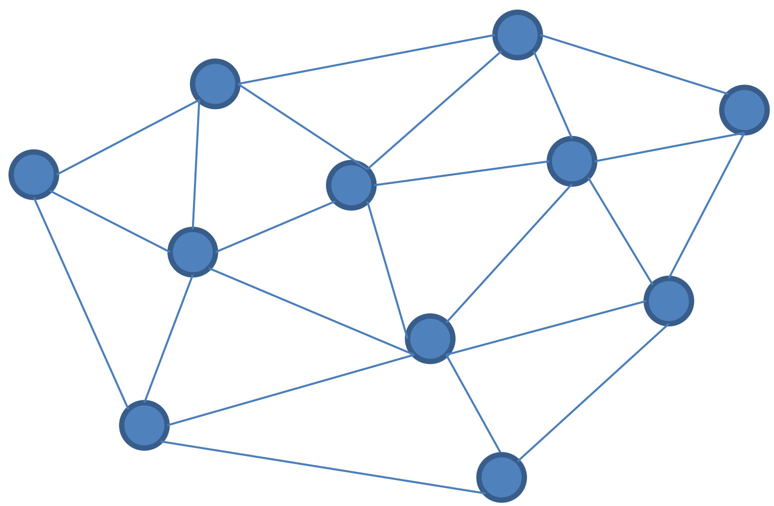
SPH – 2003



SPH – 2010

From mass-springs to fluids

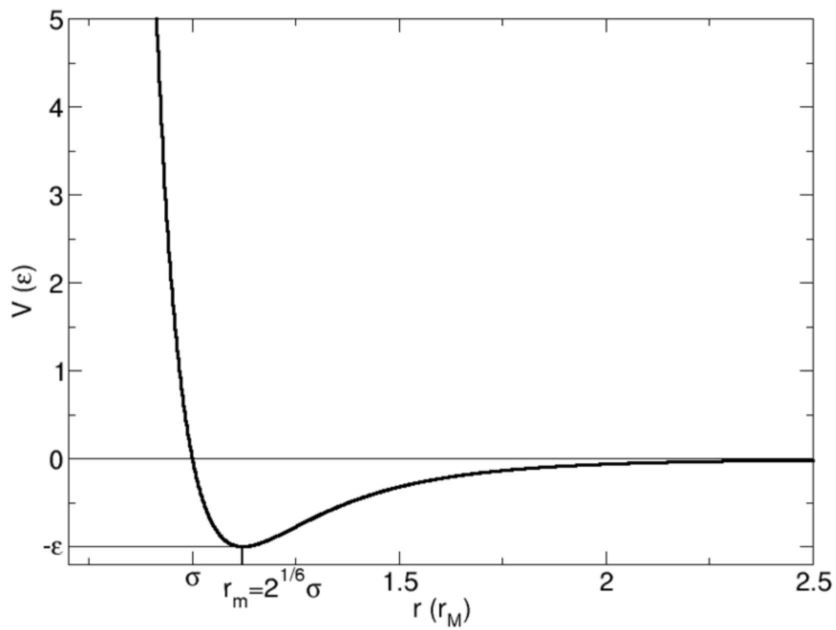
How might we move from a mass-spring model towards simulating fluids?



Lennard-Jones potentials

A method from molecular dynamics:

- Forces depend on inter-particle distances



Navier-Stokes equations

Instead of starting from a collection of discrete liquid molecules, consider fluid as a continuum.

Navier-Stokes equations are basically “F=ma” for continuum fluids.

F=ma

For a small parcel of fluid:

- Mass = density(ρ) x volume
- Acceleration = rate of change of velocity, $\frac{D\boldsymbol{u}}{Dt}$

Material Derivative

Per unit volume:

$\rho\frac{D\boldsymbol{u}}{Dt}$ = net force on fluid

Material Derivative

Consider how a quantity (eg. temp.) at a fixed point in a body of fluid can change:

1) Forces local to the point may cause changes, independent of the flow’s velocity.
   - eg. heating will increase temperature
2) The quantity has a higher or lower value upstream, so the point’s value changes as the flow goes past.

Material Derivative

$$\frac{D\varphi}{Dt} = \overset{(1)}{\frac{\partial\varphi}{\partial t}} + \overset{(2)}{u \cdot \nabla\varphi}$$

$\boxed{\text{Advection}}$

In SPH, quantities are stored on moving particles, that travel with the flow.

– advection is effectively automatic.

Navier-Stokes equations

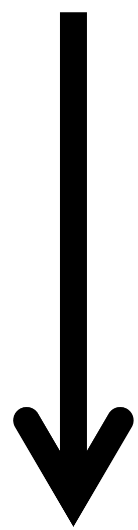
Forces on fluid:

- Pressure
- Viscosity
- Gravity
- Surface tension
- Any other external forces…

Gravity

Simple: Just a constant downward force.

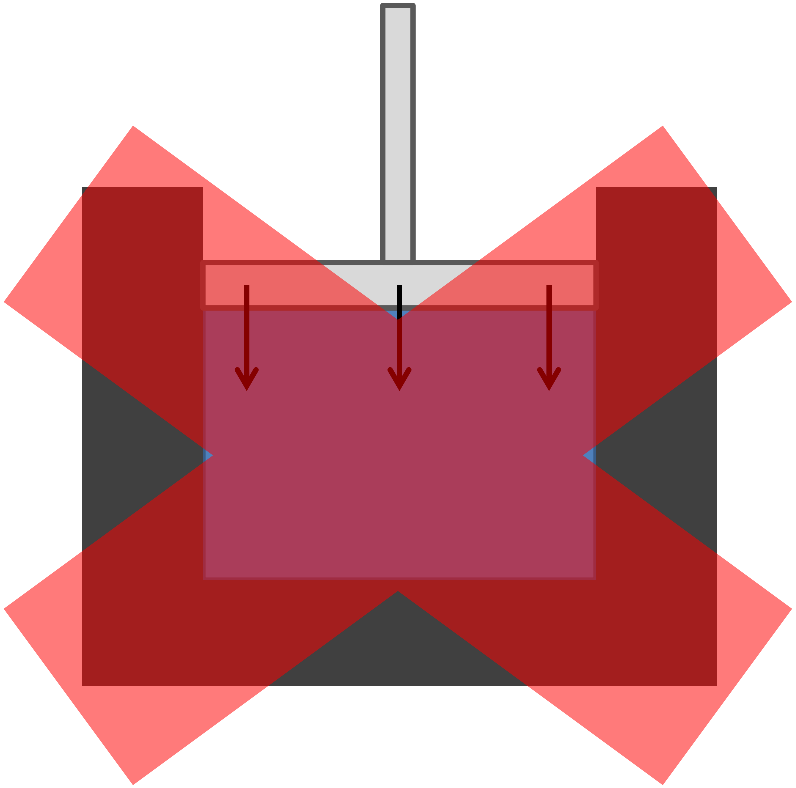
(At least on Earth, at scales we typically care about...)



$$f_{gravity} = \rho g$$

Pressure

Most fluids we encounter are effectively incompressible – they resist volume change.



Pressure

Gives a constraint, $\nabla \cdot \boldsymbol{u} = 0$.

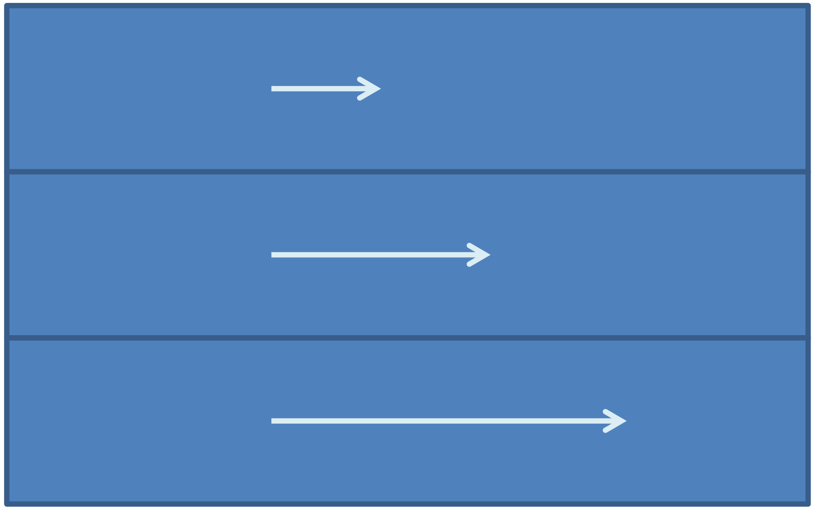
Pressure is the force that opposes compression (and expansion).

Pressure differences yield changes in velocity, so force is the *gradient* of pressure:

$$f_{pressure} = -\nabla p$$

Viscosity

Loss of energy due to internal friction.



Molecules diffuse between layers of fluid, thereby equalizing the velocity.

Viscosity

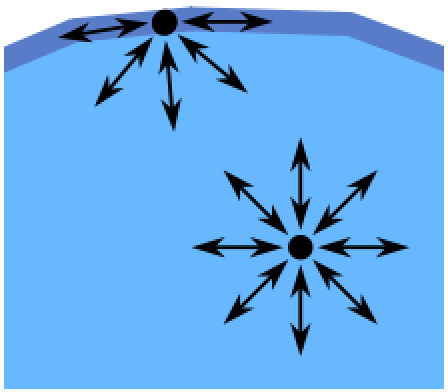
One way to model smoothing (and diffusion) is with a Laplace operator, Δ:

$$f_{viscosity} = \mu \Delta \boldsymbol{u} = \mu \nabla \cdot \nabla \boldsymbol{u}$$

…where μ is the coefficient of viscosity.

Surface Tension

Cohesion between molecules of liquid causes a force imbalance near the surface.



Net force is tangential, ie. a tension on the surface.

Surface Tension

Often approximated by a normal force, proportional to surface curvature:

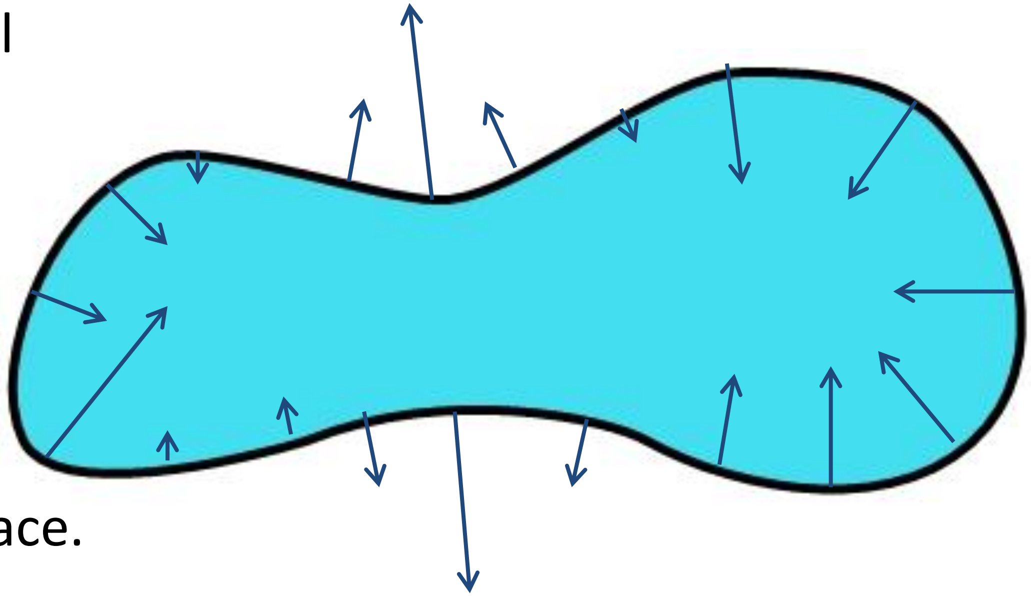
$$f_{st} = \gamma \kappa \boldsymbol{n}$$

where

γ: surface tension coefficient

κ: surface curvature

$\boldsymbol{n}$: surface normal



*Only applies AT the surface.

Navier-Stokes

Putting it all together…

Momentum equation:

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\nabla p + \mu \nabla \cdot \nabla \boldsymbol{u} + \gamma \kappa \boldsymbol{n} + \rho \boldsymbol{g} + f_{other}$$

Continuity equation:

$$\nabla \cdot \boldsymbol{u} = 0$$

SPH

Smoothed Particle Hydrodynamics (SPH):

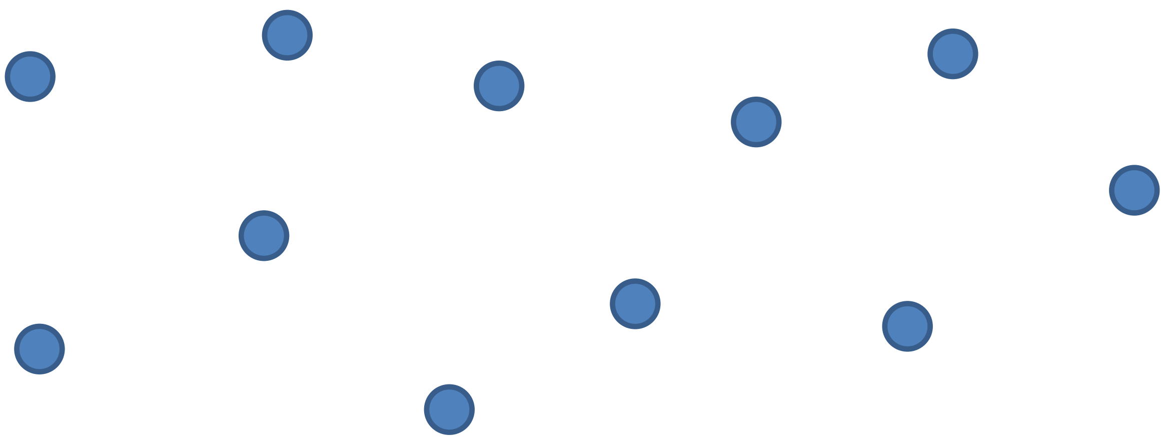
- approximate N-S equations on a set of moving fluid particles.

I will mostly follow the paper:

"**Particle-Based Fluid Simulation for Interactive Applications**" [Müller et al 2003]

SPH

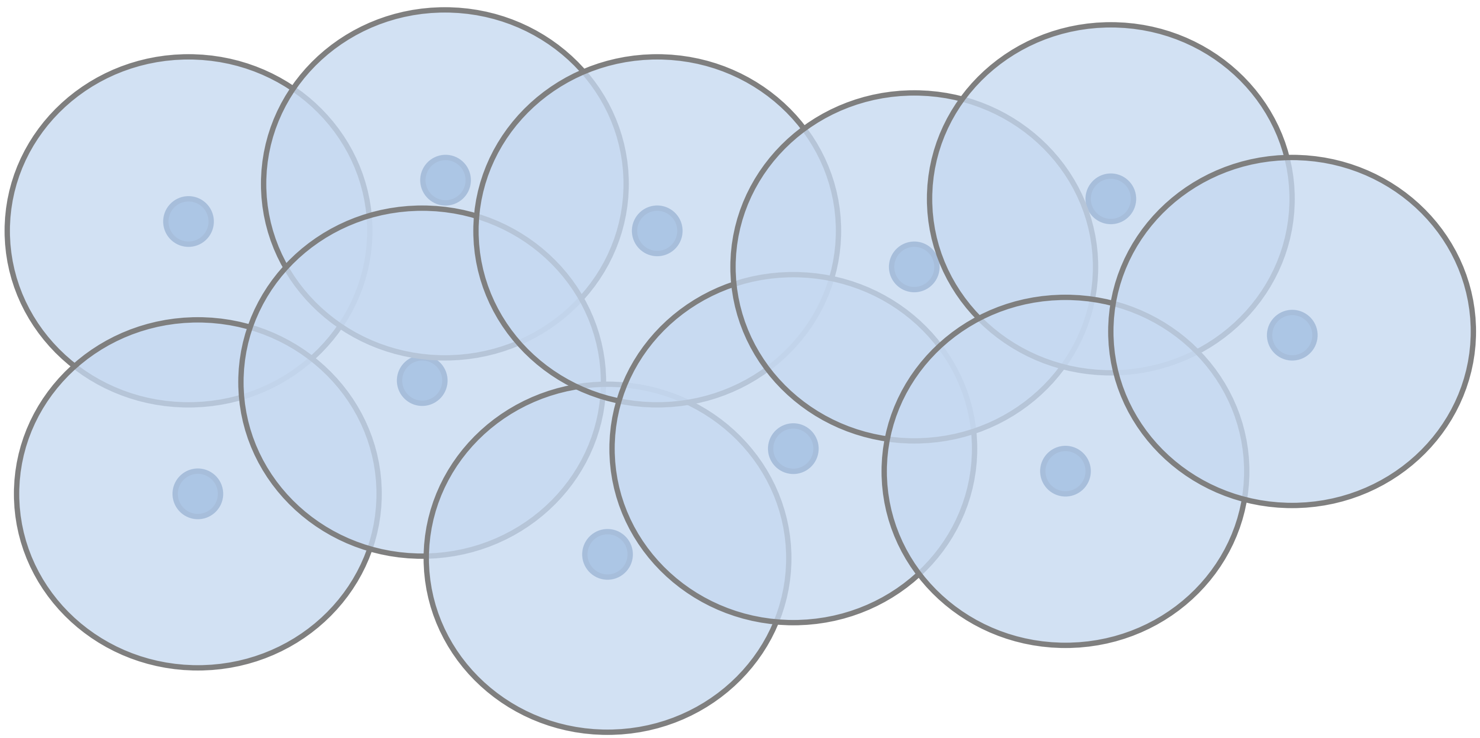
Particles have mass, velocity, & position.



To approximate a continuous fluid, we need information everywhere, not just points.

SPH

So "smooth" particle information over an area.

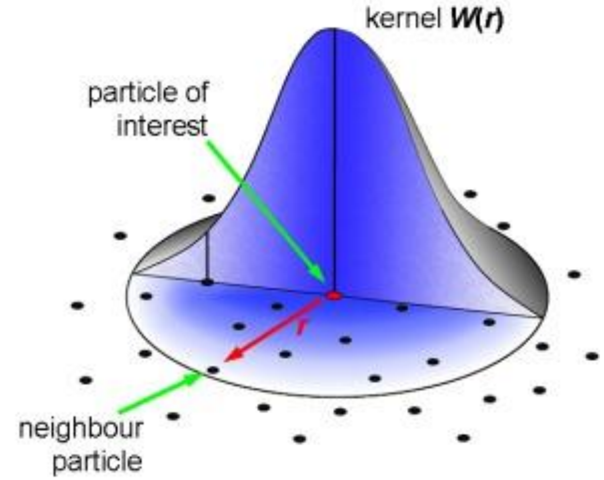


The value at a point in space is a weighted sum of values from nearby particles.

Smoothing kernels

A smoothing kernel is a weighting function *W* that spreads out the data.

$$A_S(\mathbf{r}) = \sum_j m_j \frac{A_j}{\rho_j} W(\mathbf{r} - \mathbf{r}_j, h)$$



Kernel Properties

- Symmetric
- Finite support
- Normalized (Integrates to 1)
- Smooth
- As smoothing radius -> 0, *W* approximates a delta function

Derivatives

What about derivatives of fluid quantities?

$$\nabla A_S(\mathbf{r}) = \nabla \sum_j m_j \frac{A_j}{\rho_j} W(\mathbf{r} - \mathbf{r}_j, h)$$

Product rule…

$$\nabla A_S(\mathbf{r}) = \sum_j \cancel{\nabla \left(m_j \frac{A_j}{\rho_j} \right) W(\mathbf{r} - \mathbf{r}_j, h)} + m_j \frac{A_j}{\rho_j} \nabla W(\mathbf{r} - \mathbf{r}_j, h)$$

First term is zero…

$$\nabla A_S(\mathbf{r}) = \sum_j m_j \frac{A_j}{\rho_j} \nabla W(\mathbf{r} - \mathbf{r}_j, h)$$

Discretizing Fluids

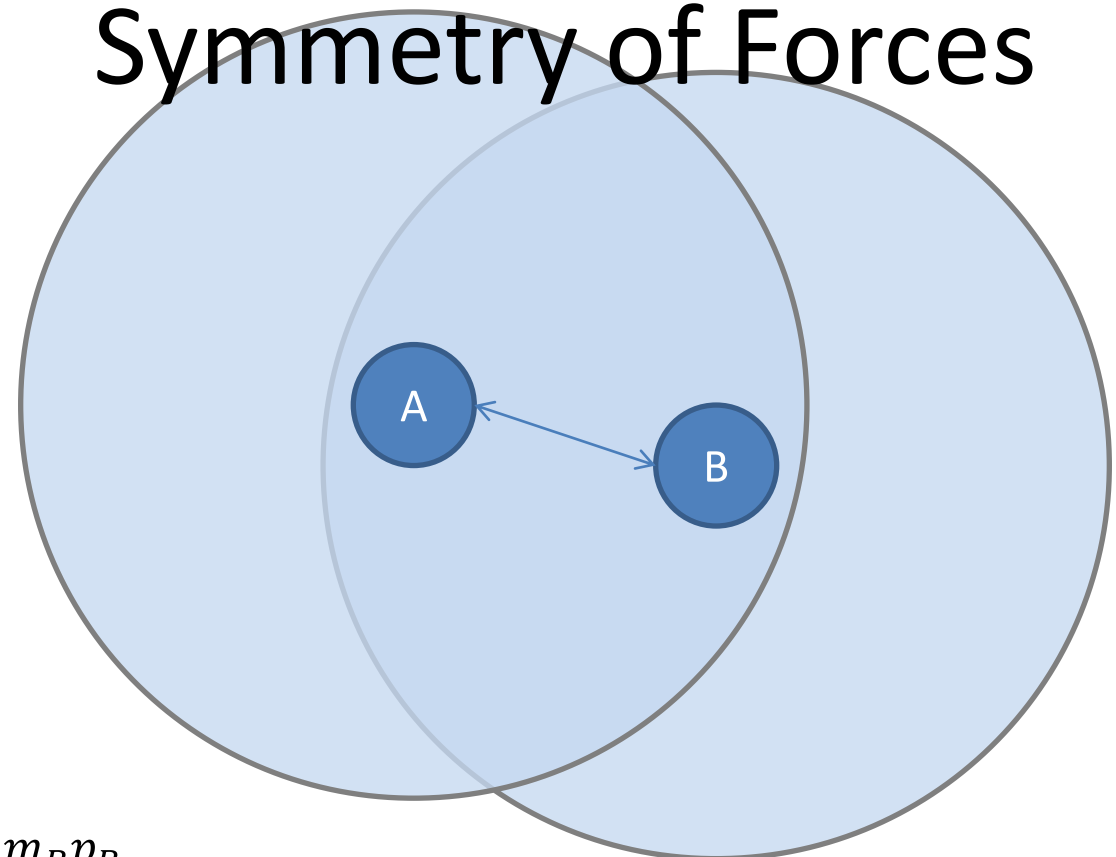
Now to apply Navier-Stokes forces to particles…

Pressure force on particle *i:*

$$\mathbf{f}_i^{\text{pressure}} = -\nabla p(\mathbf{r}_i) = -\sum_j m_j \frac{p_j}{\rho_j} \nabla W(\mathbf{r}_i - \mathbf{r}_j, h)$$

Problem: *Not symmetric!* (ie. action ≠ reaction)

Symmetry of Forces



$$f_A = 0 - \frac{m_B p_B}{\rho_B} \nabla W(r_A - r_B, h)$$

$$f_A \neq -f_B$$

$$f_B = -\frac{m_A p_A}{\rho_A} \nabla W(r_B - r_A, h) + 0$$

Discretization

One way to symmetrize it:

$$\mathbf{f}_i^{\text{pressure}} = -\sum_j m_j \frac{p_i + p_j}{2\rho_j} \nabla W(\mathbf{r}_i - \mathbf{r}_j, h)$$

Next issue: How to determine pressures?

Pressure

Enforcing perfect incompressibility is hard, so SPH assumes mild compressibility.

To compute pressure:

- Allow small particle density fluctuations
- Estimate fluid density from particles
- Use an *equation of state (EOS),* ie. a relationship between pressure and density.
    - ($\approx$ a spring equation for fluids)

Equations of State

Ideal gas law :

[Muller et al 2003]

$$p = k(\rho - \rho_0)$$

Tait equation:

[Becker & Teschner 2007]

$$p = B\left(\left(\frac{\rho}{\rho_0}\right)^{\gamma} - 1\right)$$

- Where…
  - k, B = (tuneable) constants
  - ρ_0= rest density
  - γ = 7

Comparison

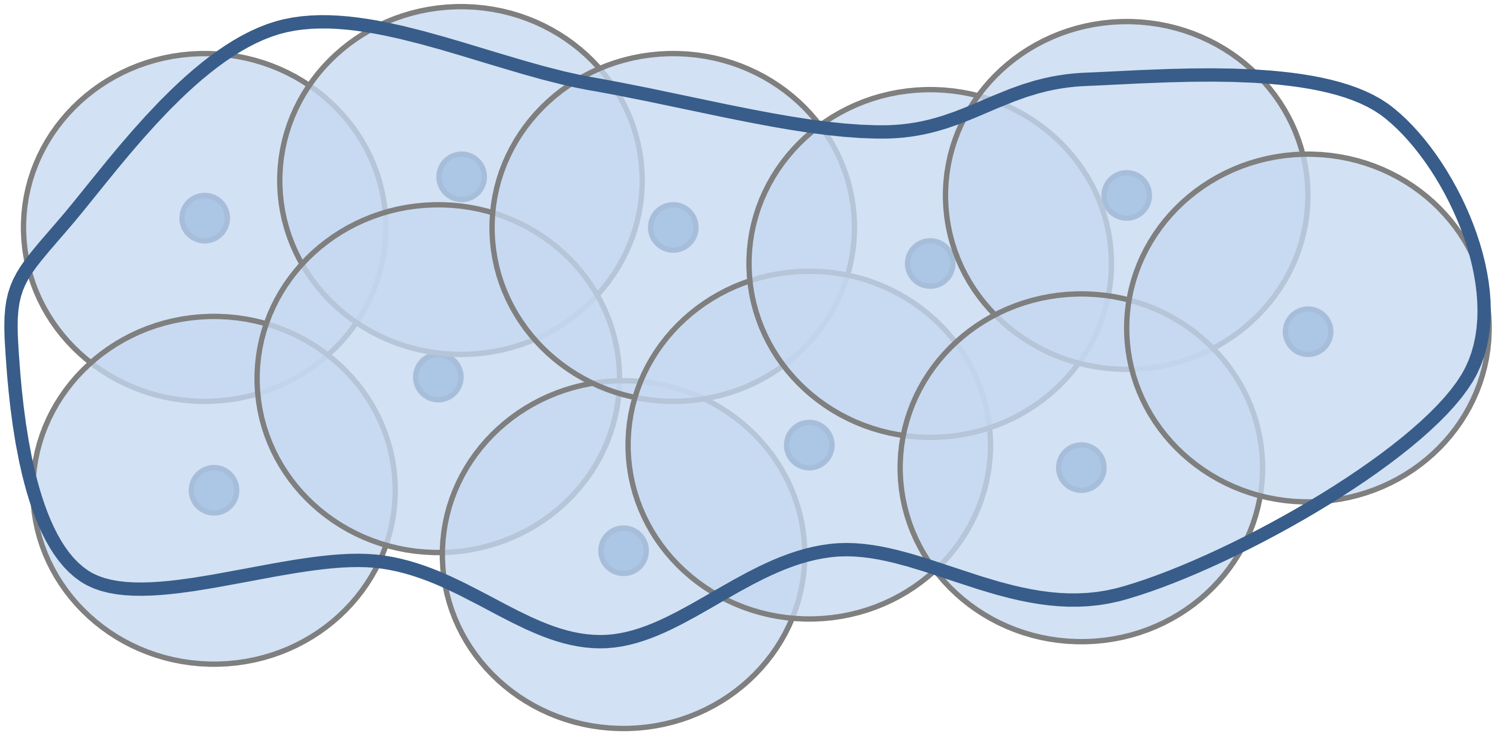

Weakly compressible
SPH for
free surface flows
Markus Becker Matthias Teschner
University of Freiburg

Other forces…

- Gravity:
  - just add ρg to particle's vertical velocity

- Viscosity:
  - Approximate $\mu \Delta \boldsymbol{u}$ with smoothing kernels
  - Symmetrize (similar to pressure)

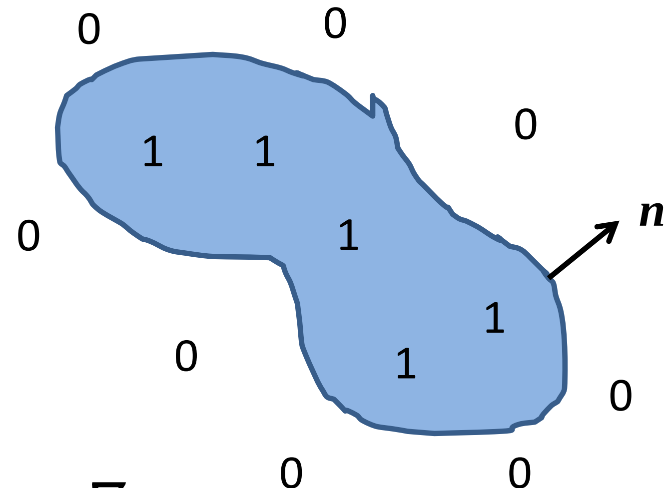
Surface Tension



- Need to estimate surface curvature κ.
  - Where is the surface?

Surface Tension

[Muller et al 2003]



Define a "color field":

$$c = \begin{cases} 1 & \text{at particles} \\ 0 & \text{outside fluid} \end{cases}$$

Estimate the normal: $\boldsymbol{n} = -\frac{\nabla c}{|\nabla c|}$

From there, curvature is: $\kappa = \nabla \cdot \boldsymbol{n}$

Only apply force where $\nabla c \neq 0$, since that implies we are near the surface.

Surface Tension

[Becker & Teschner 2007]

Take a molecularly inspired approach

- cohesion forces between all particles:

$$f^i = \frac{\gamma}{m_i} \sum_j m_j W(r_i - r_j, h)(r_i - r_j)$$

Summary

- SPH particles carry fluid data.
- Smoothing kernels provide continuous estimates of fluid quantities.
- Apply the Navier-Stokes equations to the smooth fields to determine forces, and update velocities.
- Particle positions can then be updated from the smooth velocity field.

Thoughts

What are some drawbacks of this method?

- Not truly incompressible.
- Explicit method, requires small timesteps.
- Surface representation often appears blobby.
- Tuning parameters is scene-dependent.
- Requires lots of neighbour-finding.

How about benefits?

- Reasonably intuitive, easy to code.
- Mass is never lost.
- Integrates well with existing particle systems/methods.
- Topology changes (merges/splits) are easy

Thoughts

- How might you go about coupling this fluid model to solids?
  - eg. rigid bodies, mass-spring deformables, cloth