Steiner Point Removal with Distortion $O(\log k)$

Arnold Filtser

Ben-Gurion University

April 26, 2018

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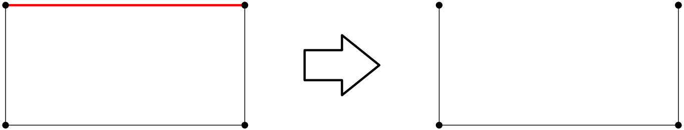

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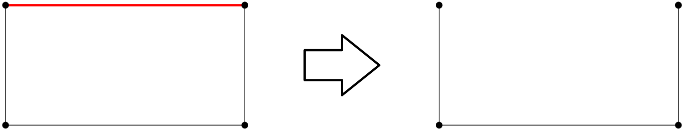

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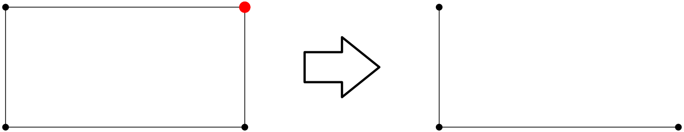

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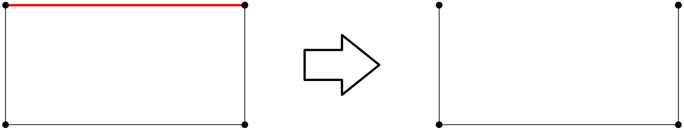

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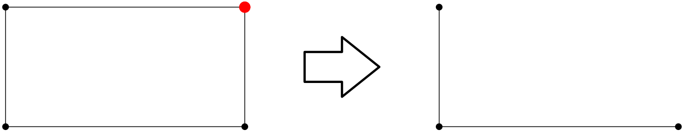
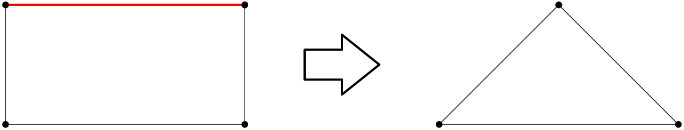


- Contracting edges.



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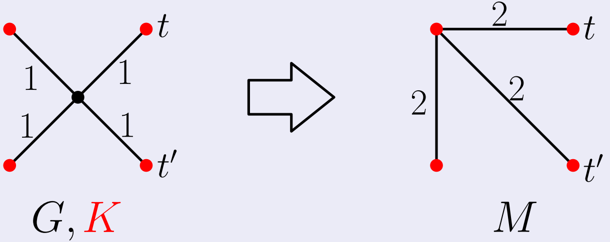

The distortion is: $\frac{d_M(t,t')}{d_G(t,t')} = \frac{4}{2} = 2$

Terminal Partitions and Induced Minor

Partition $\{V_1, \ldots, V_k\}$ of V is called a **terminal partition** if for all i,

- $t_i \in V_i$.
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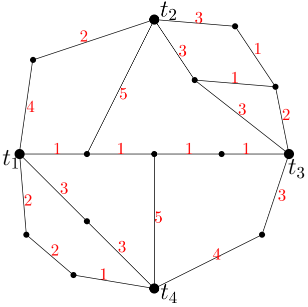

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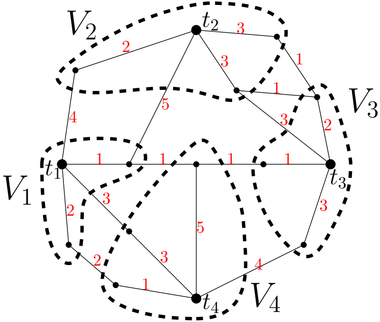

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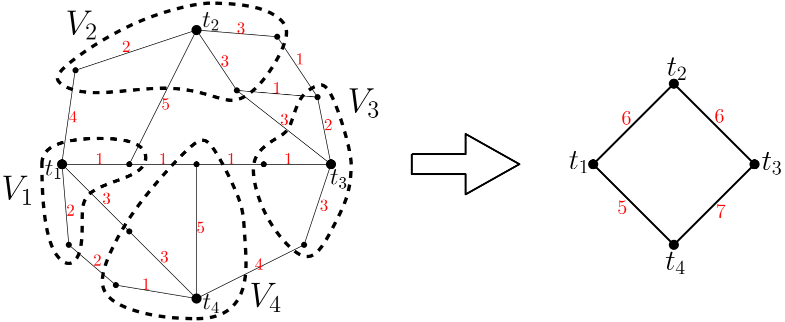

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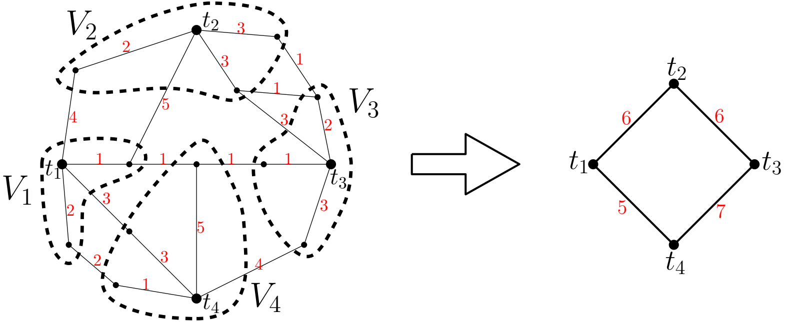

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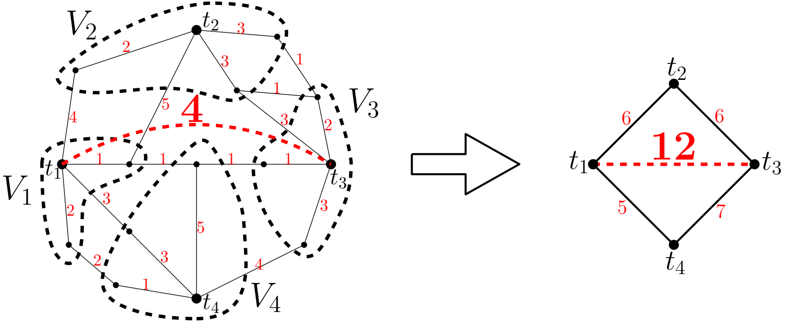

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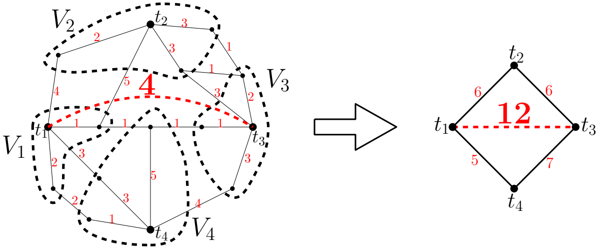

The distortion is: $\frac{d_M(t_1,t_3)}{d_G(t_1,t_3)} = \frac{12}{4} = 3.$

Induced Minor by Voronoi Cells

Natural candidate:

Let V_j be the **Voronoi cell** of t_j (breaking ties arbitrarily).

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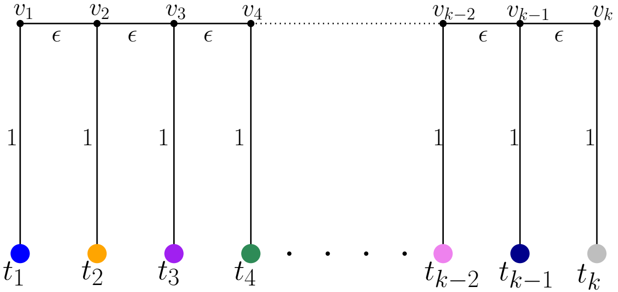

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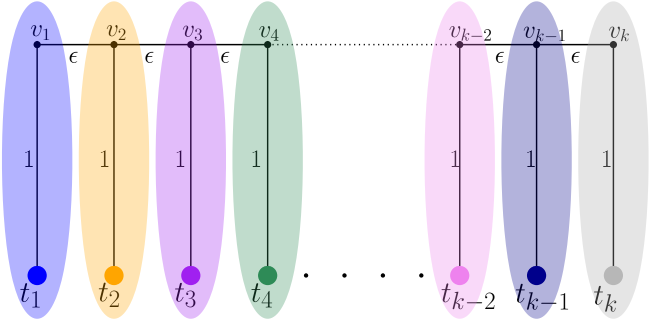

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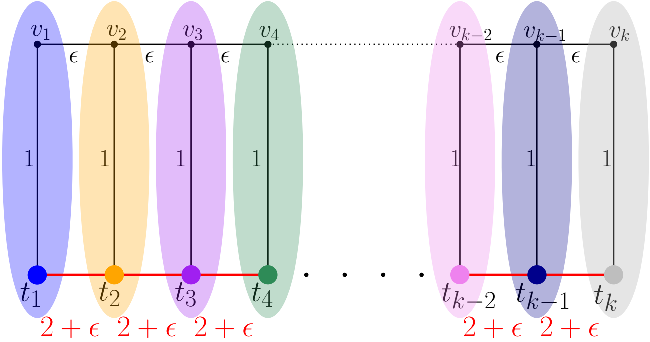

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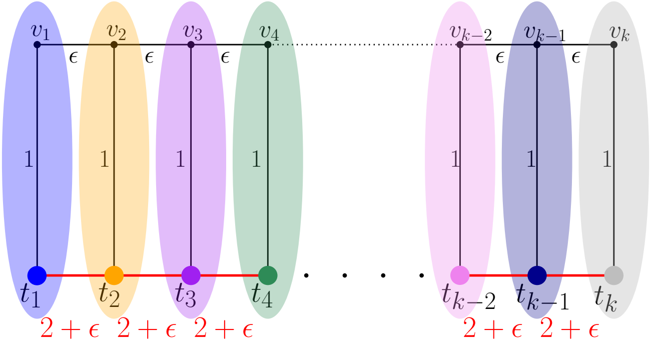

Distortion: $\frac{d_M(t_1,t_k)}{d_G(t_1,t_k)} = \frac{2+2(k-2)+(k-1)\epsilon}{2+(k-1)\epsilon} \xrightarrow[\epsilon \to 0]{} \boldsymbol{\Omega(k)}$.

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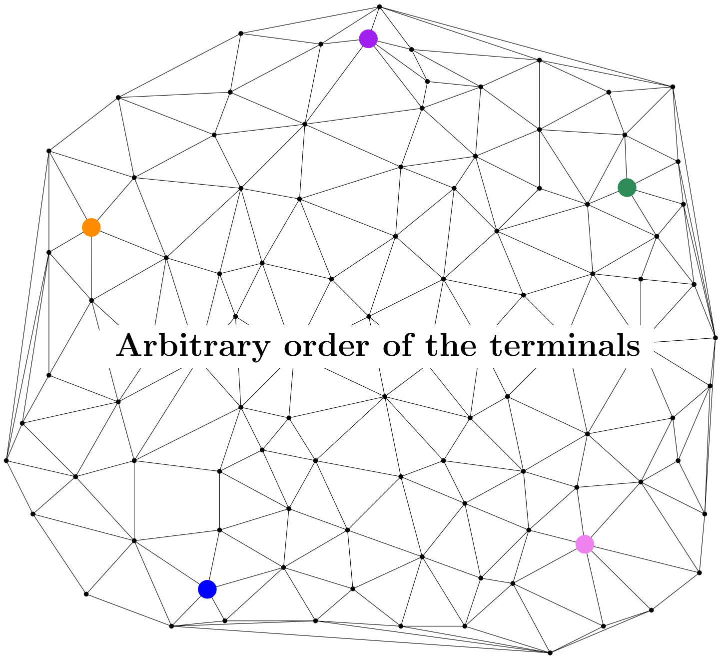
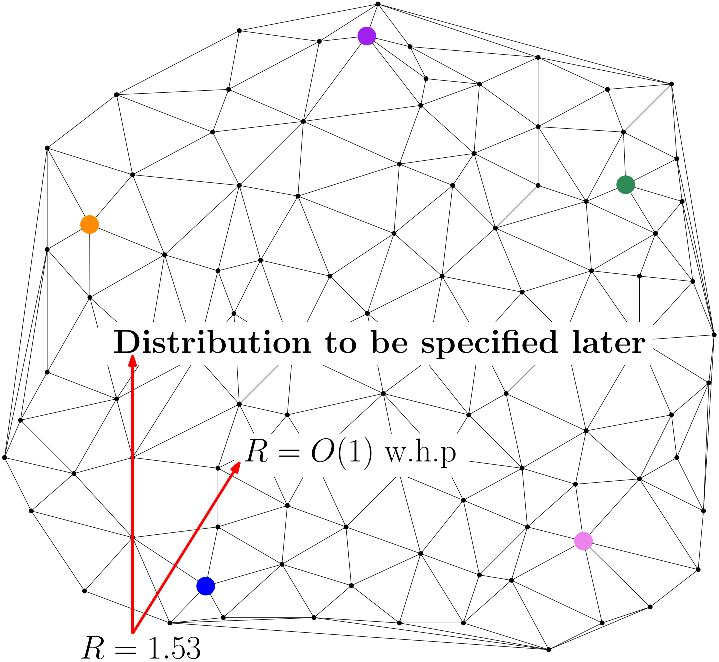

Arbitrary order of the terminals


Distribution to be specified later
$R = O(1)$ w.h.p
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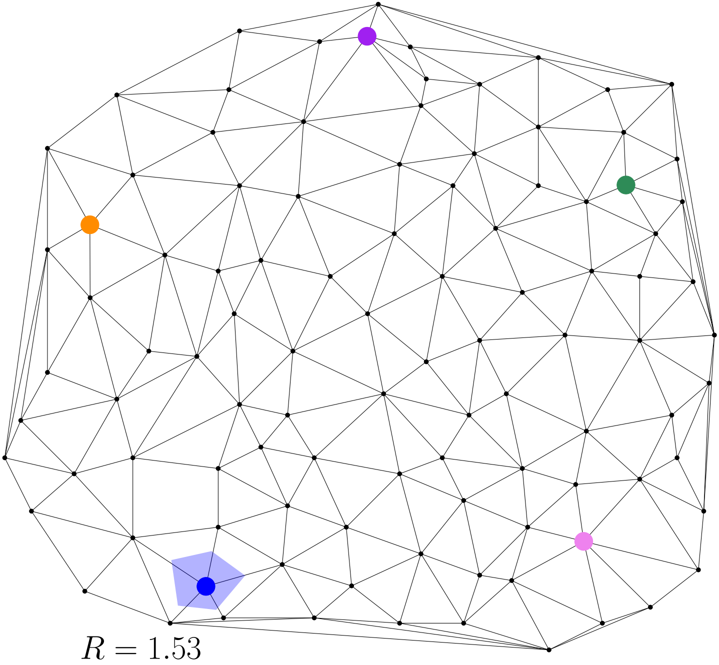
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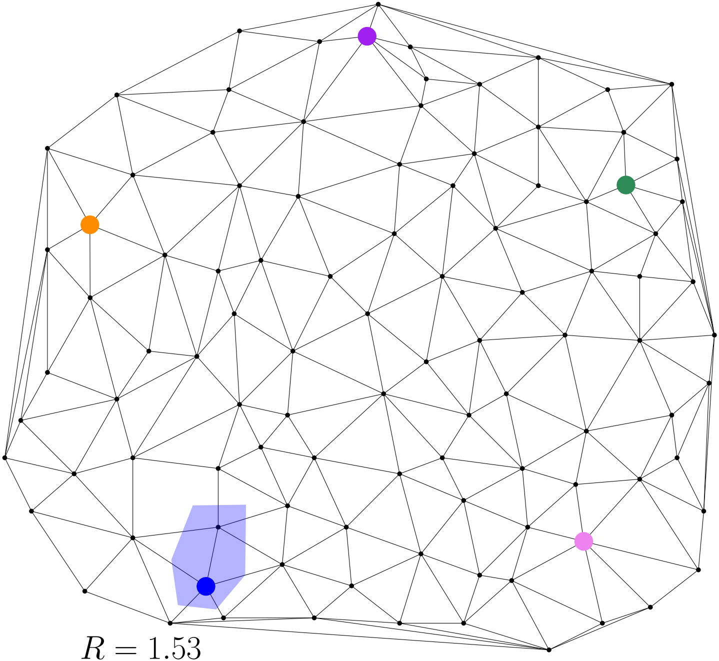

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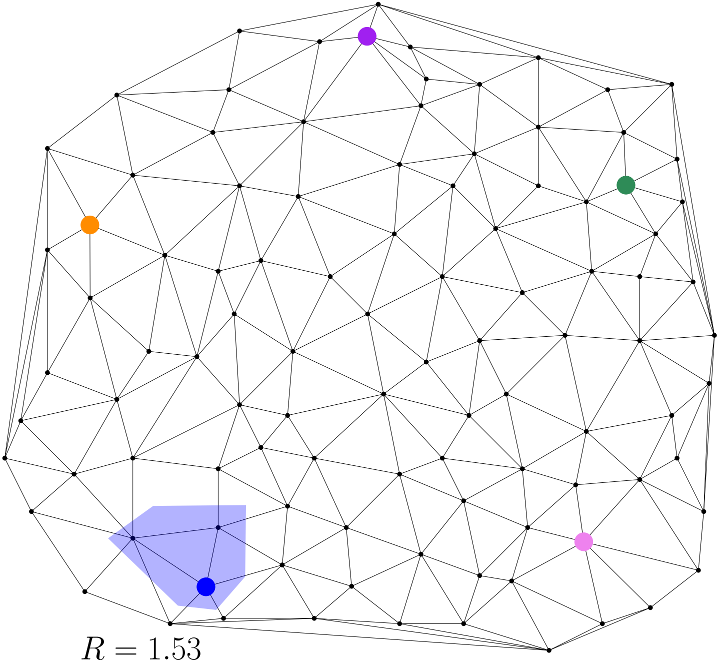
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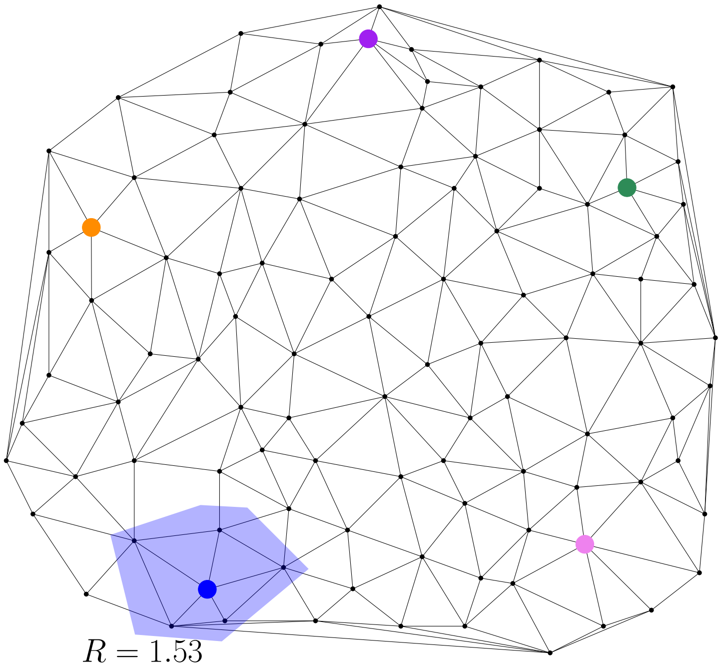
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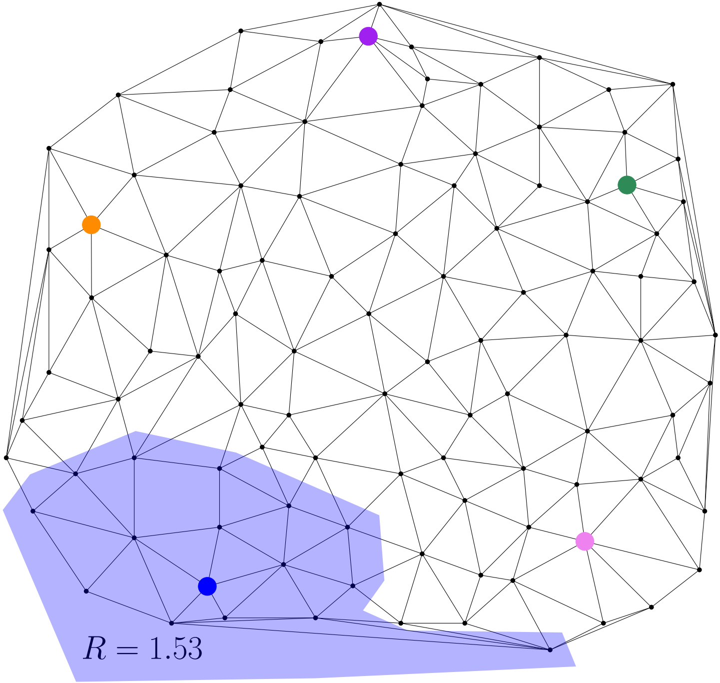
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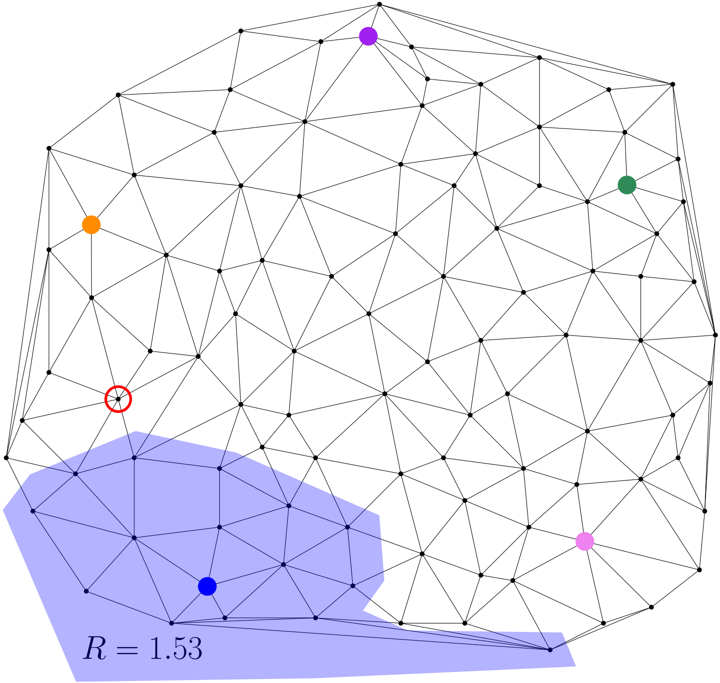
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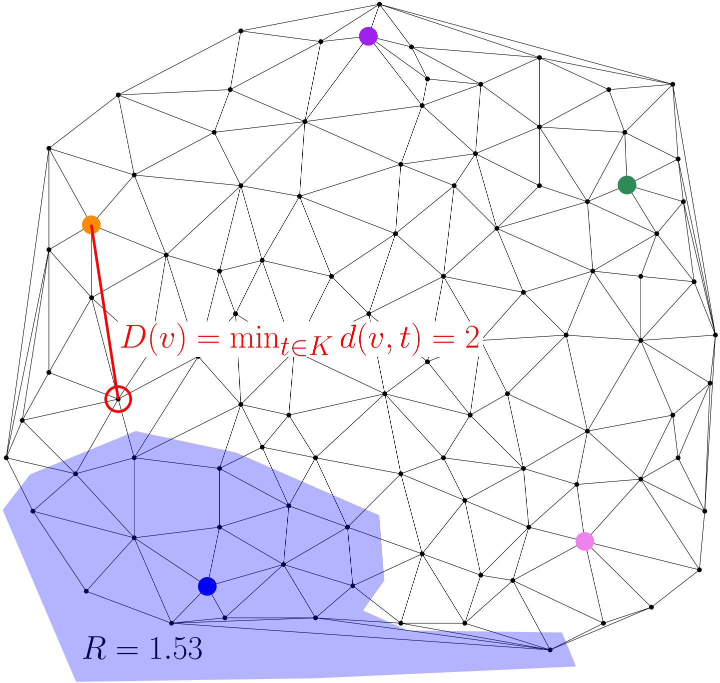

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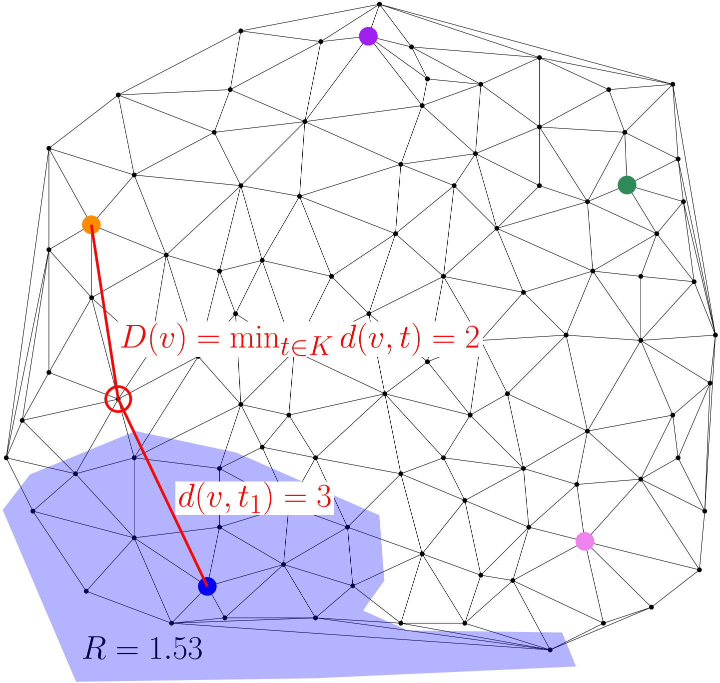
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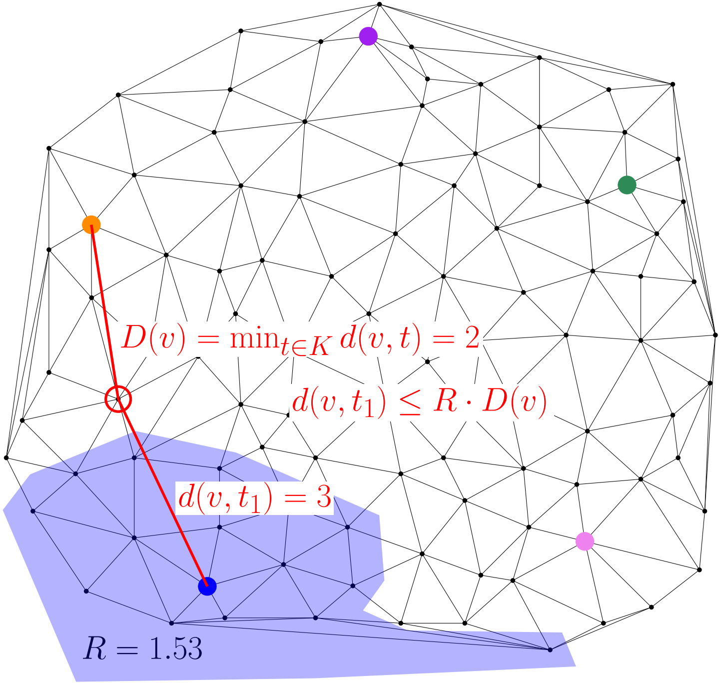

$D(v) = \min_{t \in K} d(v,t) = 2$
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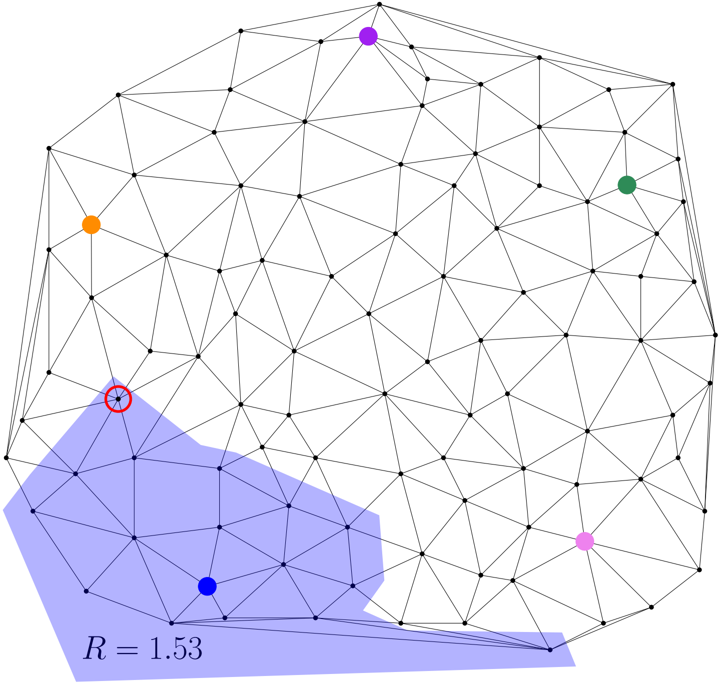
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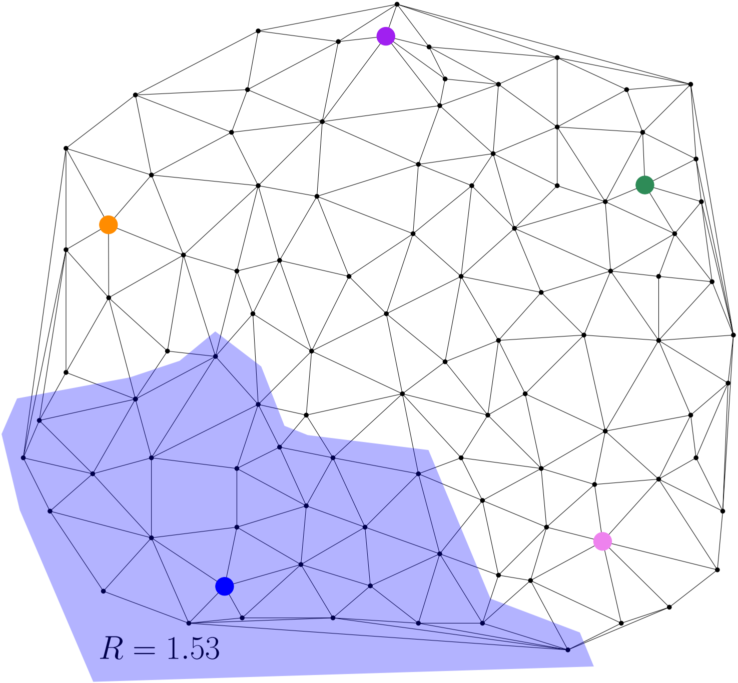
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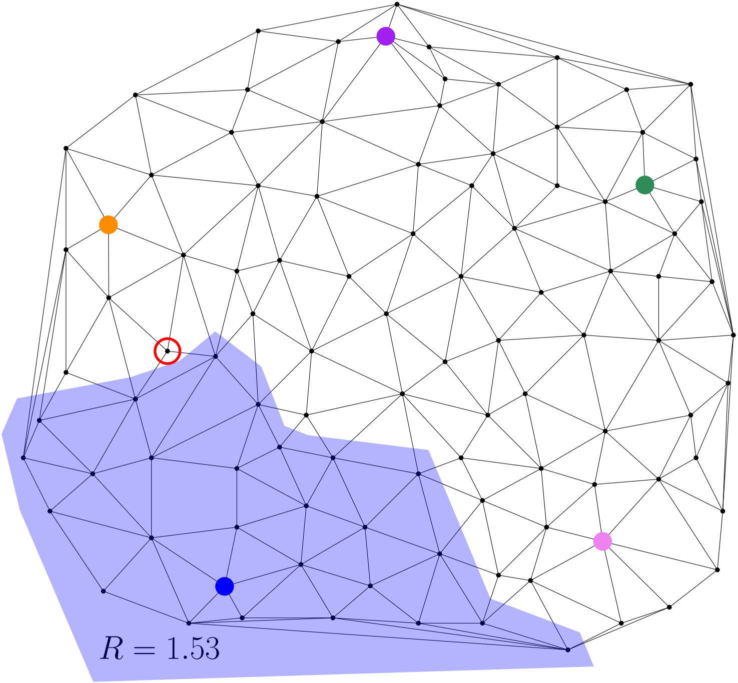
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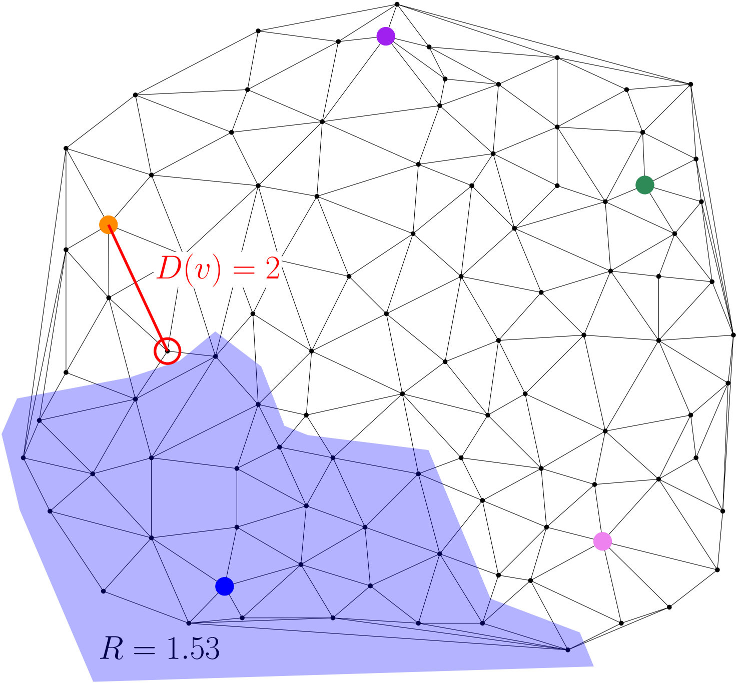
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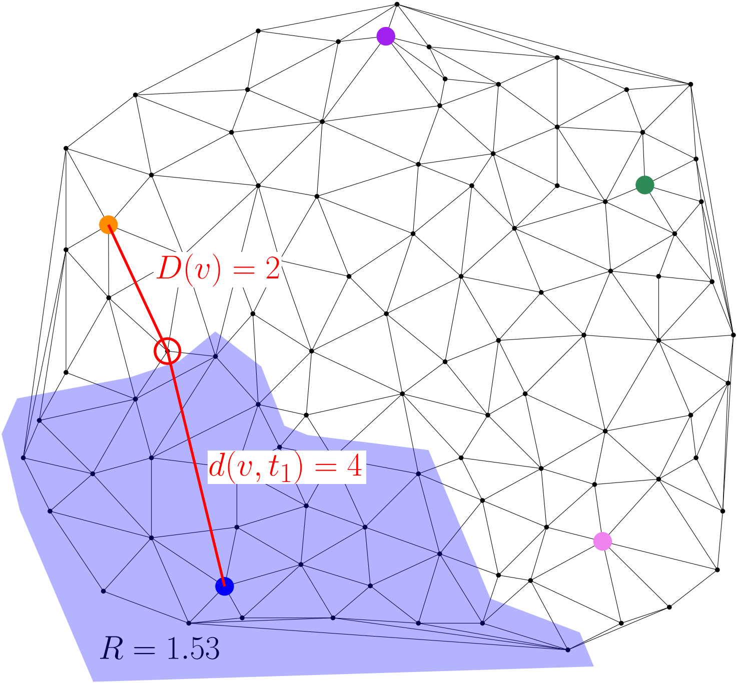
$D(v) = 2$
$d(v, t_1) = 4$
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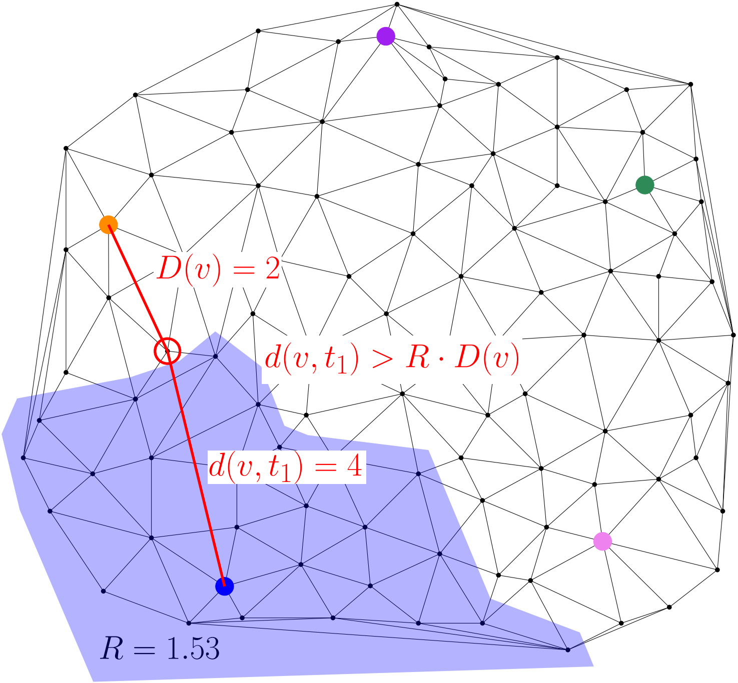
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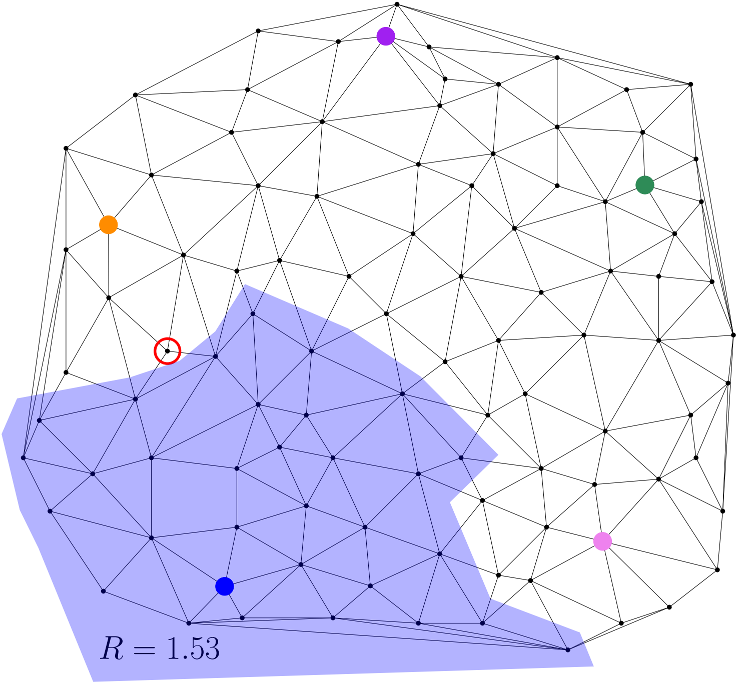
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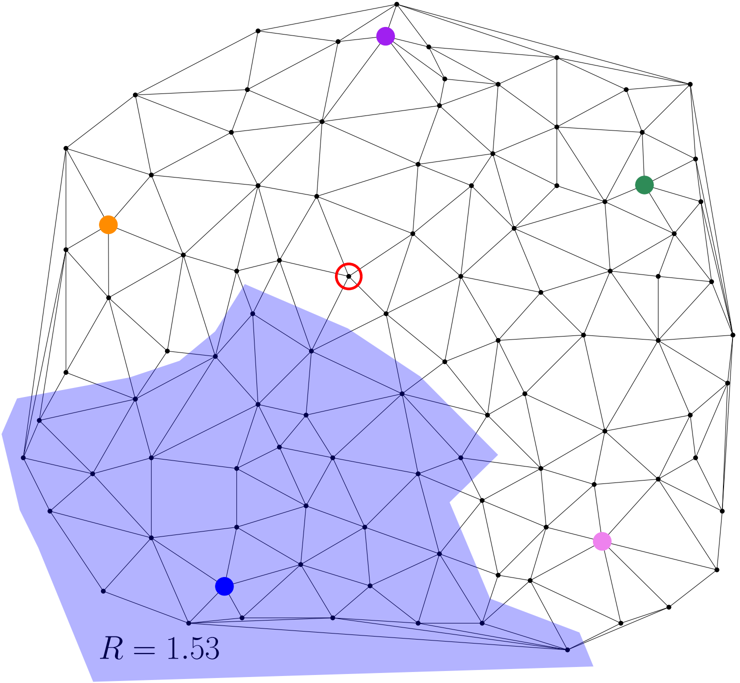
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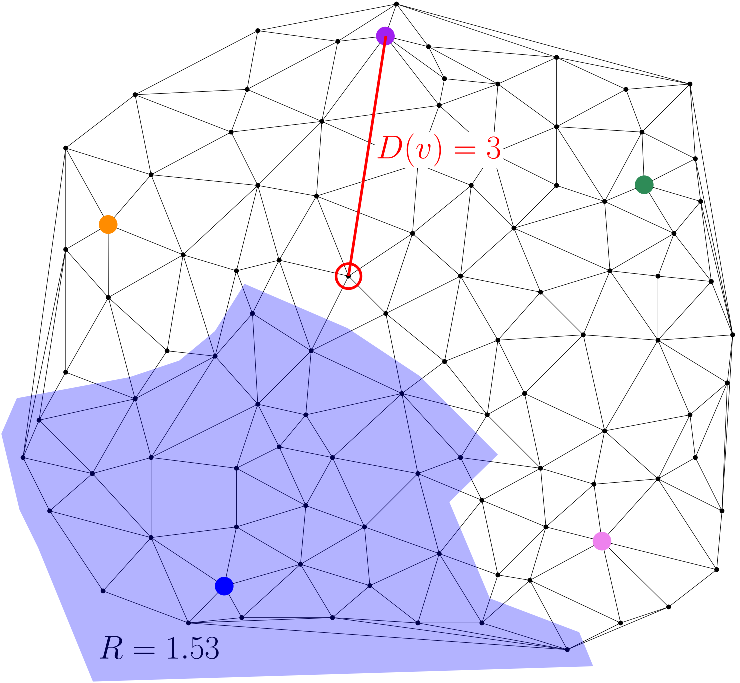
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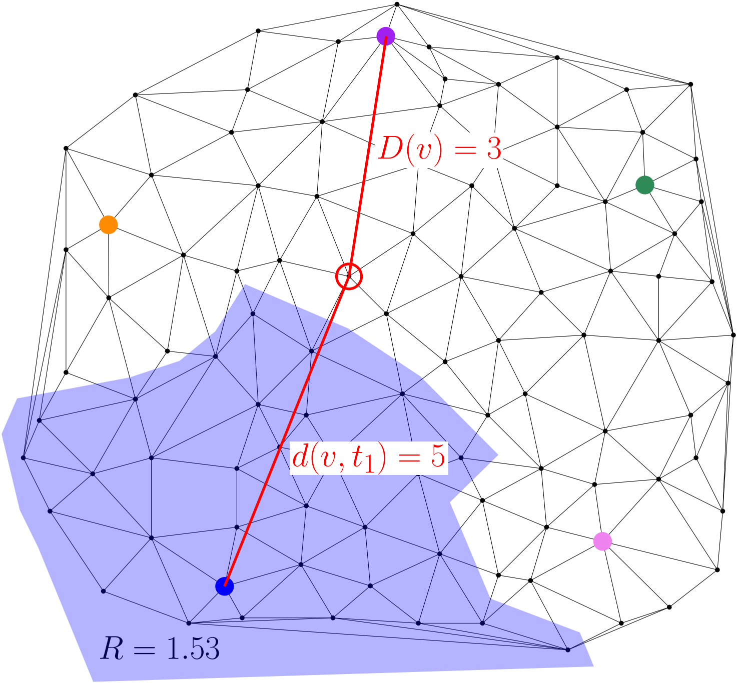
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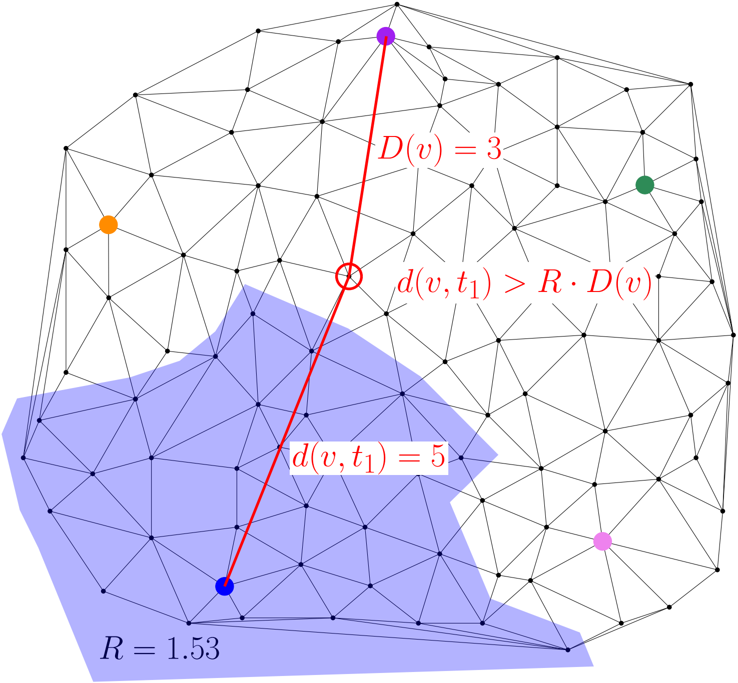

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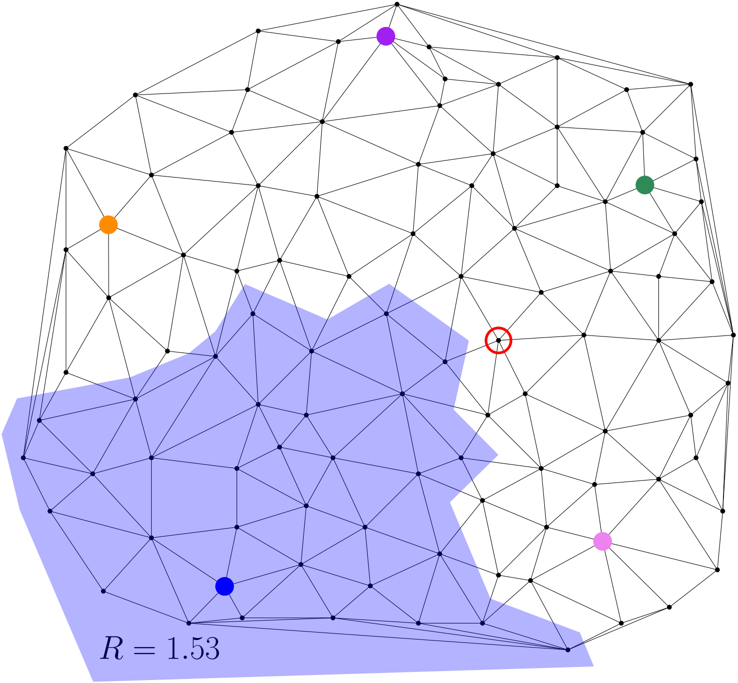
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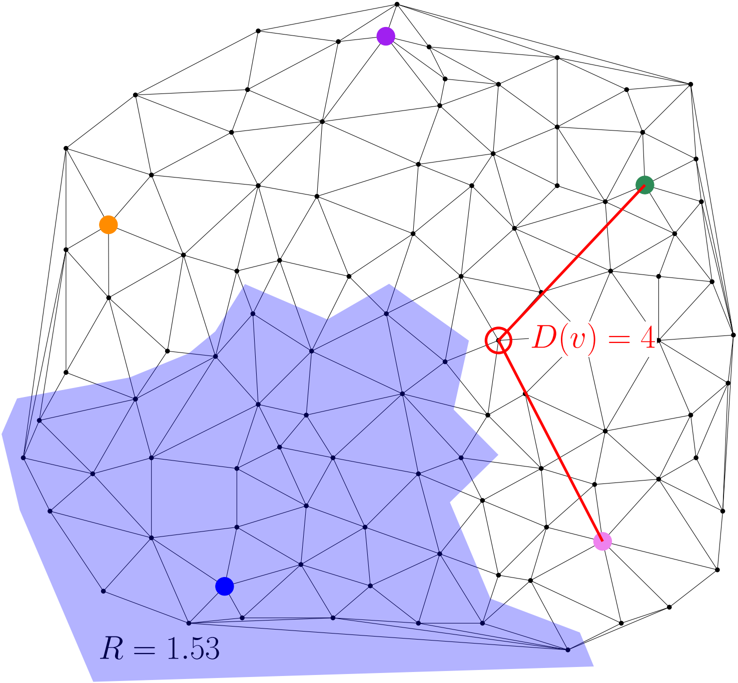

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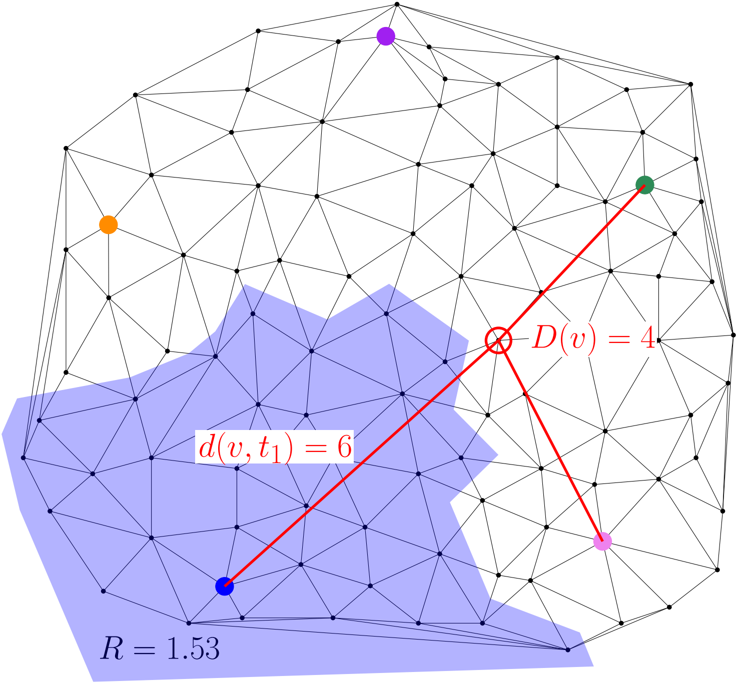

$D(v) = 4$
$d(v, t_1) = 6$
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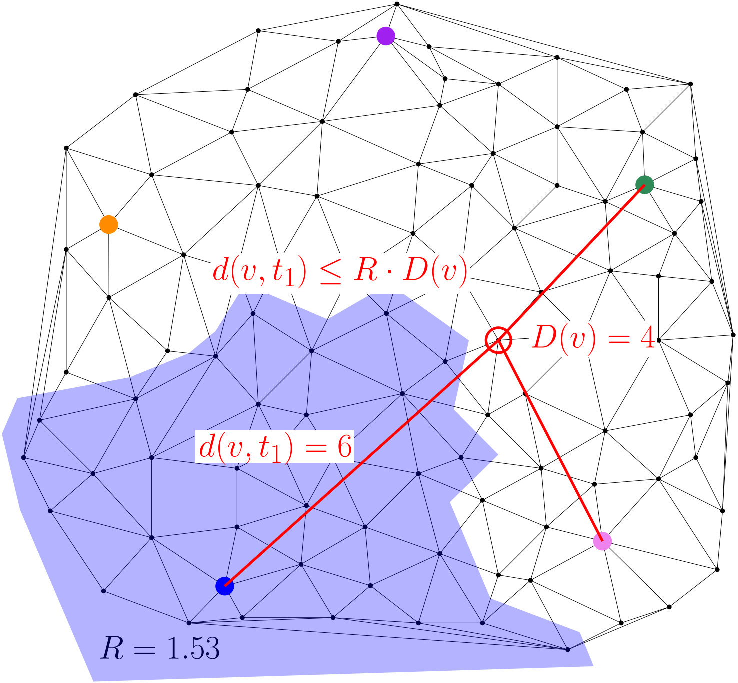

$d(v, t_1) \leq R \cdot D(v)$
$D(v) = 4$
$d(v, t_1) = 6$
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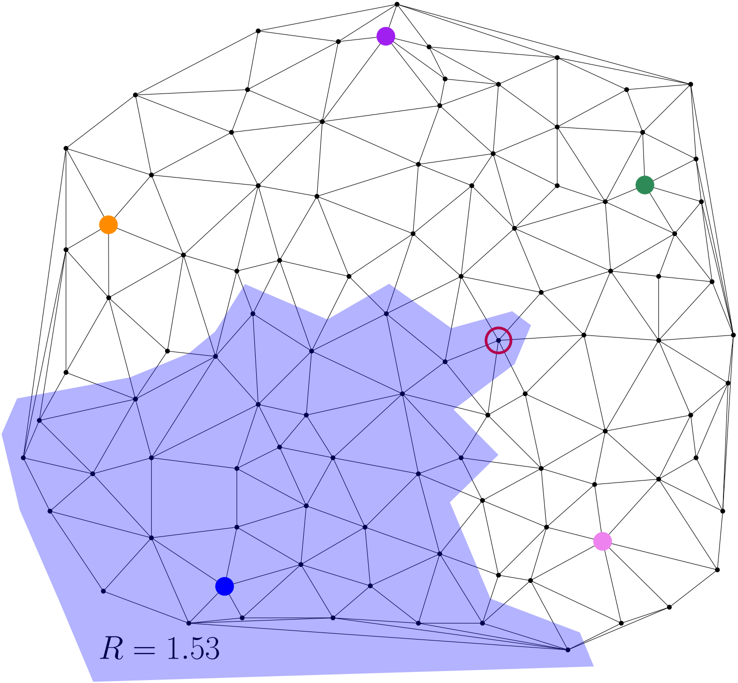
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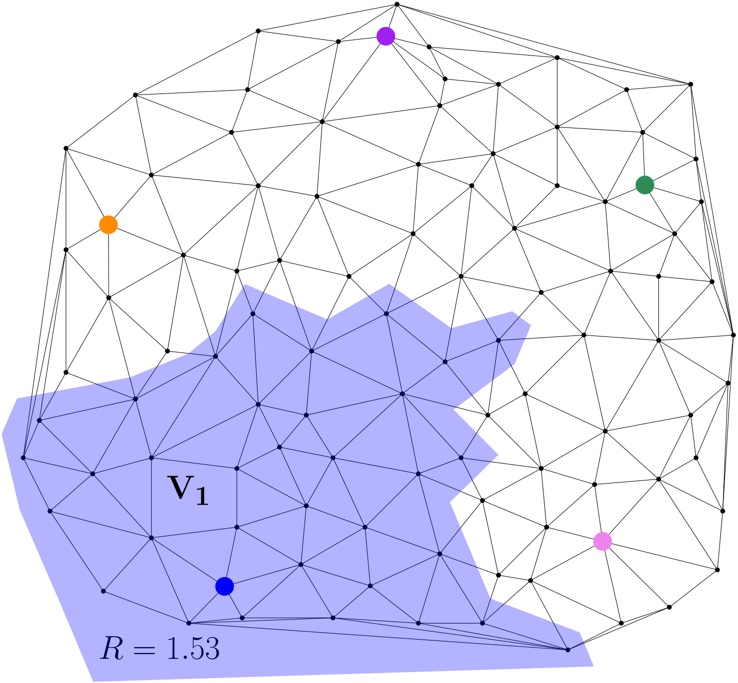
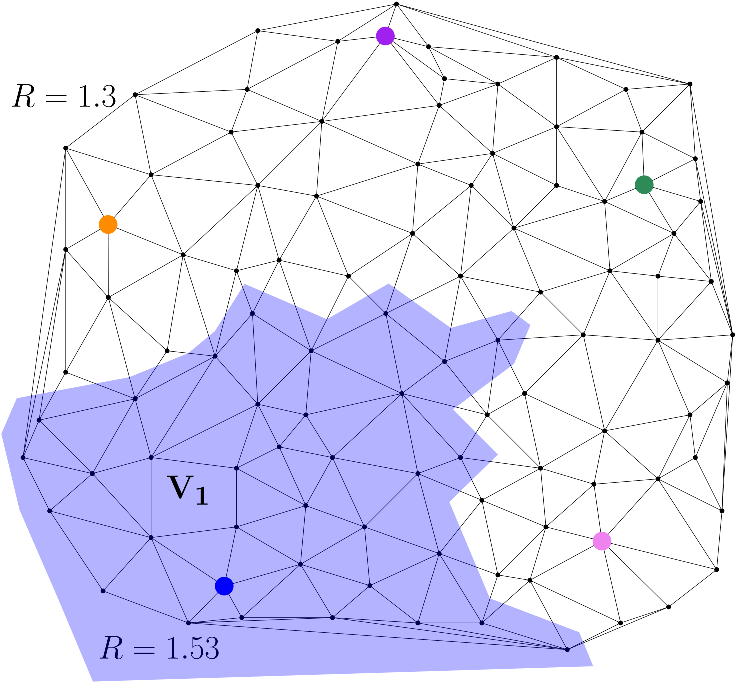
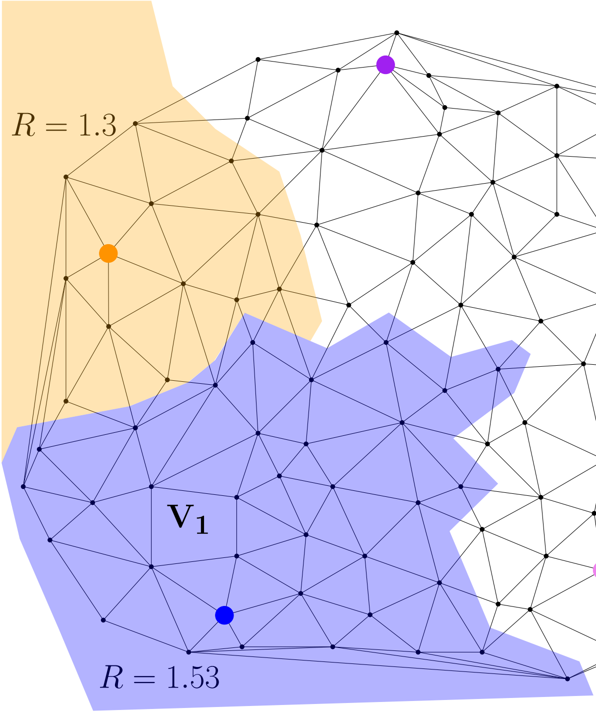
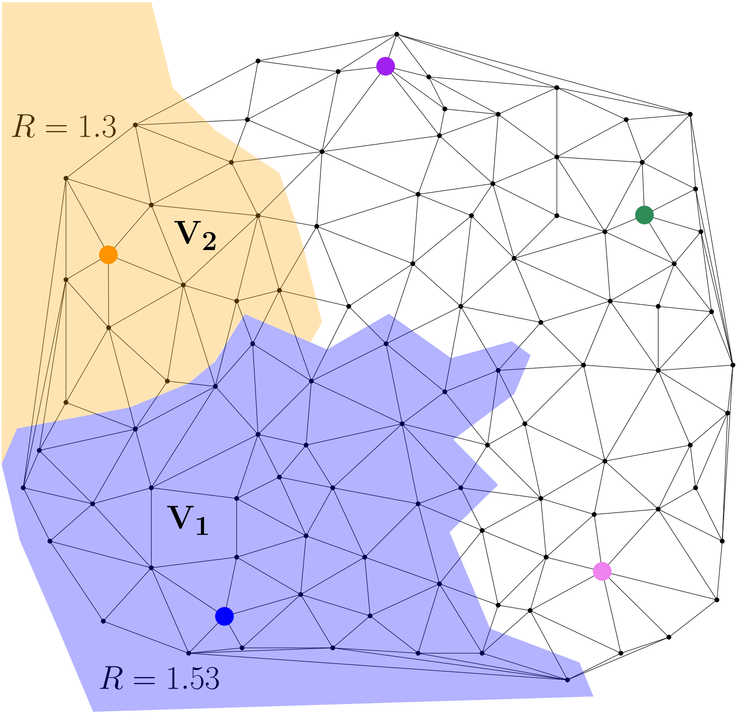
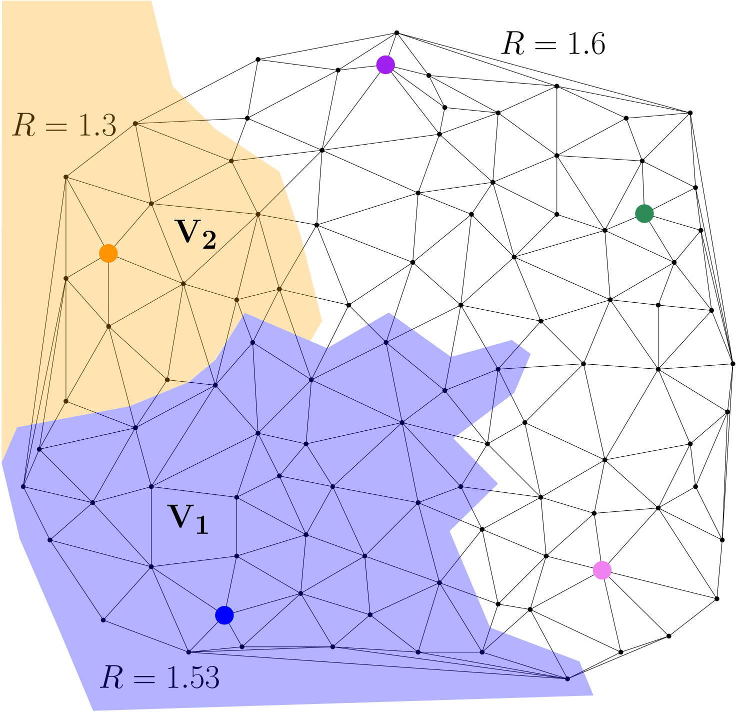

$\mathbf{V}_1$
$R = 1.53$



$R = 1.3$
$\mathbf{V_1}$
$R = 1.53$


$R = 1.3$
$\mathbf{V_1}$
$R = 1.53$


$R = 1.3$
$\mathbf{V_2}$
$\mathbf{V_1}$
$R = 1.53$



$R = 1.6$
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$\mathbf{V_2}$
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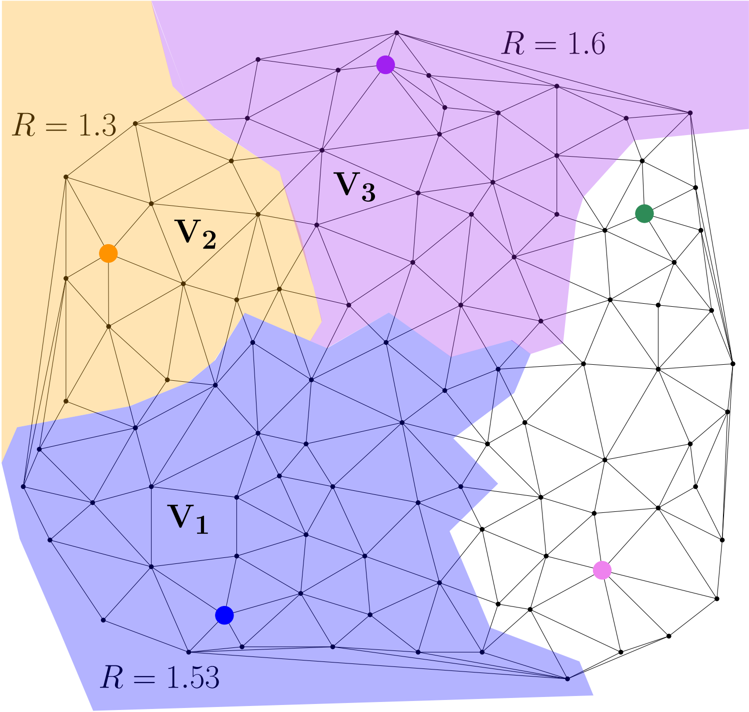

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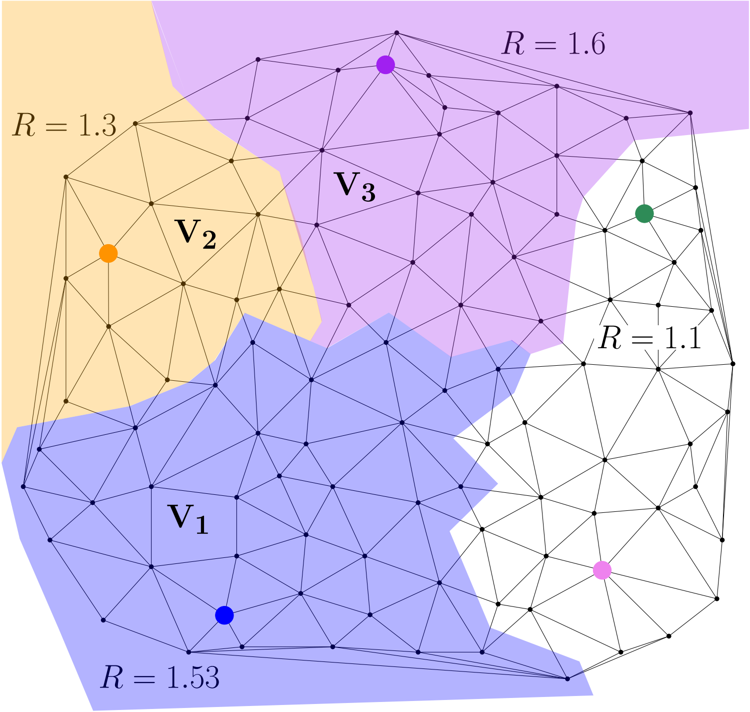
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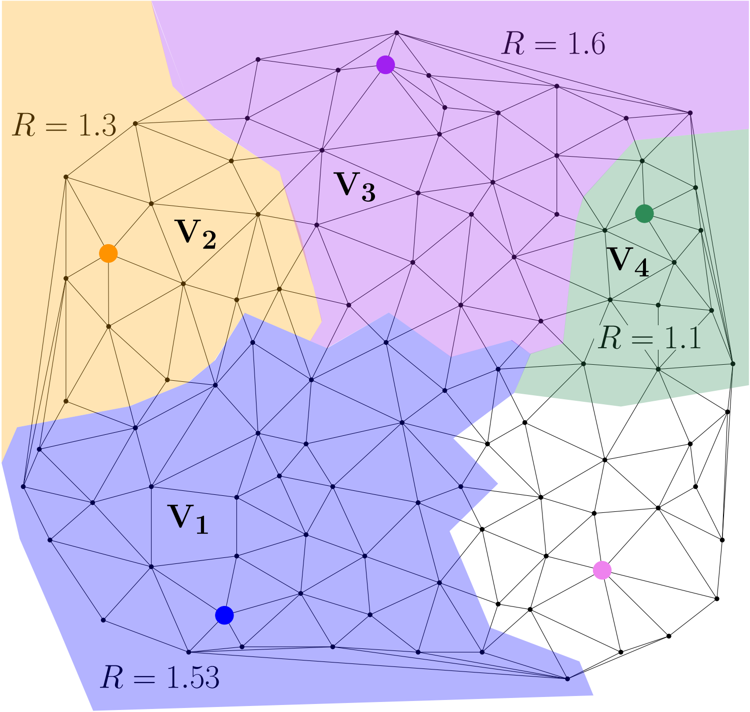

$R = 1.6$
$R = 1.3$
$\mathbf{V_3}$
$\mathbf{V_2}$
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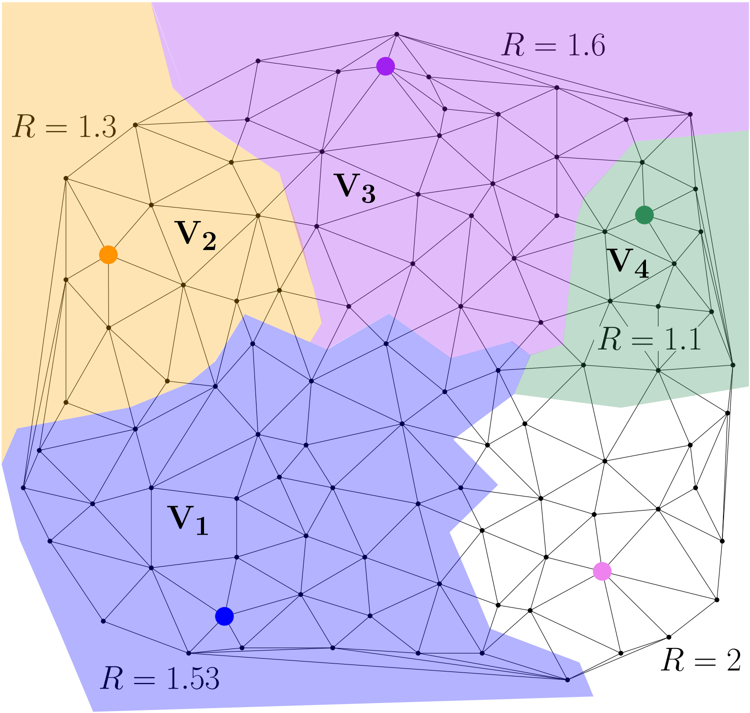
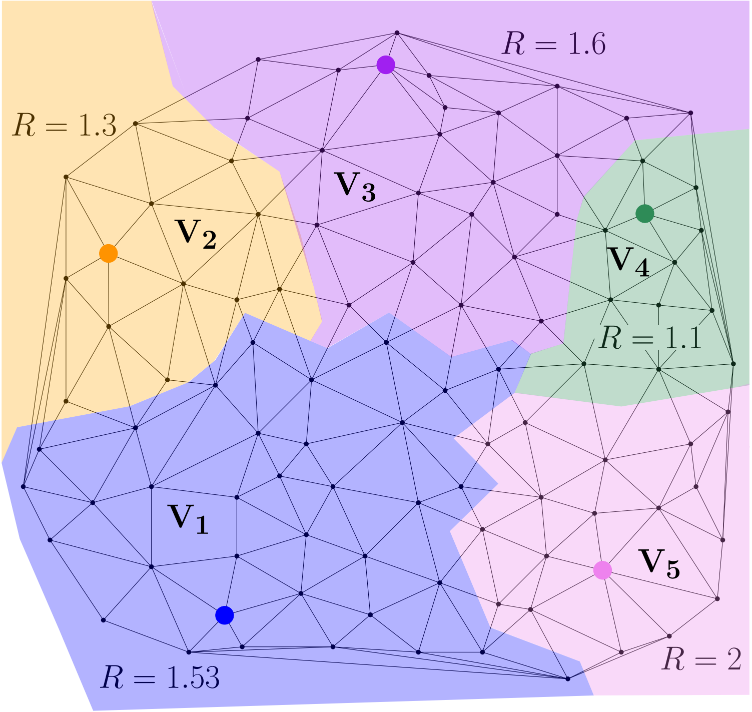


$R = 1.6$
$R = 1.3$
$\mathbf{V_3}$
$\mathbf{V_2}$
$\mathbf{V_4}$
$R = 1.1$
$\mathbf{V_1}$
$R = 2$
$R = 1.53$



$R = 1.6$
$R = 1.3$
$\mathbf{V_3}$
$\mathbf{V_2}$
$\mathbf{V_4}$
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$\mathbf{V_1}$
$\mathbf{V_5}$
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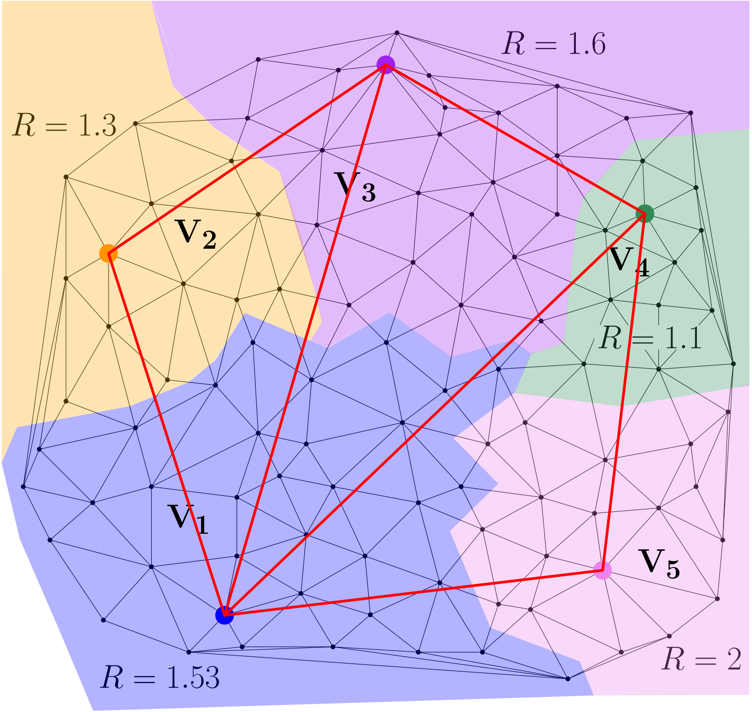
$R = 1.53$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Set $\delta = 1/20 \ln k$ and $p = \frac{1}{5}$.

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Lemma

The `Noisy Voronoi` algorithm creates a **terminal partition**.

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$t, t' \in K$, $P_{t,t'}$ is a shortest path in G.

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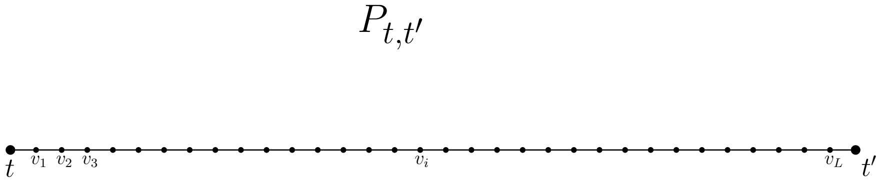

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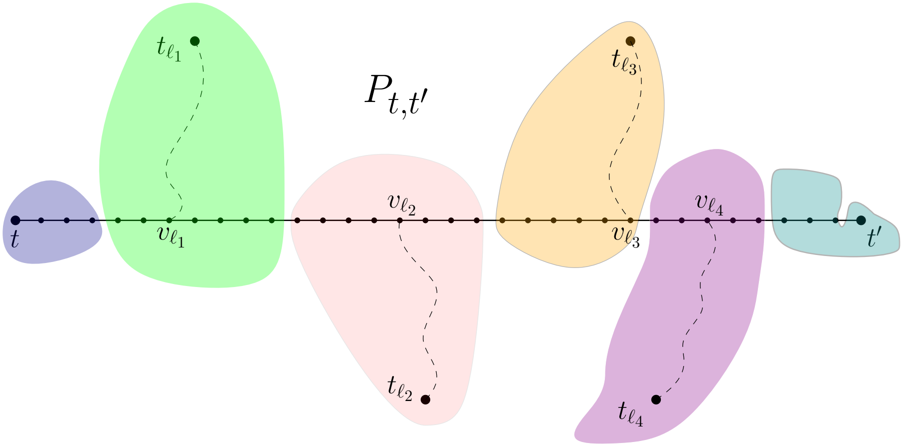

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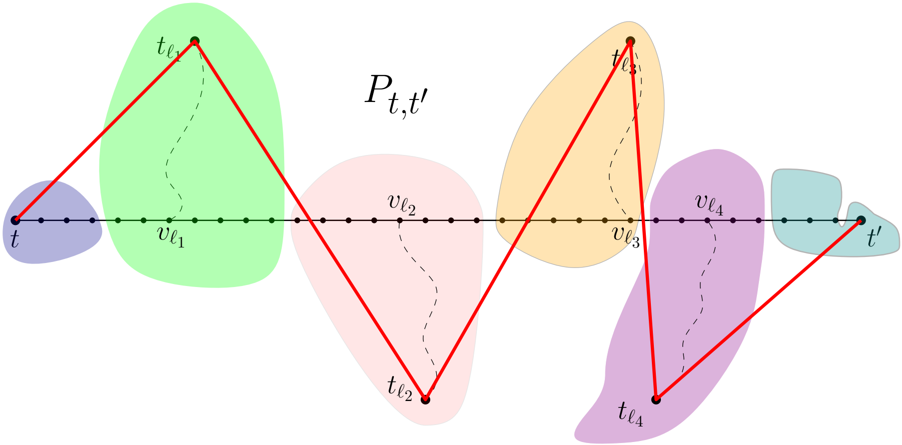

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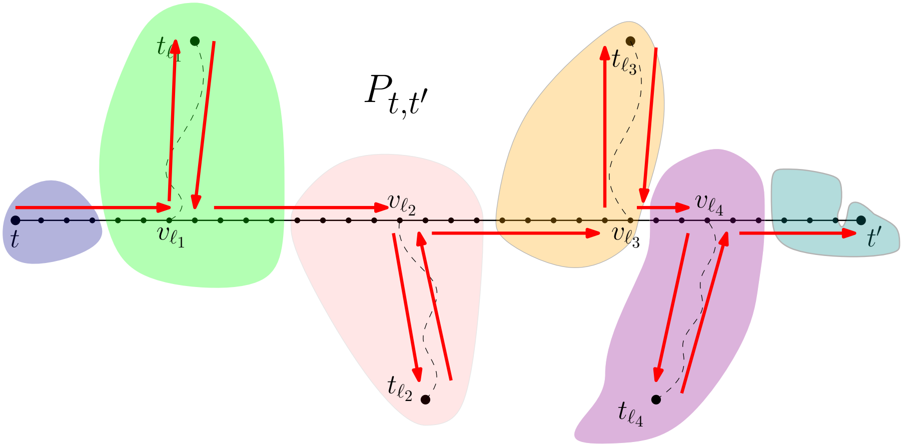

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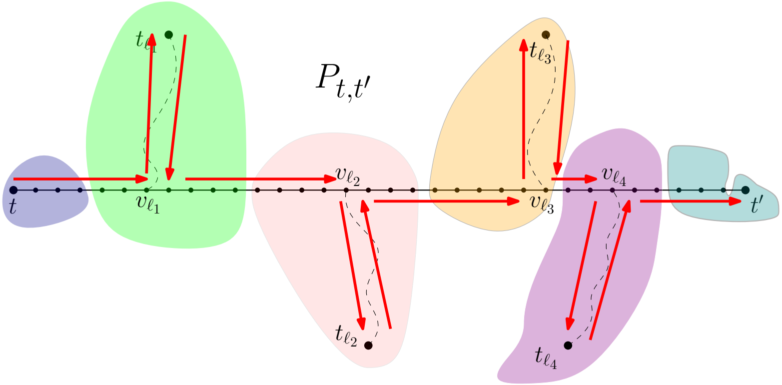

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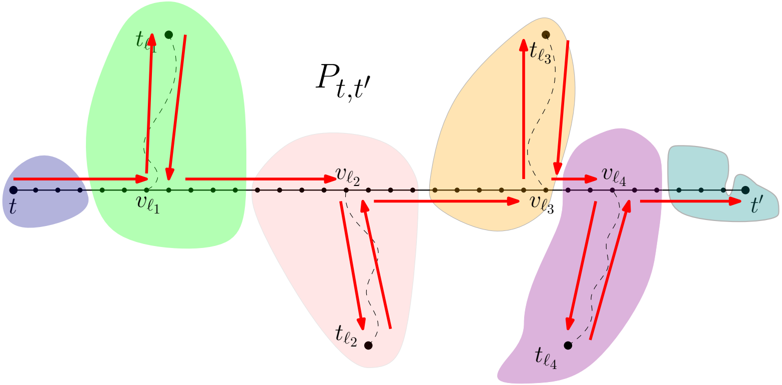

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Analyze $\sum_i d_G(t_{\ell_i}, v_{\ell_i})$!

Analyzing $\sum_i d_G(t_i, v_i)$ directly will be **tricky**,

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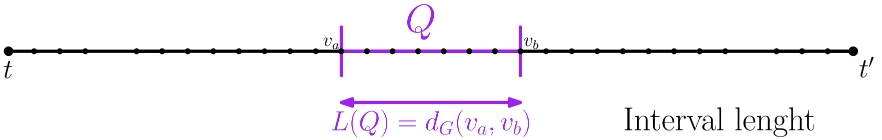
Partition of $P_{t,t'}$ to Intervals

Q is a interval of $P_{t,t'}$.



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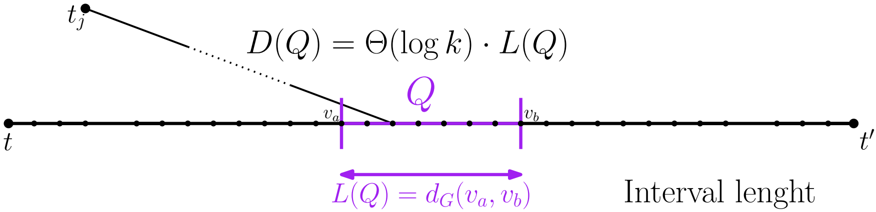

Partition $P_{t,t}$ into $\mathcal{Q}$, s.t. for each $Q \in \mathcal{Q}$

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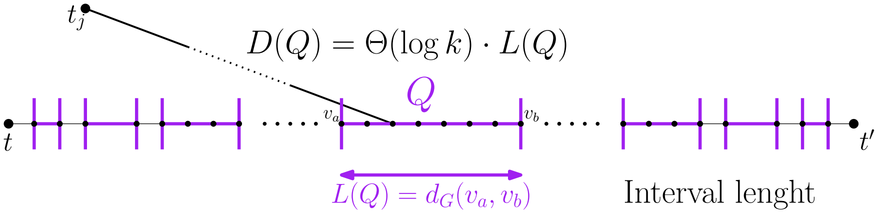

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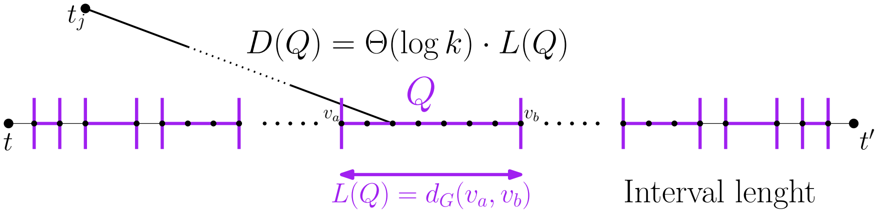

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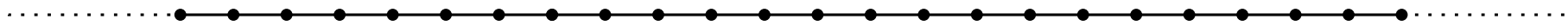
At the beginning all vertices are **active**.



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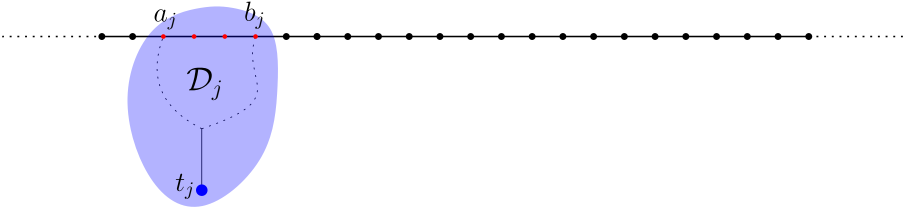

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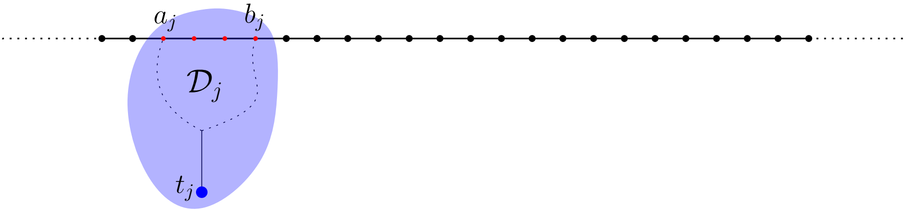

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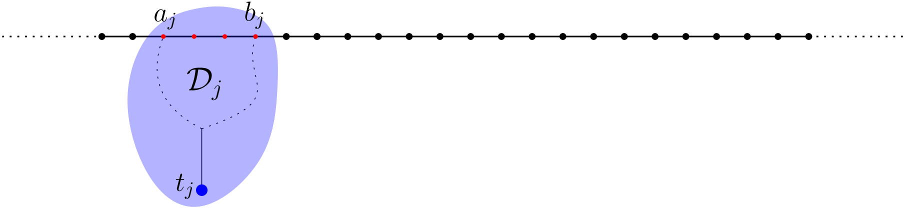

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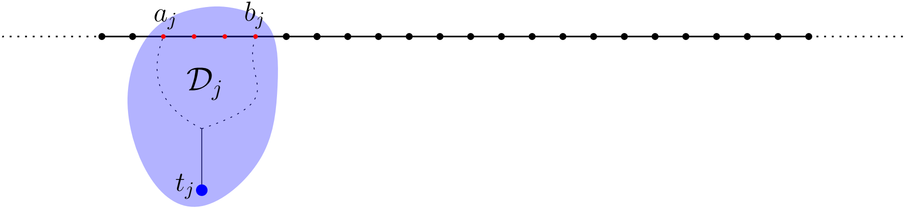

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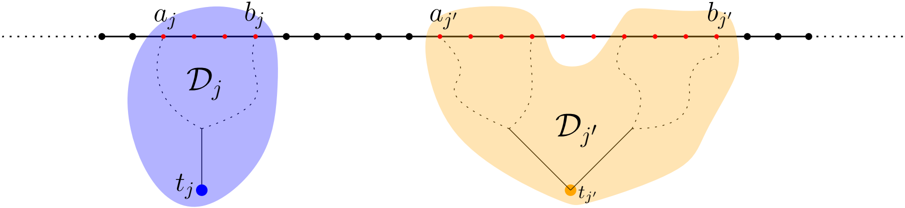

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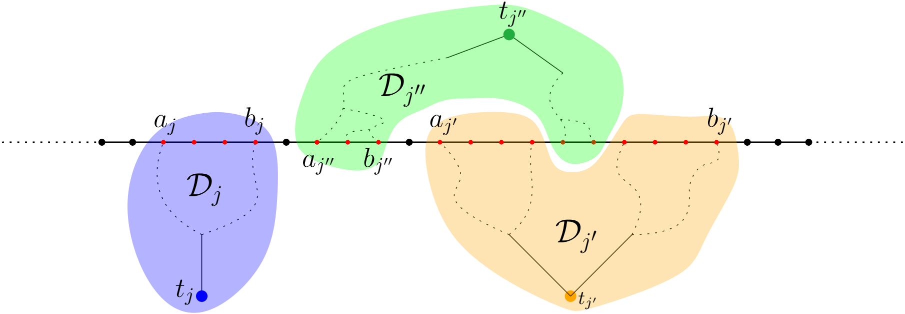

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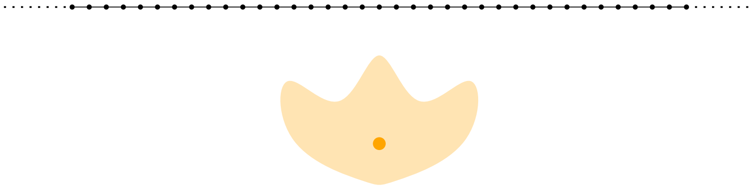
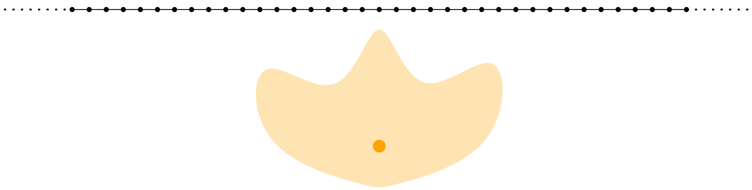
Charges



$$R_j = (1+\delta)^{g_1}$$

Detour $\mathcal{D}_j$ will be **charged** upon a single interval.

Charges



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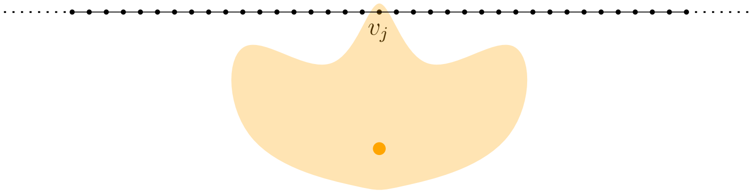
Charges



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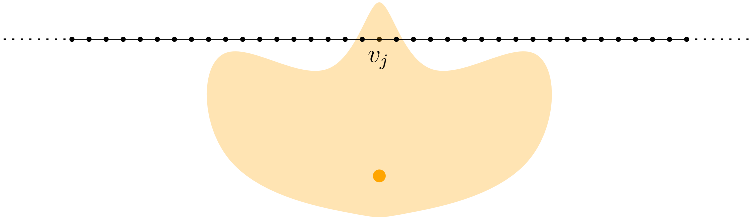

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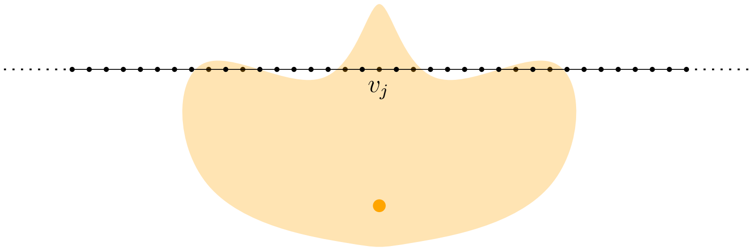

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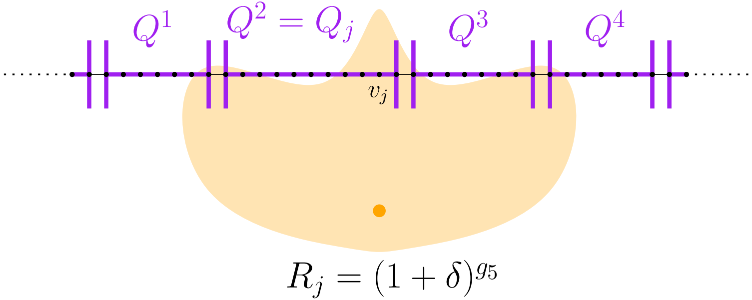

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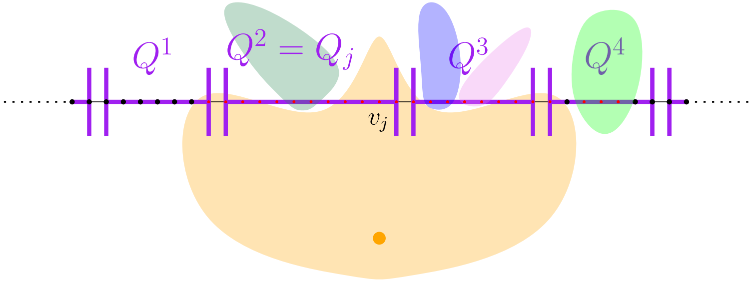

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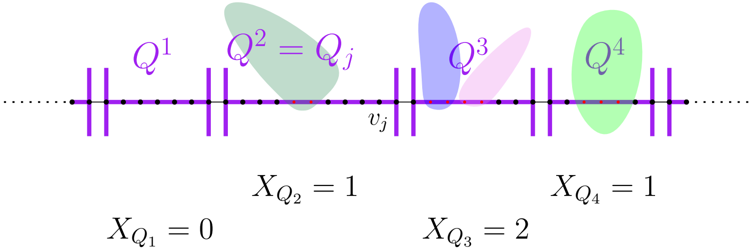

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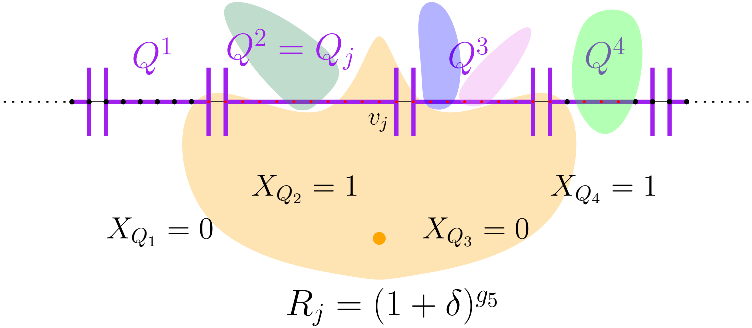

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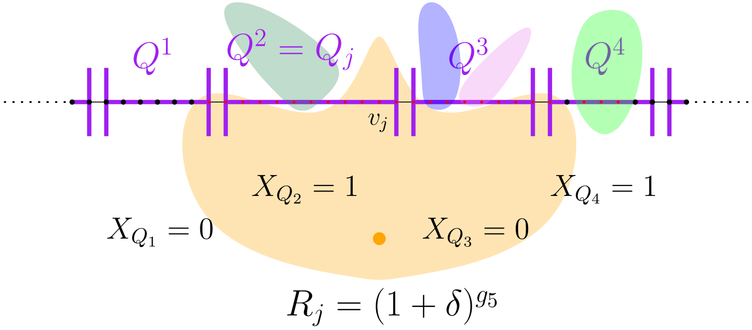
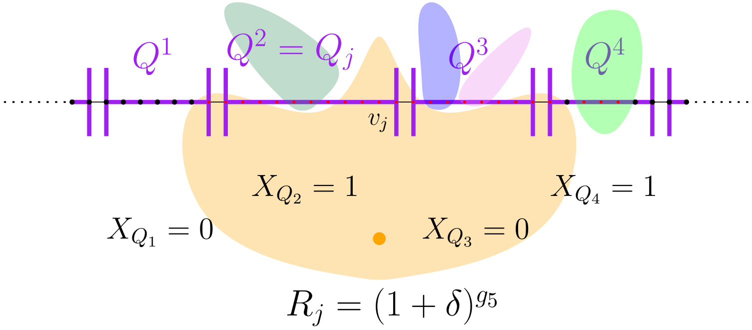
Every detour $\mathcal{D}_{j'}$ which is **contained** in $\mathcal{D}_j$ **erased**, and its charge **re-funded!**

Charges



X_{Q_j} increases by **at most** 1.

Charges



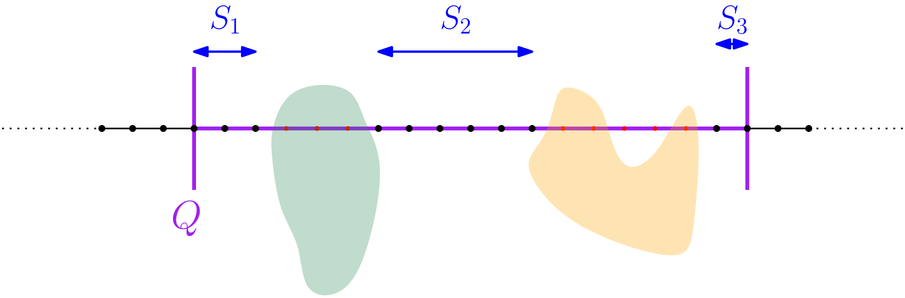
X_{Q_j} increases by **at most** 1.

For every $Q \neq Q_j$, X_Q can **only decrease**.

Slices: "The Potential to be Charged"

Within interval $Q \in \mathcal{Q}$,

maximal sub-interval of active vertices is called a **slice**.



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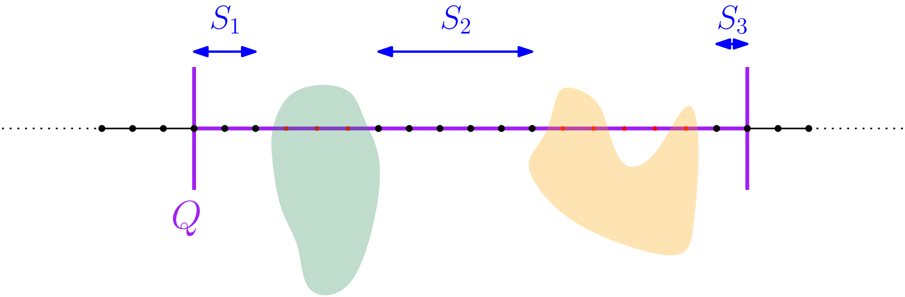
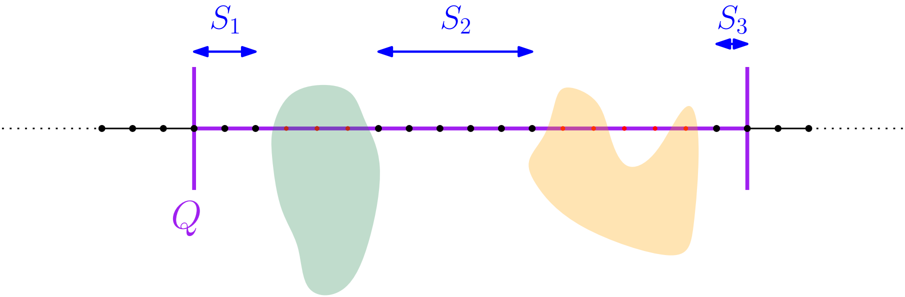


We denote by $\#\mathcal{S}(Q)$ the current **number of slices** in Q.

Slices: "The Potential to be Charged"

Within interval $Q \in \mathcal{Q}$,

maximal sub-interval of active vertices is called a **slice**.



We denote by $\#\mathcal{S}(Q)$ the current **number of slices** in Q.

At the start, $\#\mathcal{S}(Q) = 1$.

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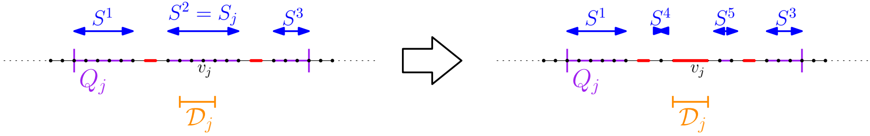

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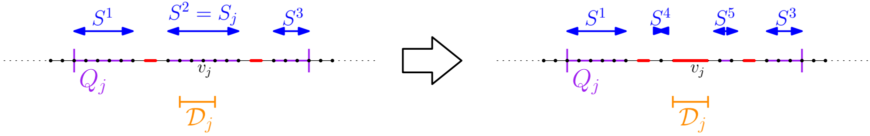

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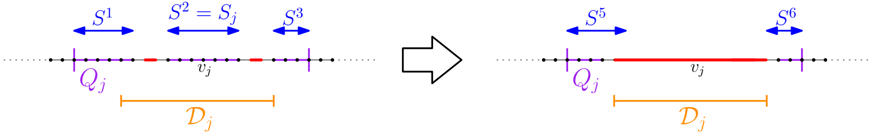

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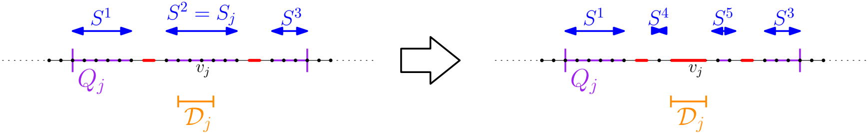

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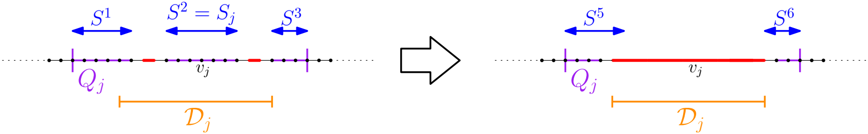
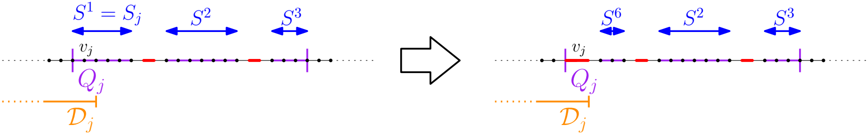


$\#S(Q_j)$ can stay **unchanged**.



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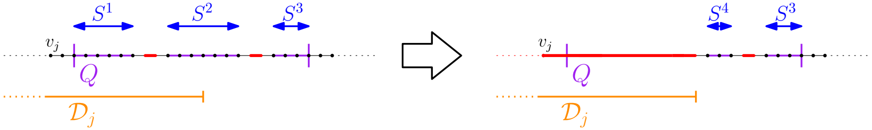

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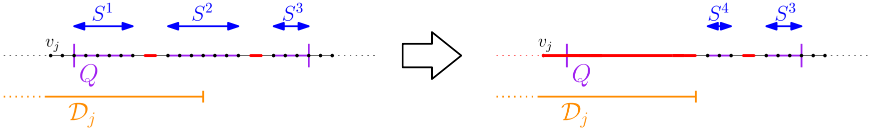

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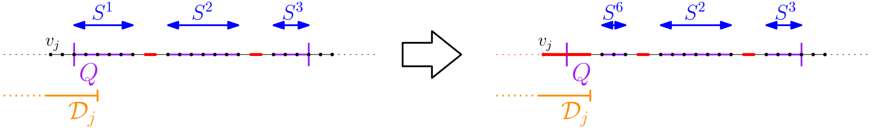

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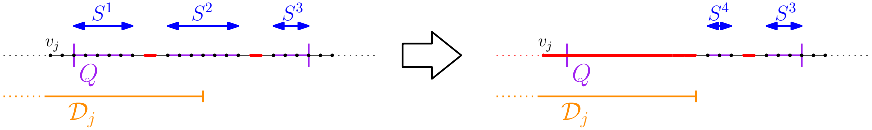

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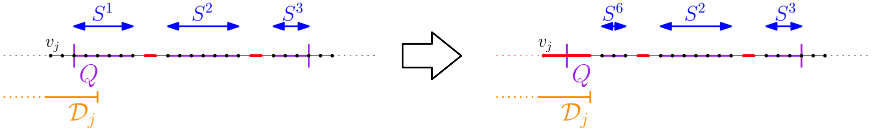

In any case, $\#S(Q)$ **cannot increase!**

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Assuming at least one active vertex joins V_j,
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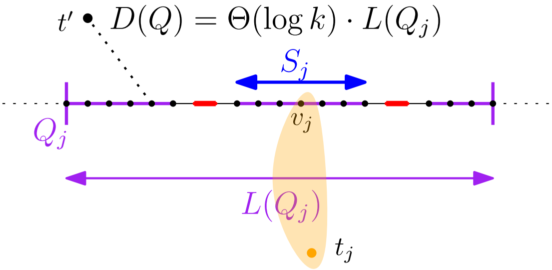

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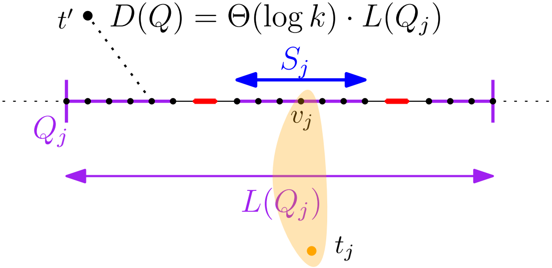

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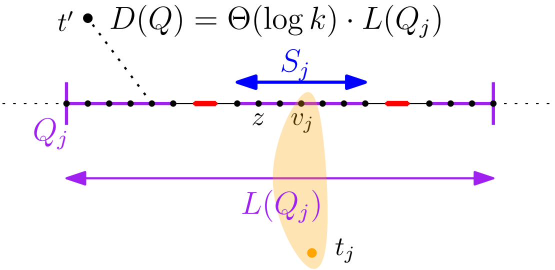

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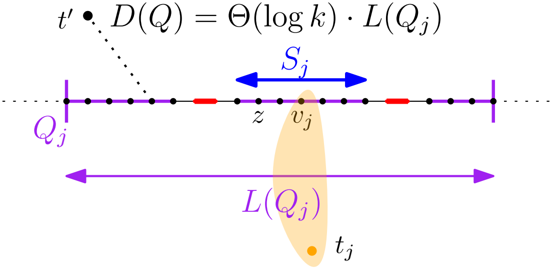

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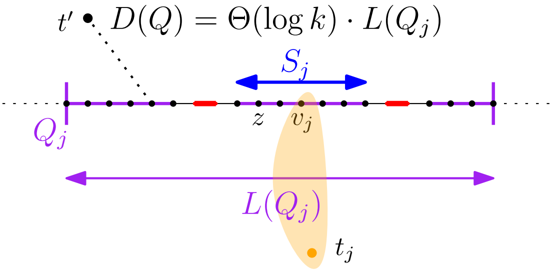

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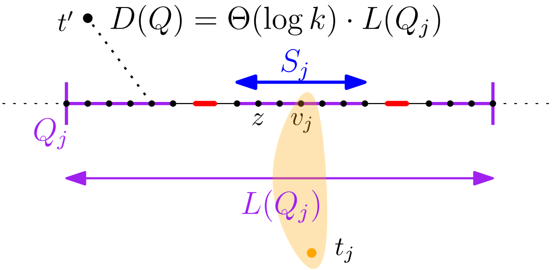

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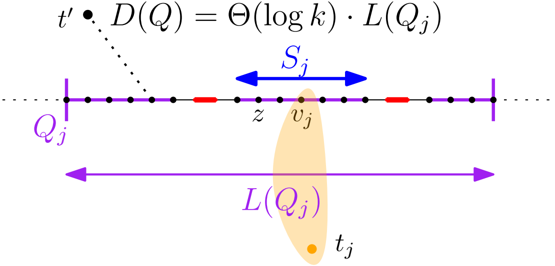

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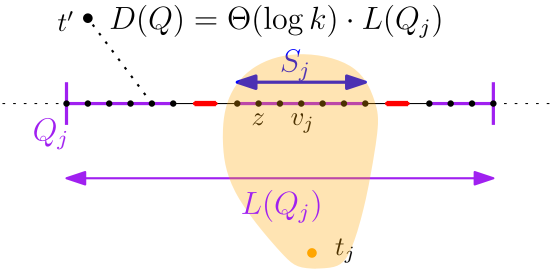

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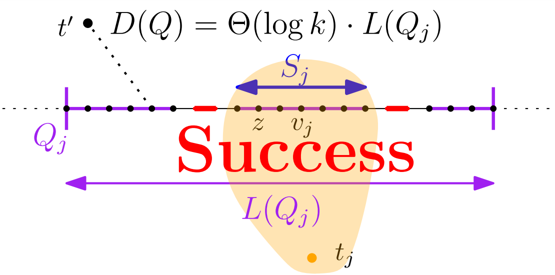

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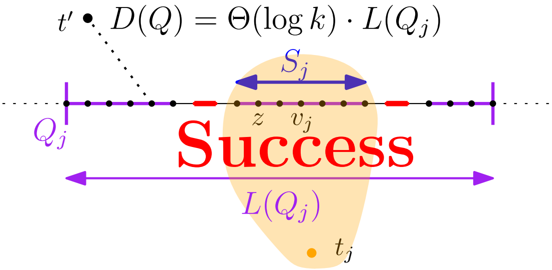

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In fact, the success probability is **either** 1 or $1 - p$.

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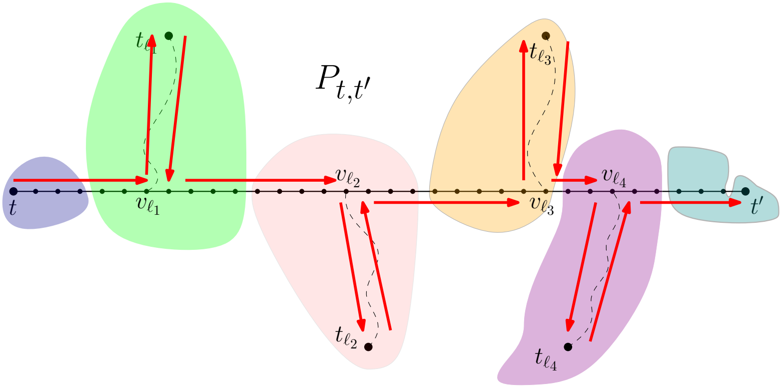

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Recall $R_j = O(1)$, thus $d_G(t_j, v_j) \leq R_j \cdot D(v_j) = O(D(v_j))$.

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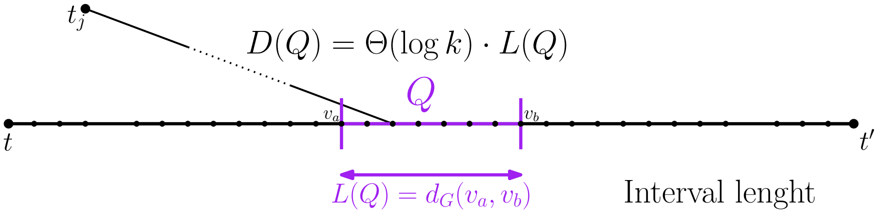

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Theorem

The **expected distortion** *of the minor M returned by the* `Noisy Voronoi` *algorithm is* $\mathbf{O(\log k)}$.

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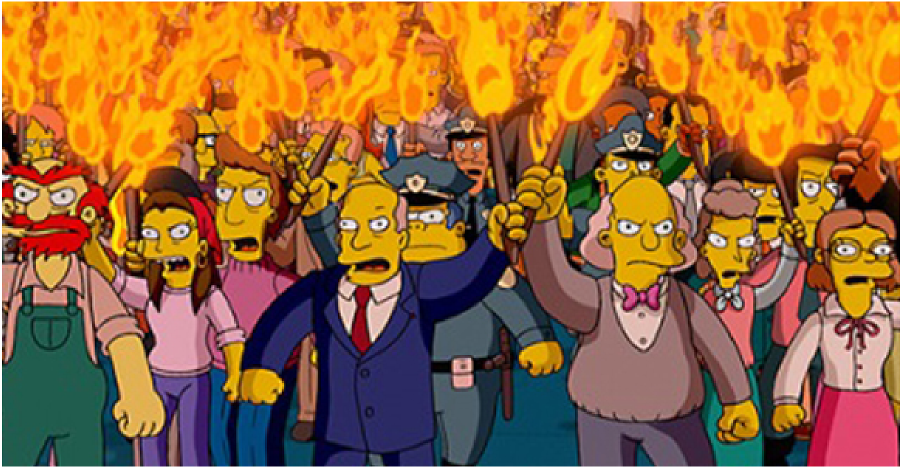
Theorem

*With **high probability**, the `Noisy Voronoi` algorithm returns a minor M with distortion $\mathbf{O(\log^2 k)}$.*



But you promised distortion $O(\log k)$!

Analyze $f(X_{Q^1}, \ldots, X_{Q^\varphi}) = \sum_{Q\in\mathcal{Q}} X_Q \cdot L(Q)$ **better**.

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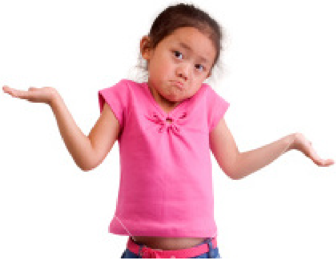

Analyze $f(X_{Q^1},\ldots,X_{Q^\varphi})=\sum_{Q\in\mathcal{Q}}X_Q\cdot L(Q)$ **better**.

But $X_{Q^1},\ldots,X_{Q^\varphi}$ are **dependent**. What can we do?

They maybe dependent, but in a "**positive**" way!

Idea

We will introduce new **series** of **i**ndependent **r**andom **v**ariables and show that they **dominate** $X_{Q^1}, \ldots, X_{Q^\varphi}$.

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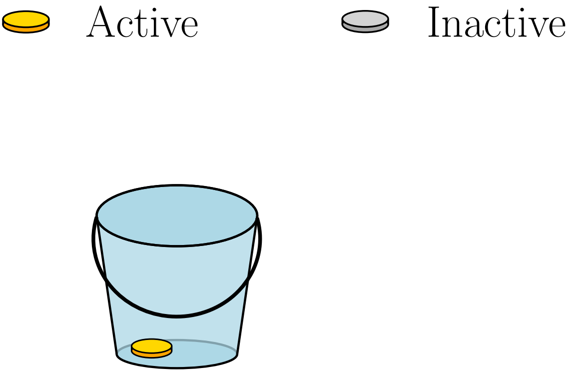

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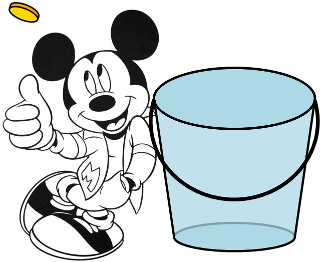
Active Inactive



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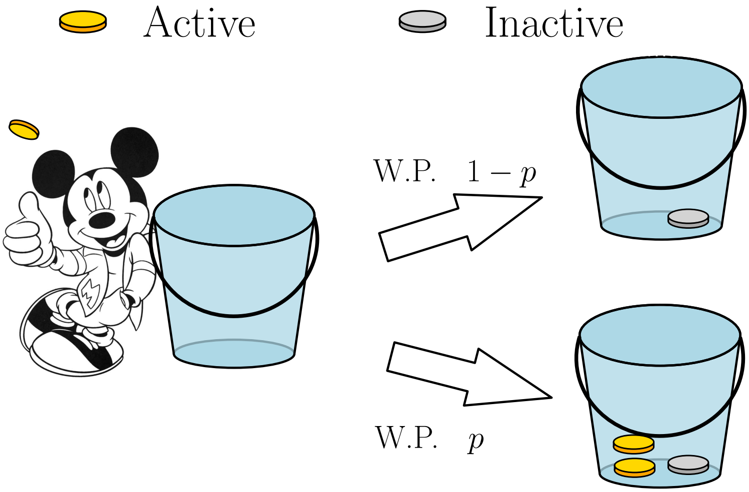
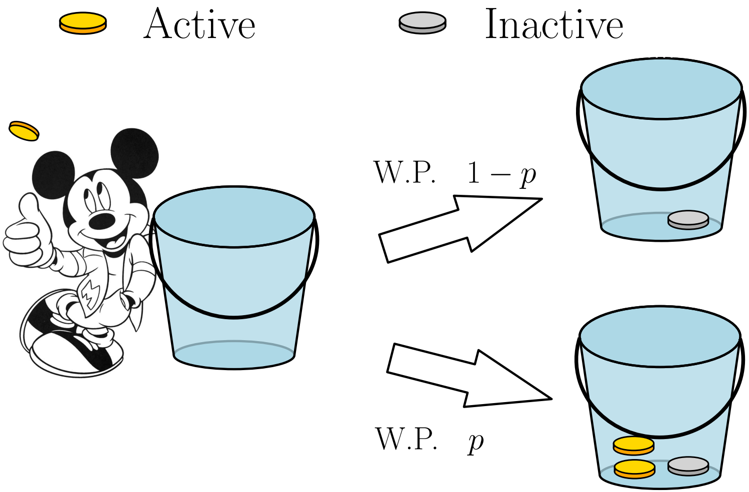
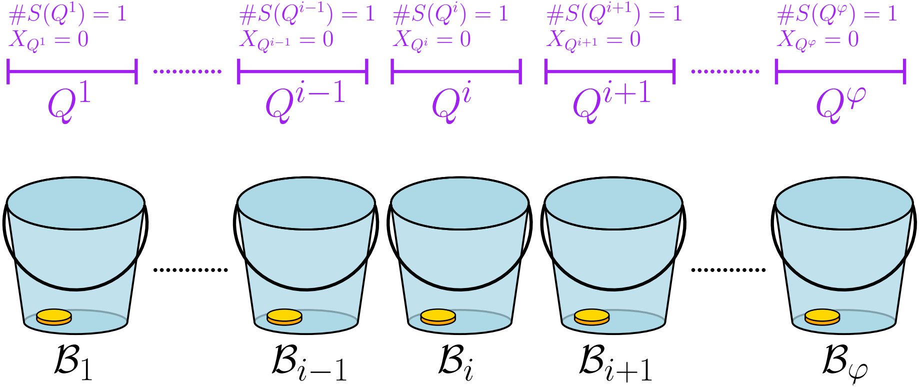




Denote by $A(B)$ the number of **active** Coins in the bucket B.
Denote by $IN(B)$ the number of **inactive** Coins in the bucket B.

Coupling



$\#S(Q^1) = 1$
$X_{Q^1} = 0$
$\#S(Q^{i-1}) = 1$
$X_{Q^{i-1}} = 0$
$\#S(Q^i) = 1$
$X_{Q^i} = 0$
$\#S(Q^{i+1}) = 1$
$X_{Q^{i+1}} = 0$
$\#S(Q^\varphi) = 1$
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Q^1
Q^{i-1}
Q^i
Q^{i+1}
Q^φ
$\mathcal{B}_1$
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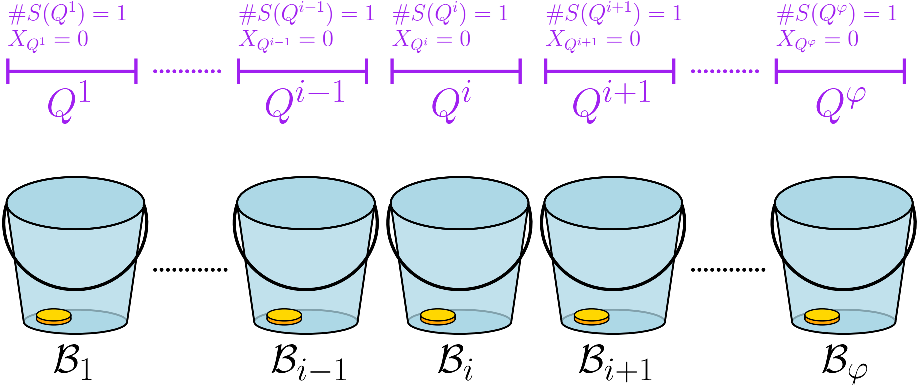
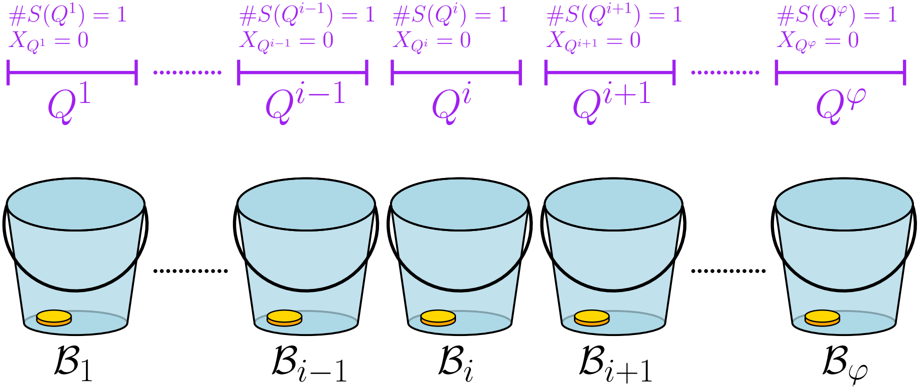
$\mathcal{B}_\varphi$

Coupling



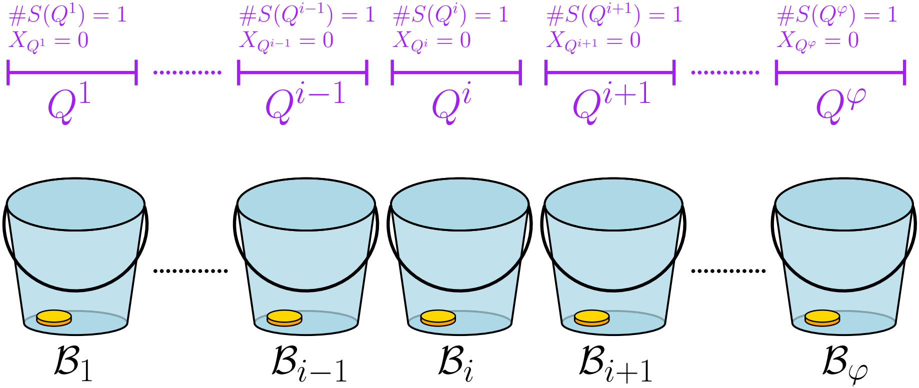
$\mathcal{B}_1, \ldots, \mathcal{B}_\varphi$ are independent buckets.

Coupling



$\mathcal{B}_1, \ldots, \mathcal{B}_\varphi$ are independent buckets.
We execute `Noisy Voronoi` algorithm
and use it in order to determine $IN(\mathcal{B}_1), \ldots, IN(\mathcal{B}_\varphi)$.

Coupling



Maintain, for all i,

$$X_{Q^i} \leq IN(\mathcal{B}_i) \quad \& \quad \#S(Q^i) \leq A(\mathcal{B}_i)$$

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The **marginal distribution** on the buckets is correct!

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Corollary (The buckets **dominate** the detour charges)

For all $\alpha \geq 0$,

$$\mathbf{Pr}\left[f\left(X_{Q^1}, \ldots, X_{Q^\varphi}\right) \geq \alpha\right] \leq \mathbf{Pr}\left[f\left(IN(\mathcal{B}_1), \ldots, IN(\mathcal{B}_\varphi)\right) \geq \alpha\right]$$

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For all $\alpha \geq 0$,

$$\Pr\left[IN(\mathcal{B}) \geq \alpha\right] \leq \Pr\left[Exp(10) + 1 \geq \alpha\right]$$

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Proof.

Meh. Too Technical. □

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Proof.

You know the drill... (f is linear and monotone coordinatewise.) □

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Note that

$$\begin{aligned} f\left(\text{Exp}(10)+1, \ldots, \text{Exp}(10)+1\right) = &\ f\left(\text{Exp}(10), \ldots, \text{Exp}(10)\right) \\ &+ \quad f(1, \ldots, 1) \end{aligned}$$

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Thus, in order to bound $f\left(X_{Q_1}, \ldots, X_{Q_\varphi}\right)$ it will be enough to bound

$$\begin{aligned} f\left(\text{Exp}(10), \ldots, \text{Exp}(10)\right) &= \sum_{i=1}^{\varphi} \text{Exp}(10) \cdot L(Q_i) \\ &= \sum_{i=1}^{\varphi} \text{Exp}\left(10 \cdot L(Q_i)\right) \end{aligned}$$

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Lemma (**Concentration Bound** for Exp)

$X_1, \ldots, X_n$ are i.r.v, where $X_i \sim \text{Exp}(\lambda_i)$.
Set: $X = \sum_i X_i$, $\lambda_M = \max_i \lambda_i$, $\mu = \mathbb{E}[X] = \sum_i \lambda_i$.

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In our case, $X_i \sim \text{Exp}\left(10 \cdot L(Q_i)\right)$. $X = \sum_i X_i$.

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$$\lambda_M = \max_i \{10 \cdot L(Q_i)\} = \max_i \left\{O\left(\frac{D(Q_i)}{\log k}\right)\right\} = O\left(\frac{d_G(t, t')}{\log k}\right)$$

$\mu \leq 10 \cdot d_G(t, t')$ $\lambda_M = O\left(\frac{d_G(t,t')}{\log k}\right)$

Thus for $a = 30 \cdot d_G(t, t')$

$$\Pr\left[X \geq a\right] \leq \exp\left(-\frac{1}{2\lambda_M}\left(a - 2\mu\right)\right) =$$

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Thus for $a = 30 \cdot d_G(t, t')$

$$\Pr\left[X \geq a\right] \leq \exp\left(-\frac{1}{2\lambda_M}\left(a - 2\mu\right)\right) = \exp\left(\Omega\left(\log k\right)\right) = \frac{1}{k^3}$$

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Thus for $a = 30 \cdot d_G(t, t')$

$$\Pr[X \geq a] \leq \exp\left(-\frac{1}{2\lambda_M}(a - 2\mu)\right) = \exp(\Omega(\log k)) = \frac{1}{k^3}$$

We conclude

$$\Pr\left[f\left(X_{Q_1}, \dots, X_{Q_\varphi}\right) \geq O(d_G(t, t'))\right]$$

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We conclude

$$\Pr\left[f\left(X_{Q_1}, \ldots, X_{Q_\varphi}\right) \geq O(d_G(t, t'))\right]$$
$$\leq \Pr[f(IN(\mathcal{B}_1), \ldots, IN(\mathcal{B}_\varphi)) \geq O(d_G(t, t'))]$$

$\mu \leq 10 \cdot d_G(t, t')$ $\lambda_M = O\left(\frac{d_G(t,t')}{\log k}\right)$

Thus for $a = 30 \cdot d_G(t, t')$

$$\Pr[X \geq a] \leq \exp\left(-\frac{1}{2\lambda_M}(a - 2\mu)\right) = \exp(\Omega(\log k)) = \frac{1}{k^3}$$

We conclude

$$\begin{aligned} &\Pr\left[f\left(X_{Q_1}, \ldots, X_{Q_\varphi}\right) \geq O(d_G(t, t'))\right] \\ &\leq \Pr\left[f\left(IN(\mathcal{B}_1), \ldots, IN(\mathcal{B}_\varphi)\right) \geq O(d_G(t, t'))\right] \\ &\leq \Pr\left[f\left(\text{Exp}(10), \ldots, \text{Exp}(10)\right) \geq O(d_G(t, t'))\right] \end{aligned}$$

$\mu \leq 10 \cdot d_G(t,t')$ $\qquad$ $\lambda_M = O\left(\frac{d_G(t,t')}{\log k}\right)$

Thus for $a = 30 \cdot d_G(t,t')$

$$\Pr[X \geq a] \leq \exp\left(-\frac{1}{2\lambda_M}(a-2\mu)\right) = \exp(\Omega(\log k)) = \frac{1}{k^3}$$

We conclude

$$\begin{aligned}
&\Pr\left[f\left(X_{Q_1},\dots,X_{Q_\varphi}\right) \geq O(d_G(t,t'))\right] \\
&\leq \Pr\left[f\left(IN(\mathcal{B}_1),\dots,IN(\mathcal{B}_\varphi)\right) \geq O(d_G(t,t'))\right] \\
&\leq \Pr\left[f\left(\mathrm{Exp}(10),\dots,\mathrm{Exp}(10)\right) \geq O(d_G(t,t'))\right] \\
&= \Pr[X \geq a] \leq \frac{1}{k^3}
\end{aligned}$$

$\mu \leq 10 \cdot d_G(t, t')$ $\lambda_M = O\left(\frac{d_G(t,t')}{\log k}\right)$

Thus for $a = 30 \cdot d_G(t, t')$

$$\Pr[X \geq a] \leq \exp\left(-\frac{1}{2\lambda_M}(a - 2\mu)\right) = \exp\left(\Omega\left(\log k\right)\right) = \frac{1}{k^3}$$

We conclude

$$\begin{aligned}
&\Pr\left[f\left(X_{Q_1}, \ldots, X_{Q_\varphi}\right) \geq O(d_G(t, t'))\right] \\
&\leq \Pr\left[f\left(IN(\mathcal{B}_1), \ldots, IN(\mathcal{B}_\varphi)\right) \geq O(d_G(t, t'))\right] \\
&\leq \Pr\left[f\left(\mathrm{Exp}(10), \ldots, \mathrm{Exp}(10)\right) \geq O(d_G(t, t'))\right] \\
&= \Pr\left[X \geq a\right] \leq \frac{1}{k^3}
\end{aligned}$$

If this event indeed occurs

$\mu \leq 10 \cdot d_G(t, t')$ $\qquad$ $\lambda_M = O\left(\frac{d_G(t,t')}{\log k}\right)$

Thus for $a = 30 \cdot d_G(t, t')$

$$\Pr[X \geq a] \leq \exp\left(-\frac{1}{2\lambda_M}(a - 2\mu)\right) = \exp(\Omega(\log k)) = \frac{1}{k^3}$$

We conclude

$$\begin{aligned}
&\Pr\left[f\left(X_{Q_1}, \ldots, X_{Q_\varphi}\right) \geq O(d_G(t, t'))\right] \\
&\leq \Pr\left[f\left(IN(\mathcal{B}_1), \ldots, IN(\mathcal{B}_\varphi)\right) \geq O(d_G(t, t'))\right] \\
&\leq \Pr\left[f\left(\text{Exp}(10), \ldots, \text{Exp}(10)\right) \geq O(d_G(t, t'))\right] \\
&= \Pr[X \geq a] \leq \frac{1}{k^3}
\end{aligned}$$

If this event indeed occurs

$$d_M(t, t') \leq d_G(t, t') + O(\log k) \cdot f\left(X_{Q_1}, \ldots, X_{Q_\varphi}\right)$$

$\mu \le 10 \cdot d_G(t, t')$ $\lambda_M = O\left(\frac{d_G(t,t')}{\log k}\right)$

Thus for $a = 30 \cdot d_G(t, t')$

$$\Pr\left[X \ge a\right] \le \exp\left(-\frac{1}{2\lambda_M}\left(a - 2\mu\right)\right) = \exp\left(\Omega\left(\log k\right)\right) = \frac{1}{k^3}$$

We conclude

$$\begin{aligned}
&\Pr\left[f\left(X_{Q_1}, \ldots, X_{Q_\varphi}\right) \ge O(d_G(t, t'))\right] \\
&\le \Pr\left[f\left(IN(\mathcal{B}_1), \ldots, IN(\mathcal{B}_\varphi)\right) \ge O(d_G(t, t'))\right] \\
&\le \Pr\left[f\left(\mathrm{Exp}(10), \ldots, \mathrm{Exp}(10)\right) \ge O(d_G(t, t'))\right] \\
&= \Pr\left[X \ge a\right] \le \frac{1}{k^3}
\end{aligned}$$

If this event indeed occurs

$$\begin{aligned}
d_M(t, t') &\le d_G(t, t') + O(\log k) \cdot f\left(X_{Q_1}, \ldots, X_{Q_\varphi}\right) \\
&= \boldsymbol{O(\log k)} \cdot d_G(t, t')
\end{aligned}$$

By **union bound**, w.h.p for all t, t', $d_M(t, t') = \boldsymbol{O(\log k)} \cdot d_G(t, t')$.

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Open Question

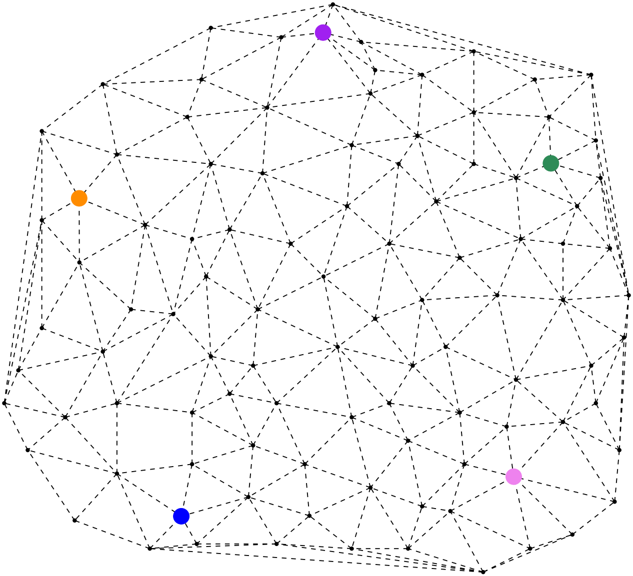
Close the gap between 8 to $\log k$!

Open Question

Close the gap between 8 to $\log k$!

Thank You!

We can assume that edges has infinitesimally small weights.
Otherwise we simply subdivide.



The set of minors and the geometry of the terminals remain the same!

Ball Growing Algorithm [KKN14]

Algorithm 1 $M = \texttt{Ball-Growing}(G=(V,E), w, K=\{t_1,\dots,t_k\})$

Set $r \leftarrow 1+\delta/\ln k$, where $\delta = 1/80$.
Set $D \leftarrow \frac{\delta}{\ln k}$.
For each $j \in [k]$, set $V_j \leftarrow \{t_j\}$, and set $R_j \leftarrow 0$.
Set $V_\perp \leftarrow V \setminus \left(\cup_{j=1}^{k} V_j\right)$.
Set $\ell \leftarrow 0$.
while $\left(\cup_{j=1}^{k} V_j\right) \neq V$ **do**
  for j from 1 to k **do**
    Choose independently at random q_j^ℓ distributed according to $\mathsf{Exp}(D \cdot r^\ell)$.
    Set $R_j \leftarrow R_j + q_j^\ell$.
    Set $V_j \leftarrow B_{G[V_\perp \cup V_j]}(t_j, R_j)$.
    Set $V_\perp \leftarrow V \setminus \left(\cup_{j=1}^{k} V_j\right)$.
  end for
  $\ell \leftarrow \ell + 1$.
end while
return the terminal-centered minor M of G induced by $V_1,\dots,V_k$.

Ball Growing Algorithm

Arbitrary order.

Expand cluster in every round.

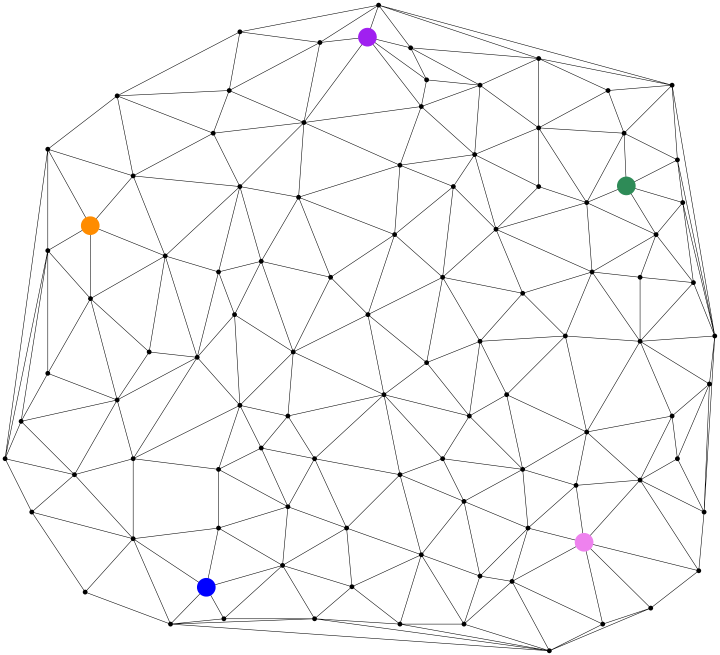
$R_1 = 0$

$R_2 = 0$

$R_3 = 0$

$R_4 = 0$

$R_5 = 0$



Ball Growing Algorithm

Arbitrary order.

Expand cluster in every round.

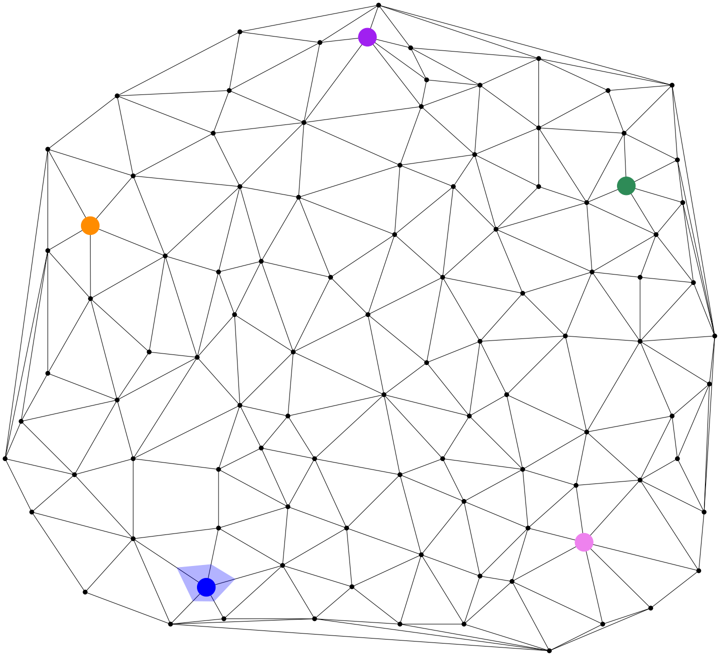
R_1 =0.2

R_2 =0

R_3 =0

R_4 =0

R_5 =0



Ball Growing Algorithm

Arbitrary order.

Expand cluster in every round.

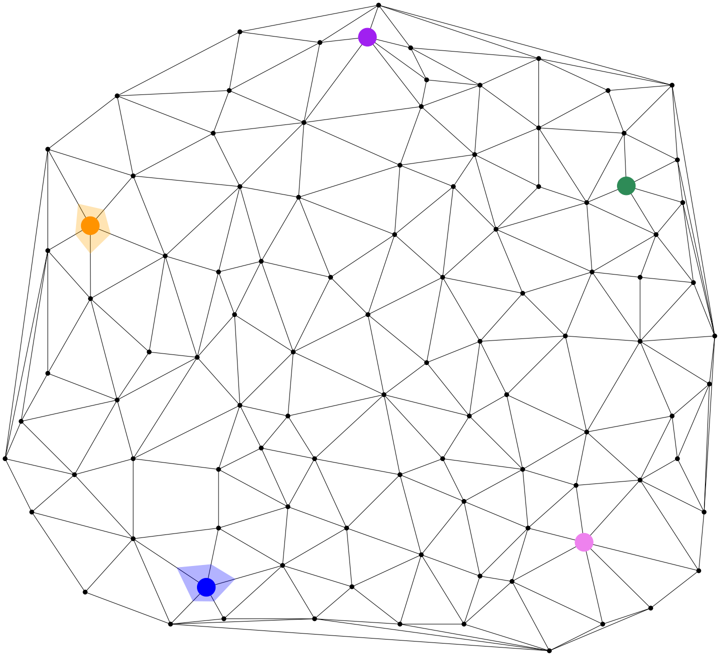
R_1 =0.2

R_2 =0.1

R_3 =0

R_4 =0

R_5 =0



Ball Growing Algorithm

Arbitrary order.

Expand cluster in every round.

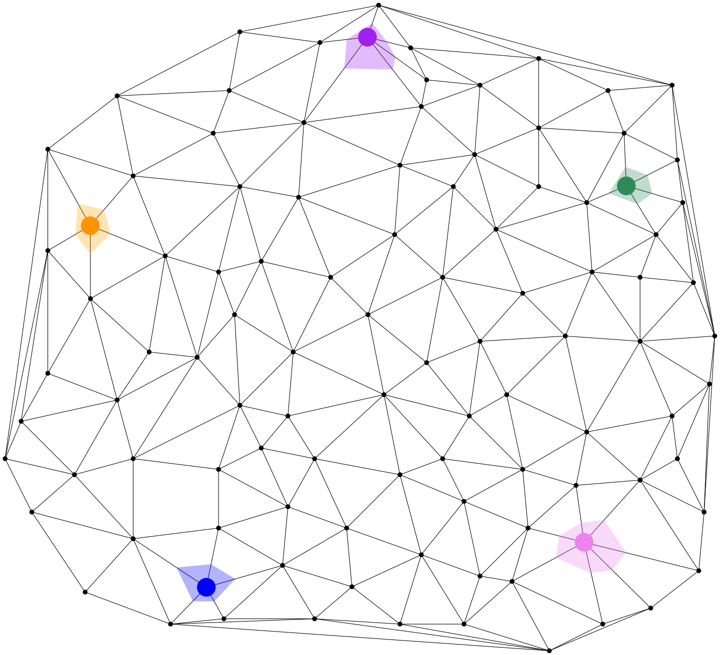
$R_1 = 0.2$

$R_2 = 0.1$

$R_3 = 0.3$

$R_4 = 0.1$

$R_5 = 0.25$



Ball Growing Algorithm

Arbitrary order.

Expand cluster in every round.

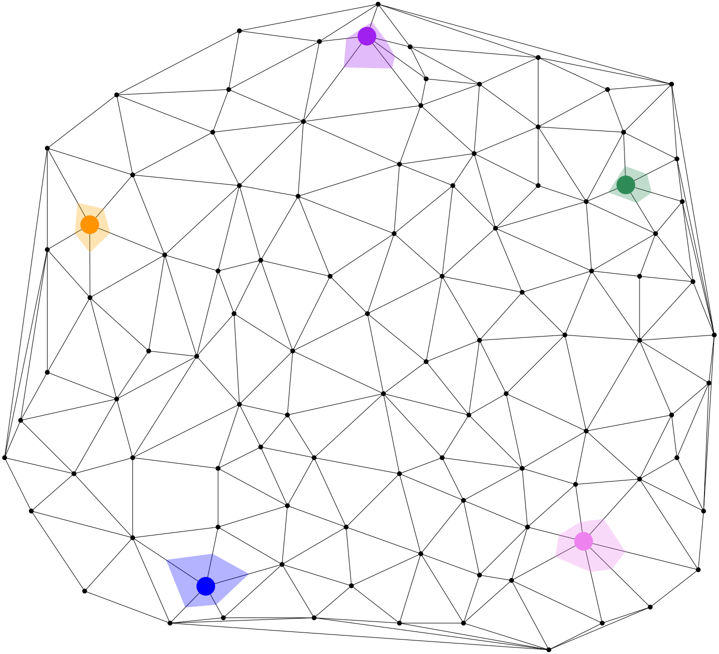
$R_1 = 0.5$

$R_2 = 0.1$

$R_3 = 0.3$

$R_4 = 0.1$

$R_5 = 0.25$



Ball Growing Algorithm

Arbitrary order.

Expand cluster in every round.

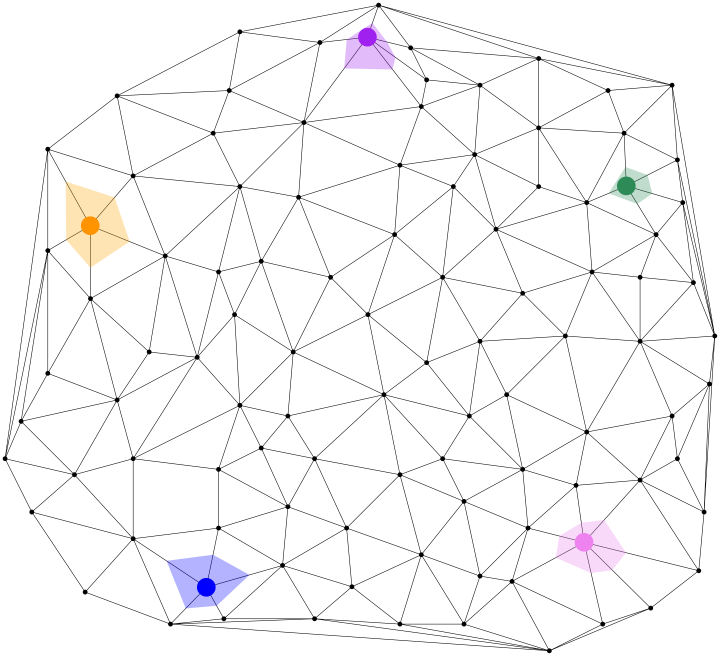
R_1 =0.5

R_2 =0.55

R_3 =0.3

R_4 =0.1

R_5 =0.25



Ball Growing Algorithm

Arbitrary order.

Expand cluster in every round.

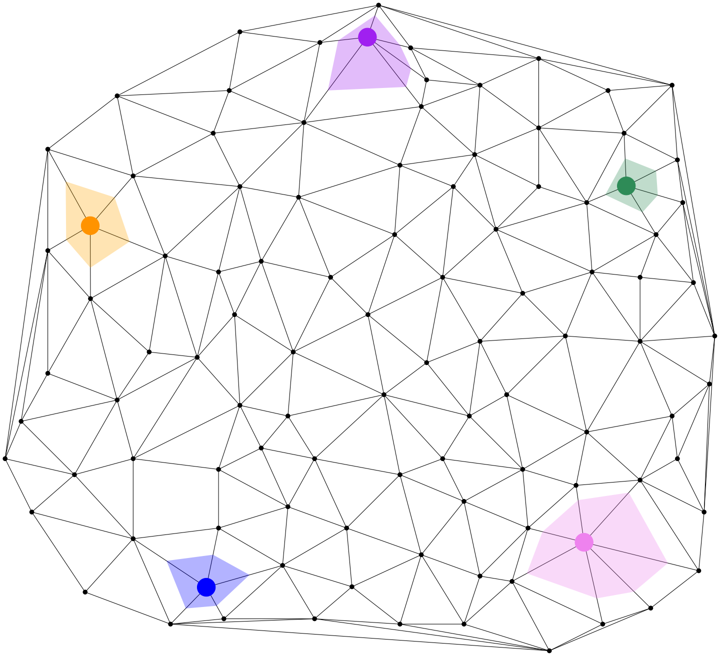
$R_1 = 0.5$

$R_2 = 0.55$

$R_3 = 0.6$

$R_4 = 0.2$

$R_5 = 0.8$



Ball Growing Algorithm

Arbitrary order.

Expand cluster in every round.

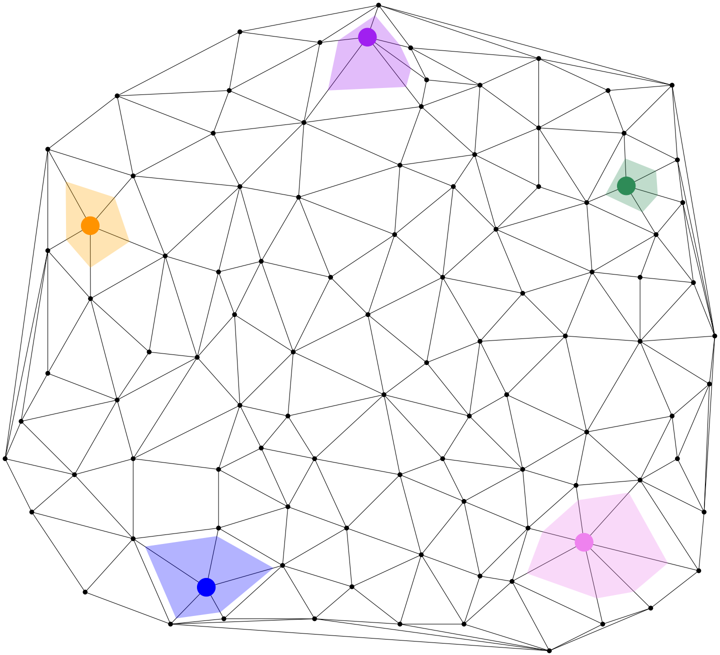
$R_1 = 0.9$

$R_2 = 0.55$

$R_3 = 0.6$

$R_4 = 0.2$

$R_5 = 0.8$



Ball Growing Algorithm

Arbitrary order.

Expand cluster in every round.

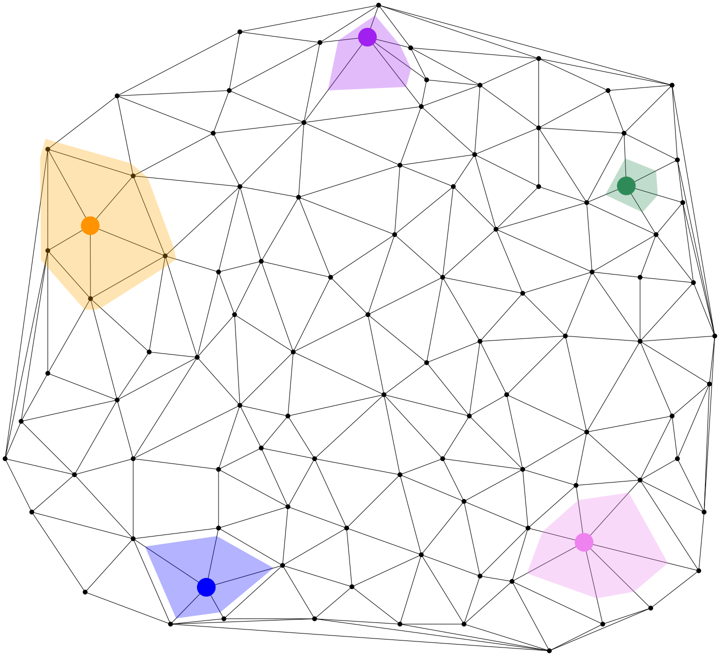
R_1 =0.9

R_2 =1.05

R_3 =0.6

R_4 =0.2

R_5 =0.8



Ball Growing Algorithm

Arbitrary order.

Expand cluster in every round.

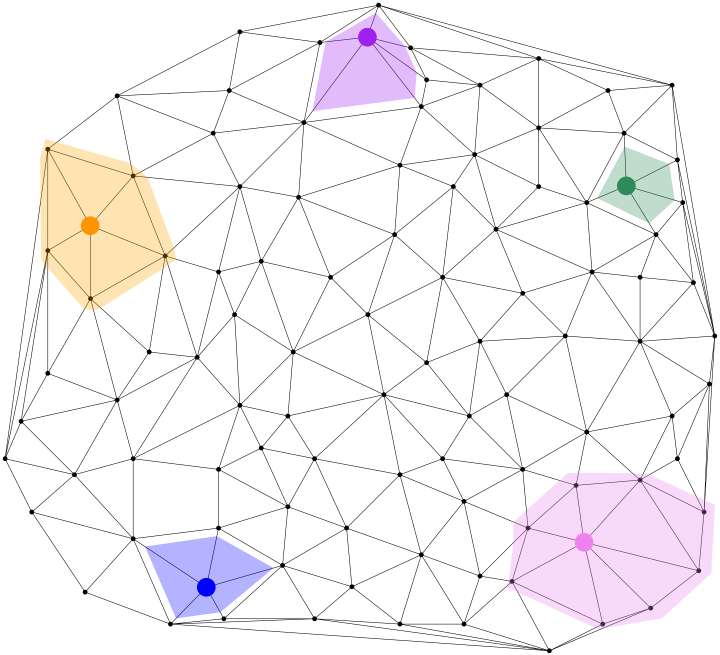
$R_1 = 0.9$

$R_2 = 1.05$

$R_3 = 0.85$

$R_4 = 0.7$

$R_5 = 1.1$



Ball Growing Algorithm

Arbitrary order.

Expand cluster in every round.

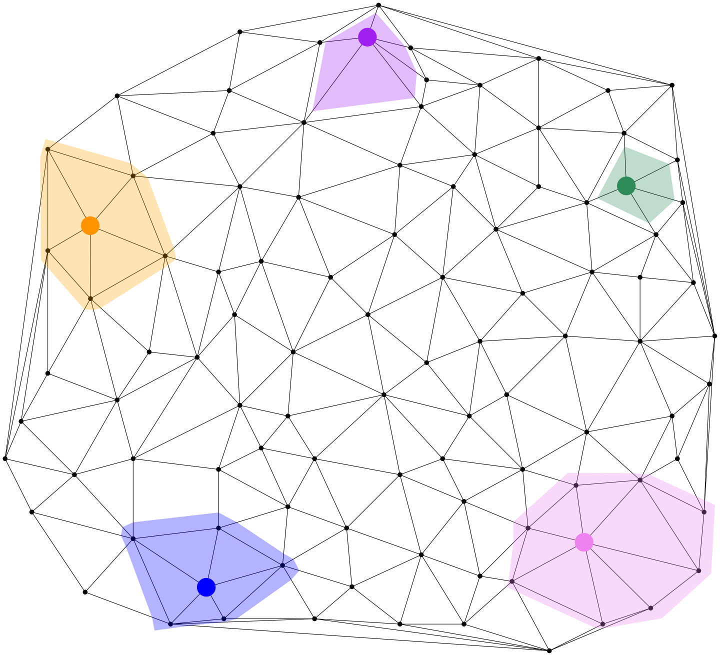
R_1 =1.1

R_2 =1.05

R_3 =0.85

R_4 =0.7

R_5 =1.1



Ball Growing Algorithm

Arbitrary order.

Expand cluster in every round.

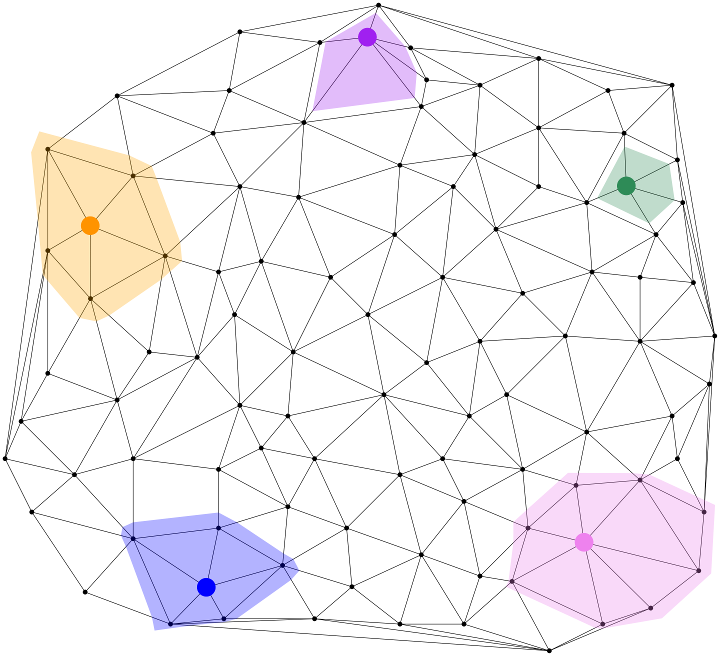
R_1 =1.1

R_2 =1.2

R_3 =0.85

R_4 =0.7

R_5 =1.1



Ball Growing Algorithm

Arbitrary order.

Expand cluster in every round.

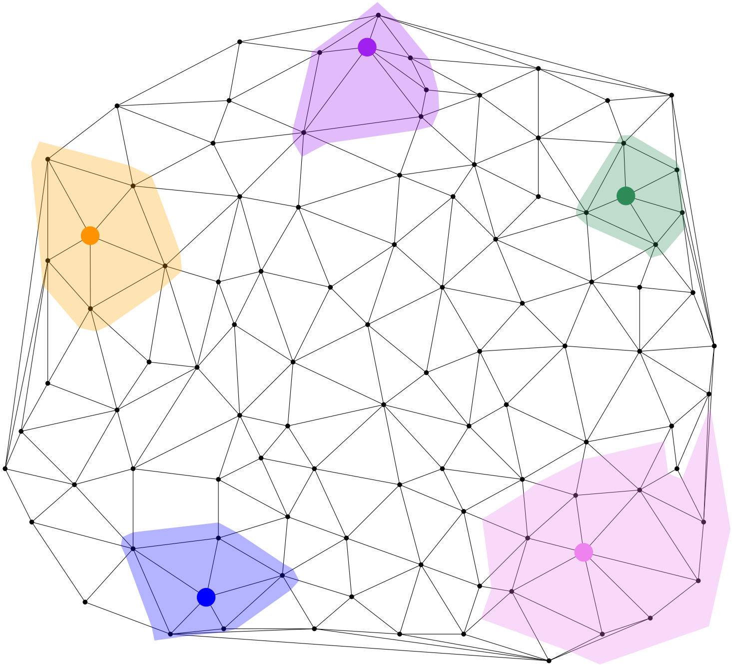
R_1 =1.1

R_2 =1.2

R_3 =1.1

R_4 =1.05

R_5 =1.9



Ball Growing Algorithm

Arbitrary order.

Expand cluster in every round.

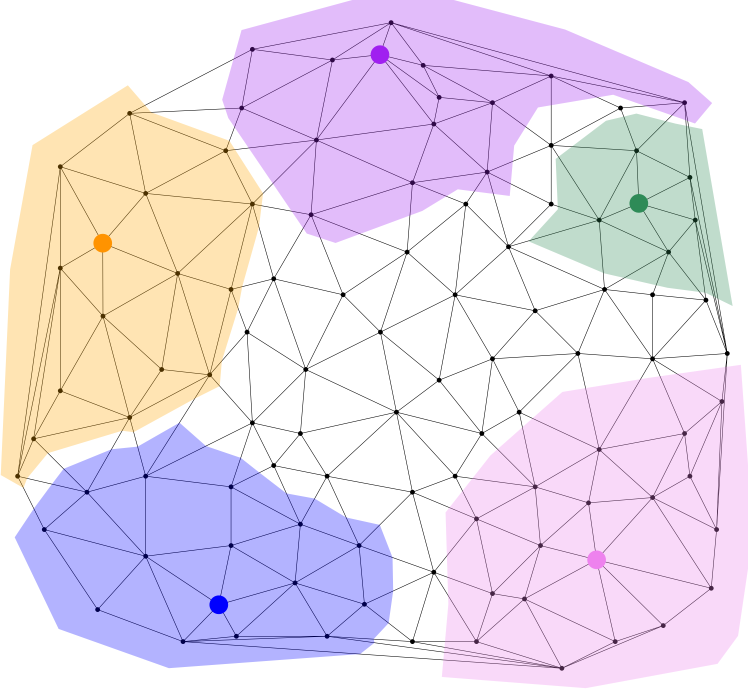
$R_1 = 2.5$

$R_2 = 2.2$

$R_3 = 2.3$

$R_4 = 1.8$

$R_5 = 2.8$



Ball Growing Algorithm

Arbitrary order.

Expand cluster in every round.

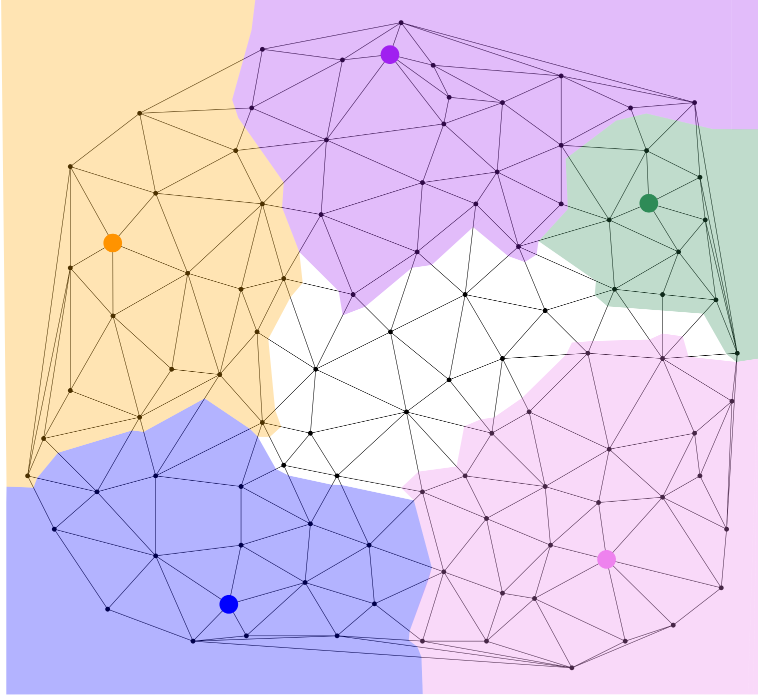
$R_1 = 2.9$

$R_2 = 3.2$

$R_3 = 3.15$

$R_4 = 2.2$

$R_5 = 3.2$



Ball Growing Algorithm

Arbitrary order.

Expand cluster in every round.

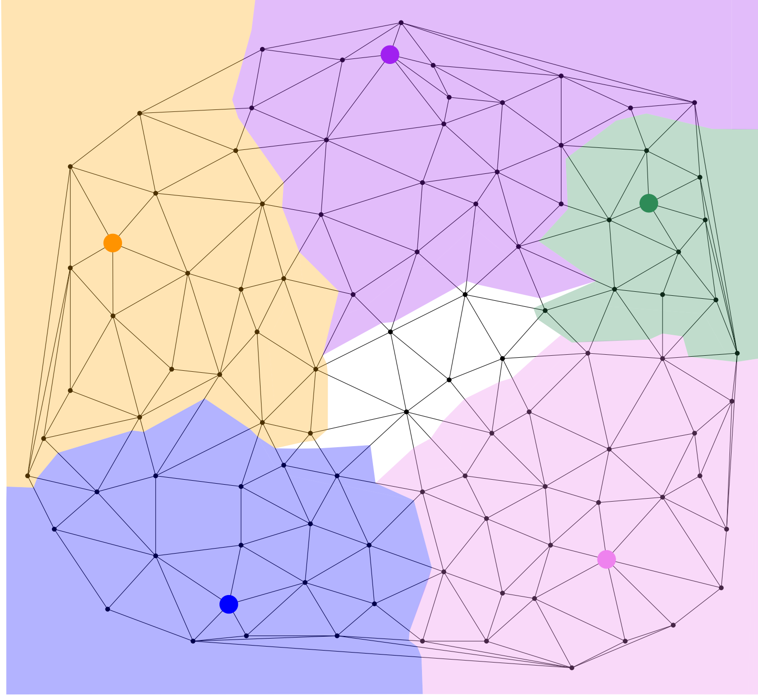
$R_1 = 3.4$

$R_2 = 4.1$

$R_3 = 3.8$

$R_4 = 3.1$

$R_5 = 3.6$



Ball Growing Algorithm

Arbitrary order.

Expand cluster in every round.

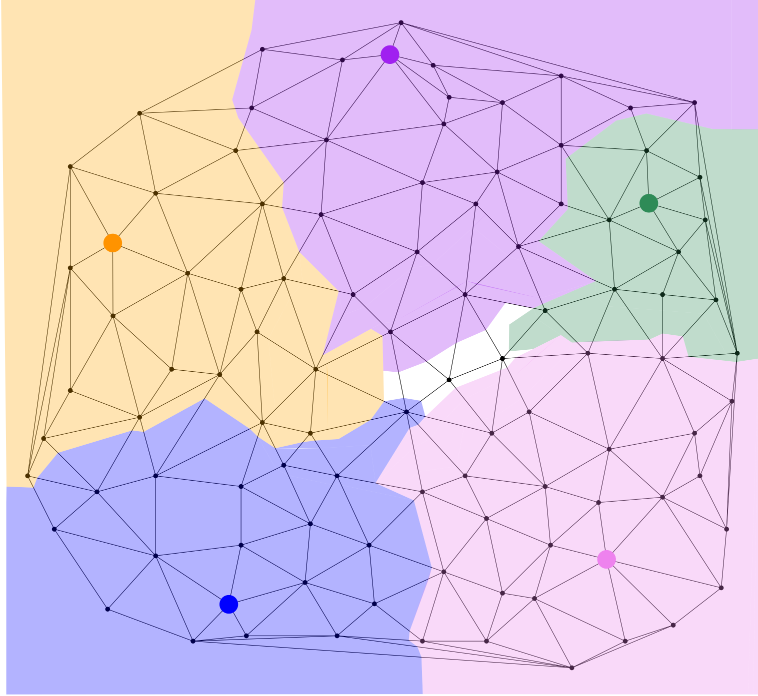
$R_1 = 4.2$

$R_2 = 4.8$

$R_3 = 4.5$

$R_4 = 3.7$

$R_5 = 3.8$



Ball Growing Algorithm

Arbitrary order.

Expand cluster in every round.

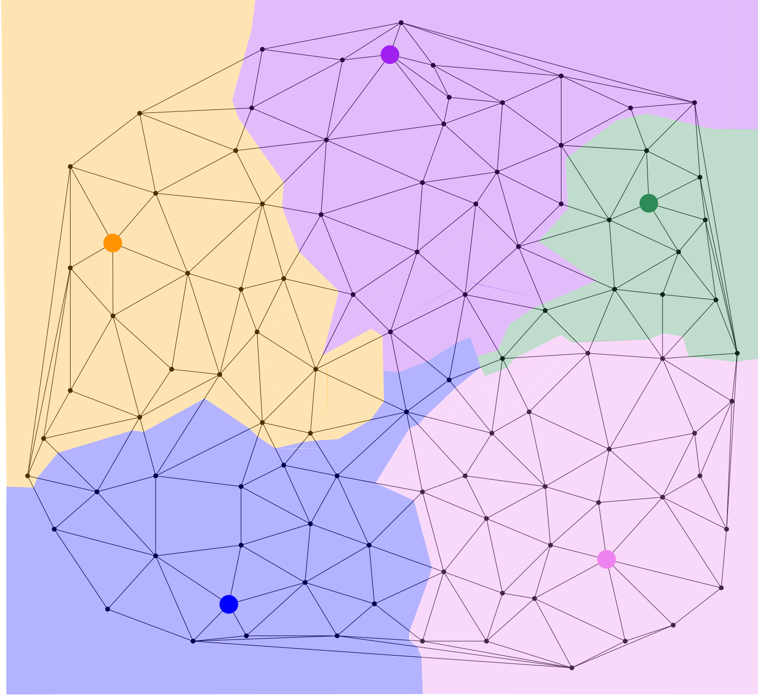
$R_1 = 5.5$

$R_2 = 6$

$R_3 = 4.9$

$R_4 = 4.5$

$R_5 = 5.1$



Ball Growing Algorithm

Arbitrary order.

Expand cluster in every round.

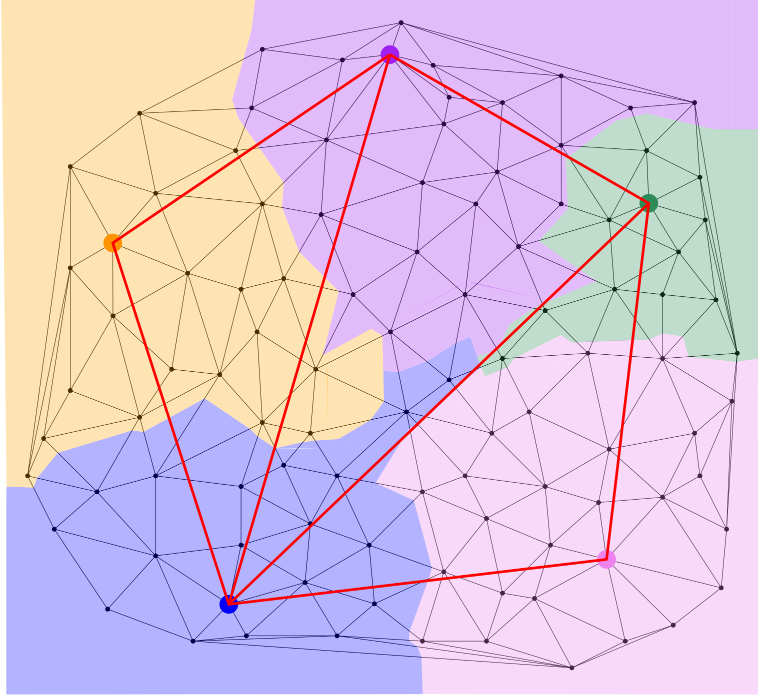
$R_1 = 5.5$

$R_2 = 6$

$R_3 = 4.9$

$R_4 = 4.5$

$R_5 = 5.1$



Ball Growing Algorithm

Arbitrary order.

Expand cluster in every round.

R_1 =5.5

R_2 =6

R_3 =4.9

R_4 = 4.5

R_5 =5.1

Noisy Voronoi

Algorithm 2 $M = \texttt{Noisy-Voronoi}(G = (V, E, w), K = \{t_1, \ldots, t_k\})$

Set $\delta = 1/20 \ln k$ and $p = \frac{1}{5}$.
Set $V_\perp \leftarrow V \setminus K$.
for j from 1 to k **do**
    Choose independently at random g_j distributed according to $\mathsf{Geo}(p)$.
    Set $R_j \leftarrow (1+\delta)^{g_j}$.
    Set $V_j \leftarrow \texttt{Create-Cluster}(G, V_\perp, t_j, R_j)$.
    Remove all the vertices in V_j from $V_\perp$.
end for
return the terminal-centered minor M of G induced by $V_1, \ldots, V_k$.

Noisy Voronoi

Algorithm 2 $M = \texttt{Noisy-Voronoi}(G = (V, E, w), K = \{t_1, \ldots, t_k\})$

```
Set δ = 1/20 ln k and p = 1/5.
Set V⊥ ← V \ K.
for j from 1 to k do
    Choose independently at random g_j distributed according to Geo(p)
    Set R_j ← (1 + δ)^{g_j}.
    Set V_j ← Create-Cluster(G, V⊥, t_j, R_j).
    Remove all the vertices in V_j from V⊥.
end for
return the terminal-centered minor M of G induced by V_1, ..., V_k.
```