Ramsey Spanning Trees and their Applications

Arnold Filtser

Ben-Gurion University

Co-authors: Ittai Abraham, Shiri Chechik, Michael Elkin, Ofer Neiman

25 April 2018
Metric Embeddings

$$(X, d_X)$$

$$(R^d, \|\cdot\|_2)$$

Embedding $f : X \rightarrow R^d$ has distortion $\alpha$ if for all $x, y \in X$

$$d_X(x, y) \leq \|f(x) - f(y)\|_2 \leq \alpha \cdot d_X(x, y)$$
Metric Embeddings

\[(X, d_X) \rightarrow (R^d, \|\cdot\|_2)\]

Embedding \(f : X \rightarrow R^d\) has distortion \(\alpha\) if for all \(x, y \in X\)

\[d_X(x, y) \leq \|f(x) - f(y)\|_2 \leq \alpha \cdot d_X(x, y)\]

**Theorem (Bourgain, 85)**

*Every \(n\)-point metric \((X, d_X)\) is embeddable into Euclidean space with distortion \(O(\log n)\).*
**Metric Embeddings**

A metric embedding $f : X \rightarrow \mathbb{R}^d$ has distortion $\alpha$ if for all $x, y \in X$

$$d_X(x, y) \leq \|f(x) - f(y)\|_2 \leq \alpha \cdot d_X(x, y)$$

*Theorem (Bourgain, 85)*

Every $n$-point metric $(X, d_X)$ is embeddable into Euclidean space with distortion $O(\log n)$.

Asymptotically tight.
Metric Ramsey-Type Problem

For a fixed distortion $k > 1$, what is the largest subset $M \subset X$, s.t. $(M, d_X)$ is embeddable into Euclidean space with distortion $k$?

\[(X, d_X) \rightarrow (R^d, \|\cdot\|_2)\]

\[f : X \rightarrow R^d\]
Metric Ramsey-Type Problem

For a fixed distortion $k > 1$, what is the largest subset $M \subset X$, s.t. $(M, d_X)$ is embeddable into Euclidean space with distortion $k$?

$M$ $(X, d_X)$

$(R^d, \|\cdot\|_2)$

$f : M \rightarrow R^d$

$\forall x, y \in M, \quad d_X(x, y) \leq \|f(x) - f(y)\|_2 \leq k \cdot d_X(x, y)$
Metric Ramsey-Type Problem

For a fixed distortion $k > 1$, what is the largest subset $M \subset X$, s.t. $(M, d_X)$ is embeddable into Euclidean space with distortion $k$?

Theorem (Mendel, Naor 07, following BFM86, BLMN05)

For every $n$-point metric space and $k \geq 1$, there exists a subset $M$ of size $n^{1-1/k}$ that can be embedded into Euclidean space with distortion $O(k)$. 

$$\forall x, y \in M, \quad d_X(x, y) \leq \|f(x) - f(y)\|_2 \leq k \cdot d_X(x, y)$$
Metric Ramsey-Type Problem

**Theorem (Mendel, Naor 07, following BFM86, BLMN05)**

For every $n$-point metric space and $k \geq 1$, there exists a subset $M$ of size $n^{1-1/k}$ that can be embedded into Euclidean space with distortion $O(k)$. Asymptotically tight. Euclidean space can be replaced here by an ultrametric $U$ (a.k.a. HST). Ultrametric is a spatial kind of tree which is:

1. Very useful for divide and conquer algorithms.
2. Isometrically embeds into Euclidean space (i.e. distortion 1).
Metric Ramsey-Type Problem

Theorem (Mendel, Naor 07, following BFM86, BLMN05)

For every $n$-point metric space and $k \geq 1$, there exists a subset $M$ of size $n^{1-1/k}$ that can be embedded into Euclidean space with distortion $O(k)$.

Asymptotically tight.
Metric Ramsey-Type Problem

Theorem (Mendel, Naor 07, following BFM86, BLMN05)

For every \( n \)-point metric space and \( k \geq 1 \), there exists a subset \( M \) of size \( n^{1-1/k} \) that can be embedded into Euclidean space with distortion \( O(k) \).

Asymptotically tight.

Euclidean space can be replace here by an ultrametric \( U \) (a.k.a HST)
Metric Ramsey-Type Problem

**Theorem (Mendel, Naor 07, following BFM86, BLMN05)**

For every $n$-point metric space and $k \geq 1$, there exists a subset $M$ of size $n^{1-1/k}$ that can be embedded into an ultrametric with distortion $O(k)$.

Asymptotically tight.

Euclidean space can be replace here by an ultrametric $U!$ (a.k.a HST)

Ultrametric is a spacial kind of tree which is:

1. Very useful for divide an conquer algorithms.
2. Isometrically embeds into Euclidean space (i.e. distortion 1).
Our Second Result: Metric Ramsey-Type Problem

Theorem (Mendel, Naor 07, following BFM86, BLMN05)

For every $n$-point metric space and $k \geq 1$, there exists a subset $M$ of size $n^{1-1/k}$ that can be embedded into ultrametric with distortion $O(k)$. 
Our Second Result: Metric Ramsey-Type Problem

Theorem (Mendel, Naor 07, following BFM86, BLMN05)

For every $n$-point metric space and $k \geq 1$, there exists a subset $M$ of size $n^{1-1/k}$ that can be embedded into ultrametric with distortion $O(128 \cdot k)$.

The constant in the distortion important as it in the exponent.
Our Second Result: Metric Ramsey-Type Problem

**Theorem (Mendel, Naor 07, following BFM86, BLMN05)**

For every \( n \)-point metric space and \( k \geq 1 \), there exists a subset \( M \) of size \( n^{1-1/k} \) that can be embedded into ultrametric with distortion \( O(128 \cdot k) \).

The constant in the distortion important as it in the exponent.

<table>
<thead>
<tr>
<th>Paper</th>
<th>Distortion</th>
<th>Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>BFM06</td>
<td>( O(k \log \log n) )</td>
<td>( n^{1-1/k} )</td>
</tr>
<tr>
<td>BLMN04</td>
<td>( O(k \log k) )</td>
<td>( n^{1-1/k} )</td>
</tr>
<tr>
<td>MN07</td>
<td>( 128 \cdot k )</td>
<td>( n^{1-1/k} )</td>
</tr>
<tr>
<td>BGS16</td>
<td>( 33 \cdot k )</td>
<td>( n^{1-1/k} )</td>
</tr>
<tr>
<td>NT12</td>
<td>( 2e \cdot k )</td>
<td>( n^{1-1/k} )</td>
</tr>
</tbody>
</table>
Our Second Result: Metric Ramsey-Type Problem

Theorem (Mendel, Naor 07, following BFM86, BLMN05)

For every $n$-point metric space and $k \geq 1$, there exists a subset $M$ of size $n^{1-1/k}$ that can be embedded into ultrametric with distortion $O(128 \cdot k)$.

The constant in the distortion important as it in the exponent.

<table>
<thead>
<tr>
<th>Paper</th>
<th>Distortion</th>
<th>Size</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>BFM06</td>
<td>$O(k \log \log n)$</td>
<td>$n^{1-1/k}$</td>
<td>Randomized</td>
</tr>
<tr>
<td>BLMN04</td>
<td>$O(k \log k)$</td>
<td>$n^{1-1/k}$</td>
<td>Randomized</td>
</tr>
<tr>
<td>MN07</td>
<td>$128 \cdot k$</td>
<td>$n^{1-1/k}$</td>
<td>Randomized</td>
</tr>
<tr>
<td>BGS16</td>
<td>$33 \cdot k$</td>
<td>$n^{1-1/k}$</td>
<td>Randomized</td>
</tr>
<tr>
<td>NT12</td>
<td>$2e \cdot k$</td>
<td>$n^{1-1/k}$</td>
<td>Randomized</td>
</tr>
</tbody>
</table>
Our Second Result: Metric Ramsey-Type Problem

Theorem (Mendel, Naor 07, following BFM86, BLMN05)

For every $n$-point metric space and $k \geq 1$, there exists a subset $M$ of size $n^{1-1/k}$ that can be embedded into an ultrametric with distortion $O(128 \cdot k)$.

The constant in the distortion important as it in the exponent.

<table>
<thead>
<tr>
<th>Paper</th>
<th>Distortion</th>
<th>Size</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>BFM06</td>
<td>$O(k \log \log n)$</td>
<td>$n^{1-1/k}$</td>
<td>Randomized</td>
</tr>
<tr>
<td>BLMN04</td>
<td>$O(k \log k)$</td>
<td>$n^{1-1/k}$</td>
<td>Randomized</td>
</tr>
<tr>
<td>MN07</td>
<td>$128 \cdot k$</td>
<td>$n^{1-1/k}$</td>
<td>Randomized</td>
</tr>
<tr>
<td>BGS16</td>
<td>$33 \cdot k$</td>
<td>$n^{1-1/k}$</td>
<td>Randomized</td>
</tr>
<tr>
<td>NT12</td>
<td>$2e \cdot k$</td>
<td>$n^{1-1/k}$</td>
<td>Randomized</td>
</tr>
<tr>
<td><strong>This Paper</strong></td>
<td><strong>$8 \cdot k - 2$</strong></td>
<td>$n^{1-1/k}$</td>
<td>Deterministic</td>
</tr>
</tbody>
</table>
Our Second Result: Metric Ramsey-Type Problem

Theorem (Mendel, Naor 07, following BFM86, BLMN05)

For every $n$-point metric space and $k \geq 1$, there exists a subset $M$ of size $n^{1-1/k}$ that can be embedded into ultrametric with distortion $O(128 \cdot k)$.

The constant in the distortion important as it in the exponent.

<table>
<thead>
<tr>
<th>Paper</th>
<th>Distortion</th>
<th>Size</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>BFM06</td>
<td>$O(k \log \log n)$</td>
<td>$n^{1-1/k}$</td>
<td>Randomized</td>
</tr>
<tr>
<td>BLMN04</td>
<td>$O(k \log k)$</td>
<td>$n^{1-1/k}$</td>
<td>Randomized</td>
</tr>
<tr>
<td>MN07</td>
<td>$128 \cdot k$</td>
<td>$n^{1-1/k}$</td>
<td>Randomized</td>
</tr>
<tr>
<td>BGS16</td>
<td>$33 \cdot k$</td>
<td>$n^{1-1/k}$</td>
<td>Randomized</td>
</tr>
<tr>
<td>NT12</td>
<td>$2e \cdot k$</td>
<td>$n^{1-1/k}$</td>
<td>Randomized</td>
</tr>
<tr>
<td><strong>This Paper</strong></td>
<td><strong>$8 \cdot k - 2$</strong></td>
<td><strong>$n^{1-1/k}$</strong></td>
<td><strong>Deterministic</strong></td>
</tr>
</tbody>
</table>

*Bartal had similar (deterministic) result.*
**Theorem (Our Secondary Result)**

For every $n$-point metric space and $k \geq 1$, there is a deterministic algorithm that finds a subset $M$ of size $n^{1-1/k}$ that can be embedded into ultrametric with distortion $8 \cdot k$. 

---

**Corollary**

For every $n$-point metric space and $k \geq 1$, there is a set $U$ of $k \cdot n^{1/k}$ ultrametrics and a mapping $\text{home} : X \to U$, such that for every $x, y \in U$,

$$d_{\text{home}}(x)(x, y) \leq (16 \cdot k) \cdot d_X(x, y).$$
Theorem (Our Secondary Result)

For every \( n \)-point metric space and \( k \geq 1 \), there is a deterministic algorithm that finds a subset \( M \) of size \( n^{1-1/k} \) that can be embedded into ultrametric with distortion \( 8 \cdot k \).

Instead of preserving distance for \( M \times M \), we can preserve distances for \( M \times X \).
For every $n$-point metric space and $k \geq 1$, there is a deterministic algorithm that finds a subset $M$ of size $n^{1-1/k}$ such that the hall metric can be embedded into ultrametric with distortion $16 \cdot k$ w.r.t $M \times X$.

Instead of preserving distance for $M \times M$, we can preserve distances for $M \times X$. 
Our Second Result: Metric Ramsey-Type Problem

Theorem (Our Secondary Result)

For every $n$-point metric space and $k \geq 1$, there is a deterministic algorithm that finds a subset $M$ of size $n^{1-1/k}$ such that the hall metric can be embedded into ultrametric with distortion $16 \cdot k$ w.r.t $M \times X$. 

Corollary

For every $n$-point metric space and $k \geq 1$, there is a set $U$ of $k \cdot n^{1/k}$ ultrametrics and a mapping $\phi: X \rightarrow U$, such that for every $x, y \in U$,

$$d_{\phi}(x)(x, y) \leq (16 \cdot k) \cdot d_X(x, y)$$
Our Second Result: Metric Ramsey-Type Problem

**Theorem (Our Secondary Result)**

For every $n$-point metric space and $k \geq 1$, there is a deterministic algorithm that finds a subset $M$ of size $n^{1-1/k}$ such that the hall metric can be embedded into ultrametric with distortion $16 \cdot k$ w.r.t $M \times X$.

**Corollary**

For every $n$-point metric space and $k \geq 1$, there is a set $U$ of $k \cdot n^{1/k}$ ultrametrics and a mapping $\text{home} : X \rightarrow U$, such that for every $x, y \in U$,

$$d_{\text{home}}(x, y) \leq (16 \cdot k) \cdot d_X(x, y)$$
Corollary

For every $n$-point metric space and $k \geq 1$, there is a set $\mathcal{U}$ of $k \cdot n^{1/k}$ ultrametrics and a mapping $\text{home} : X \rightarrow \mathcal{U}$, such that for every $x, y \in U$,

$$d_{\text{home}(x)}(x, y) \leq (16 \cdot k) \cdot d_X(x, y)$$
Distance Oracle

A distance oracle is a succinct data structure that (approximately) answers distance queries.
Distance Oracle

A distance oracle is a succinct data structure that (approximately) answers distance queries.

\[ \text{dist}(x, y) \approx d_x(x, y) \]
Distance Oracle

A distance oracle is a succinct data structure that (approximately) answers distance queries.

The properties of interest are size, distortion and query time.
## Distance Oracles: State of the Art

<table>
<thead>
<tr>
<th>DO</th>
<th>Distortion</th>
<th>Size</th>
<th>Query</th>
<th>Deterministic?</th>
</tr>
</thead>
<tbody>
<tr>
<td>TZ05</td>
<td>$2k - 1$</td>
<td>$O(k \cdot n^{1+1/k})$</td>
<td>$O(k)$</td>
<td>no</td>
</tr>
<tr>
<td>MN07</td>
<td>$128k$</td>
<td>$O(n^{1+1/k})$</td>
<td>$O(1)$</td>
<td>no</td>
</tr>
<tr>
<td>W13</td>
<td>$(2 + \epsilon)k$</td>
<td>$O(k \cdot n^{1+1/k})$</td>
<td>$O(1/\epsilon)$</td>
<td>no</td>
</tr>
<tr>
<td>C14</td>
<td>$2k - 1$</td>
<td>$O(k \cdot n^{1+1/k})$</td>
<td>$O(1)$</td>
<td>no</td>
</tr>
<tr>
<td>C15</td>
<td>$2k - 1$</td>
<td>$O(n^{1+1/k})$</td>
<td>$O(1)$</td>
<td>no</td>
</tr>
<tr>
<td>RTZ05</td>
<td>$2k - 1$</td>
<td>$O(k \cdot n^{1+1/k})$</td>
<td>$O(k)$</td>
<td>yes</td>
</tr>
<tr>
<td>W13</td>
<td>$2k - 1$</td>
<td>$O(k \cdot n^{1+1/k})$</td>
<td>$O(\log k)$</td>
<td>yes</td>
</tr>
</tbody>
</table>

![Diagram](image.png)
Our contribution: Deterministic Distance Oracles

<table>
<thead>
<tr>
<th>Distance Oracle</th>
<th>Distortion</th>
<th>Size</th>
<th>Query</th>
</tr>
</thead>
<tbody>
<tr>
<td>RTZ05</td>
<td>$2k - 1$</td>
<td>$O(k \cdot n^{1+1/k})$</td>
<td>$O(k)$</td>
</tr>
<tr>
<td>W13</td>
<td>$2k - 1$</td>
<td>$O(k \cdot n^{1+1/k})$</td>
<td>$O(\log k)$</td>
</tr>
<tr>
<td>This paper</td>
<td>$8(1 + \varepsilon)k$</td>
<td>$O(n^{1+1/k})$</td>
<td>$O(1/\varepsilon)$</td>
</tr>
<tr>
<td>This paper + C14</td>
<td>$2k - 1$</td>
<td>$O(k \cdot n^{1+1/k})$</td>
<td>$O(1)$</td>
</tr>
</tbody>
</table>
Our contribution: Deterministic Distance Oracles

Corollary

For every $n$-point metric space and $k \geq 1$, there is a set $\mathcal{U}$ of $k \cdot n^{1/k}$ ultrametrics and a mapping $\text{home} : X \rightarrow \mathcal{U}$, such that for every $x, y \in U$,

$$d_{\text{home}(x)}(x, y) \leq (16 \cdot k) \cdot d_X(x, y)$$
Our contribution: Deterministic Distance Oracles

Corollary

For every $n$-point metric space and $k \geq 1$, there is a set $\mathcal{U}$ of $k \cdot n^{\frac{1}{k}}$ ultrametrics and a mapping $\text{home} : X \rightarrow \mathcal{U}$, such that for every $x, y \in U$,

$$d_{\text{home}}(x, y) \leq (16 \cdot k) \cdot d_X(x, y)$$
Our contribution: Deterministic Distance Oracles

Corollary

For every $n$-point metric space and $k \geq 1$, there is a set $\mathcal{U}$ of $k \cdot n^{1/k}$ ultrametrics and a mapping $\text{home} : X \rightarrow \mathcal{U}$, such that for every $x, y \in U$,

$$d_{\text{home}(x)}(x, y) \leq (16 \cdot k) \cdot d_X(x, y)$$

Theorem (Tree Distance Oracle, HT84, BFC00)

For every tree metric, there is an exact distance oracle of linear size and constant query time.
Our contribution: Deterministic Distance Oracles

**Corollary**

For every $n$-point metric space and $k \geq 1$, there is a set $\mathcal{U}$ of $k \cdot n^{\frac{1}{k}}$ ultrametrics and a mapping $\text{home} : X \rightarrow \mathcal{U}$, such that for every $x, y \in U$,

$$d_{\text{home}}(x, y) \leq (16 \cdot k) \cdot d_X(x, y)$$

**Theorem (Tree Distance Oracle, HT84, BFC00)**

For every tree metric, there is an exact distance oracle of linear size and constant query time.

**Theorem (Ramsey based Deterministic Distance Oracle)**

For any $n$-point metric space, there is a distance oracle with:

<table>
<thead>
<tr>
<th>Distortion</th>
<th>Size</th>
<th>Query time</th>
</tr>
</thead>
<tbody>
<tr>
<td>$16 \cdot k$</td>
<td>$O(k \cdot n^{1+1/k})$</td>
<td>$O(1)$</td>
</tr>
</tbody>
</table>
Our contribution: Deterministic Distance Oracles

**Corollary**

For every $n$-point metric space and $k \geq 1$, there is a set $U$ of $k \cdot n^{1/k}$ ultrametrics and a mapping $\text{home} : X \rightarrow U$, such that for every $x, y \in U$,

$$d_{\text{home}_x}(x, y) \leq (16 \cdot k) \cdot d_X(x, y)$$

**Theorem (Ramsey based Deterministic Distance Oracle)**

For any $n$-point metric space, there is a distance oracle with:

<table>
<thead>
<tr>
<th>Distance Oracle</th>
<th>Distortion</th>
<th>Size</th>
<th>Query</th>
</tr>
</thead>
<tbody>
<tr>
<td>This paper</td>
<td>$16 \cdot k$</td>
<td>$O(k \cdot n^{1+1/k})$</td>
<td>$O(1)$</td>
</tr>
<tr>
<td>This paper</td>
<td>$8(1 + \epsilon)k$</td>
<td>$O(n^{1+1/k})$</td>
<td>$O(1/\epsilon)$</td>
</tr>
<tr>
<td>This paper + C14</td>
<td>$2k - 1$</td>
<td>$O(k \cdot n^{1+1/k})$</td>
<td>$O(1)$</td>
</tr>
</tbody>
</table>
Our contribution: Deterministic Distance Oracles

Corollary

For every \( n \)-point metric space and \( k \geq 1 \), there is a set \( U \) of \( k \cdot n^{\frac{1}{k}} \) ultrametrics and a mapping \( \text{home} : X \rightarrow U \), such that for every \( x, y \in U \),

\[
d_{\text{home}(x)}(x, y) \leq (16 \cdot k) \cdot d_X(x, y)
\]

Theorem (Ramsey based Deterministic Distance Oracle)

For any \( n \)-point metric space, there is a distance oracle with:

<table>
<thead>
<tr>
<th>Distance Oracle</th>
<th>Distortion</th>
<th>Size</th>
<th>Query</th>
</tr>
</thead>
<tbody>
<tr>
<td>This paper</td>
<td>16 \cdot k</td>
<td>( O(k \cdot n^{1+1/k}) )</td>
<td>( O(1) )</td>
</tr>
<tr>
<td>This paper</td>
<td>8((1 + \epsilon)k)</td>
<td>( O(n^{1+1/k}) )</td>
<td>( O(1/\epsilon) )</td>
</tr>
<tr>
<td>This paper + C14</td>
<td>2(k - 1)</td>
<td>( O(k \cdot n^{1+1/k}) )</td>
<td>( O(1) )</td>
</tr>
<tr>
<td>C15 (Randomized)</td>
<td>2(k - 1)</td>
<td>( O(n^{1+1/k}) )</td>
<td>( O(1) )</td>
</tr>
</tbody>
</table>
Given a weighted graph $G = (V, E, w)$, and a fixed distortion $k > 1$, what is the largest subset $M \subset V$, such that: there is a spanning tree $T$ of $G$ with distortion $k$ w.r.t $M \times V$?
Given a weighted graph $G = (V, E, w)$, and a fixed distortion $k > 1$, what is the largest subset $M \subset V$, such that: there is a spanning tree $T$ of $G$ with distortion $k$ w.r.t $M \times V$?

For all $v \in M$ and $u \in V$, $d_T(v, u) \leq k \cdot d_G(v, u)$. 
## Main Result

### Ramsey Spanning Tree Question

Given a weighted graph \( G = (V, E, w) \), and a fixed distortion \( k > 1 \), what is the largest subset \( M \subset V \), such that:

there is a spanning tree \( T \) of \( G \) with distortion \( k \) w.r.t \( M \times V \)?

### Theorem (Main Result)

For every \( n \)-vertex weighted graph \( G = (V, E, w) \) and \( k \geq 1 \), there exists a subset \( M \) of size \( n^{1 - 1/k} \) and spanning tree \( T \) of \( G \) with distortion \( O(k \cdot \log \log n) \) w.r.t \( M \times V \).
Main Result

Ramsey Spanning Tree Question

Given a weighted graph \( G = (V, E, w) \), and a fixed distortion \( k > 1 \), what is the largest subset \( M \subset V \), such that:
there is a spanning tree \( T \) of \( G \) with distortion \( k \) w.r.t \( M \times V \)?

Theorem (Main Result)

For every \( n \)-vertex weighted graph \( G = (V, E, w) \) and \( k \geq 1 \), there exists a subset \( M \) of size \( n^{1-1/k} \) and spanning tree \( T \) of \( G \) with distortion \( O(k \cdot \log \log n) \) w.r.t \( M \times V \).

Theorem (Mendel, Naor 07)

For every \( n \)-point metric space \((X, d_X)\) and \( k \geq 1 \), there exists a subset \( M \) of size \( n^{1-1/k} \) and an ultrametric \( U \) over \( X \) with distortion \( O(k) \) w.r.t \( M \times X \).
Theorem (Main Result)

For every $n$-vertex weighted graph $G = (V, E, w)$ and $k \geq 1$, there exists a subset $M$ of size $n^{1-1/k}$ and spanning tree $T$ of $G$ with distortion $O(k \cdot \log \log n)$ w.r.t $M \times V$. 
Theorem (Main Result)

For every $n$-vertex weighted graph $G = (V, E, w)$ and $k \geq 1$, there exists a subset $M$ of size $n^{1-1/k}$ and spanning tree $T$ of $G$ with distortion $O(k \cdot \log \log n)$ w.r.t $M \times V$.

Corollary

For every $n$-vertex weighted graph $G = (V, E, w)$ and $k \geq 1$, there is a set $\mathcal{T}$ of $k \cdot n^{1/k}$ spanning trees and a mapping $\text{home} : V \to \mathcal{T}$, such that for every $u, v \in V$,

$$d_{\text{home}(v)}(v, u) \leq O(k \cdot \log \log n) \cdot d_G(v, u)$$
Corollary

For every $n$-vertex weighted graph $G = (V, E, w)$ and $k \geq 1$, there is a set $\mathcal{T}$ of $k \cdot \frac{n}{k}$ spanning trees and a mapping $\text{home} : V \rightarrow \mathcal{T}$, such that for every $u, v \in V$,

$$d_{\text{home}}(v)(v, u) \leq O(k \cdot \log \log n) \cdot d_G(v, u)$$

The union of all the trees in $\mathcal{T}$ creates an $O(k \cdot \log \log n)$-spanner with $O(k \cdot n^{1+\frac{1}{k}})$ edges.
Application: Compact Routing Scheme

- Huge network
Application: Compact Routing Scheme

- **Huge** network
- There is a **server** in each **node**.
Application: Compact Routing Scheme

- **Huge** network
- There is a **server** in each **node**.
- Task: **route** packages throughout the network.
**Application: Compact Routing Scheme**

- **Huge** network
- There is a **server** in each **node**.
- Task: **route** packages throughout the network.
- **Store** the whole network in each node is **unfeasible**.
Compact Routing Scheme

Destination
Label
Content

Initiating node
**Compact Routing Scheme**

<table>
<thead>
<tr>
<th>Label</th>
<th>Content</th>
</tr>
</thead>
<tbody>
<tr>
<td>00101</td>
<td></td>
</tr>
<tr>
<td>11000</td>
<td></td>
</tr>
<tr>
<td>10010</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

[Diagram](#) showing the flow from an initiating node to a destination node through a routing table.

- **Initiating node**
- **Routing Table**
  - 00101
  - 11000
  - 10010
  - ...

[Diagram details](#) include symbols for nodes and arrows indicating the path of data.
Compact Routing Scheme

Initiating node

Destination Label Content

Label Content

Routing Table
00101
11000
10010
...

Decision time

Routing Decision time
Compact Routing Scheme

<table>
<thead>
<tr>
<th>Label</th>
<th>Content</th>
<th>Destination</th>
<th>Initiating node</th>
</tr>
</thead>
<tbody>
<tr>
<td>00101</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10010</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>. . . . . .</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Routing Table
00101
11000
10010
. . . . . .

Decision time
## Compact Routing Scheme

In order to keep other **parameters small**, we will allow **stretch**.

<table>
<thead>
<tr>
<th>Label</th>
<th>Content</th>
<th>Destination</th>
</tr>
</thead>
<tbody>
<tr>
<td>00101</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10010</td>
<td></td>
<td></td>
</tr>
<tr>
<td>...</td>
<td></td>
<td>...</td>
</tr>
</tbody>
</table>
In order to keep other parameters small, we will allow stretch. Stretch $k$: the length of a route from $v$ to $u$ will be $\leq k \cdot d_G(v, u)$. 
In order to keep other parameters small, we will allow stretch. Stretch $k$: the length of a route from $v$ to $u$ will be $\leq k \cdot d_G(v, u)$.

**Theorem (Thorup, Zwick, 01)**

For any $n$-vertex tree $T = (V, E)$, there is a routing scheme with:

<table>
<thead>
<tr>
<th>Stretch</th>
<th>Label</th>
<th>Table</th>
<th>Decision time</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$O(\log n)$</td>
<td>$O(1)$</td>
<td>$O(1)$</td>
</tr>
</tbody>
</table>
For every $n$-vertex weighted graph $G = (V, E, w)$ and $k \geq 1$, there is a set $\mathcal{T}$ of $k \cdot n^{1/k}$ spanning trees and a mapping $\text{home} : V \rightarrow \mathcal{T}$, such that for every $u, v \in V$,

$$d_{\text{home}(v)}(v, u) \leq O(k \cdot \log \log n) \cdot d_G(v, u)$$

To route a package from $u$ to $v$, we will simply route on $\text{home}(v)$! The label of $v$ will consist of: $(\text{home}(v), \text{Label}_{\text{home}(v)}(v))$. The table of $v$ will consist of union of all tables in $T$. $G$ $T_1$ $T_i = \text{home}(v)$ $T_{k \cdot n^{1/k}}$
Routing using Ramsey Spanning Trees

For every $n$-vertex weighted graph $G = (V, E, w)$ and $k \geq 1$, there is a set $\mathcal{T}$ of $k \cdot n^{1/k}$ spanning trees and a mapping $\text{home} : V \rightarrow \mathcal{T}$, such that for every $u, v \in V$,

$$d_{\text{home}(v)}(v, u) \leq O(k \cdot \log \log n) \cdot d_G(v, u)$$

To route a package from $u$ to $v$, we will simply route on $\text{home}(v)$!
Routing using Ramsey Spanning Trees

For every $n$-vertex weighted graph $G = (V, E, w)$ and $k \geq 1$, there is a set $\mathcal{T}$ of $k \cdot n^{1/k}$ spanning trees and a mapping $\text{home} : V \to \mathcal{T}$, such that for every $u, v \in V$,

$$d_{\text{home}(v)}(v, u) \leq O(k \cdot \log \log n) \cdot d_G(v, u)$$

To route a package from $u$ to $v$, we will simply route on $\text{home}(v)$!

The label of $v$ will consist of: $\left(\text{home}(v), \text{Label}_{\text{home}(v)}(v)\right)$.
Routing using Ramsey Spanning Trees

For every $n$-vertex weighted graph $G = (V, E, w)$ and $k \geq 1$, there is a set $\mathcal{T}$ of $k \cdot n^{\frac{1}{k}}$ spanning trees and a mapping $\text{home} : V \rightarrow \mathcal{T}$, such that for every $u, v \in V$,

$$d_{\text{home}(v)}(v, u) \leq O(k \cdot \log \log n) \cdot d_G(v, u)$$

To route a package from $u$ to $v$, we will simply route on $\text{home}(v)$! The label of $v$ will consist of: $(\text{home}(v), \text{Label}_{\text{home}(v)}(v))$. The table of $v$ will consist of union of all tables in $\mathcal{T}$.
Routing using Ramsey Spanning Trees

For every $n$-vertex weighted graph $G = (V, E, w)$ and $k \geq 1$, there is a set $\mathcal{T}$ of $k \cdot n^{\frac{1}{k}}$ spanning trees and a mapping $\text{home} : V \rightarrow \mathcal{T}$, such that for every $u, v \in V$,

$$d_{\text{home}(v)}(v, u) \leq O(k \cdot \log \log n) \cdot d_G(v, u)$$

To route a package from $u$ to $v$, we will simply route on $\text{home}(v)$!

The label of $v$ will consist of: $(\text{home}(v), \text{Label}_{\text{home}(v)}(v))$.

The table of $v$ will consist of union of all tables in $\mathcal{T}$.

Theorem (Ramsey based Compact Routing Scheme)

For any $n$-vertex graph, there is a routing scheme with:

<table>
<thead>
<tr>
<th>Stretch</th>
<th>Label</th>
<th>Table</th>
<th>Decision time</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O(k \cdot \log \log n)$</td>
<td>$O(\log n)$</td>
<td>$O(k \cdot n^{\frac{1}{k}})$</td>
<td>$O(1)$</td>
</tr>
</tbody>
</table>
**Theorem (Ramsey based Compact Routing Scheme)**

For any \( n \)-vertex graph, there is a routing scheme with:

<table>
<thead>
<tr>
<th>Stretch</th>
<th>Label</th>
<th>Table</th>
<th>Decision time</th>
</tr>
</thead>
<tbody>
<tr>
<td>( O(k \cdot \log \log n) )</td>
<td>( O(\log n) )</td>
<td>( O(k \cdot n^{\frac{1}{k}}) )</td>
<td>( O(1) )</td>
</tr>
</tbody>
</table>

**Theorem (Thorup, Zwick 01, Chechik 13)**

For any \( n \)-vertex graph, there is a routing scheme with:

<table>
<thead>
<tr>
<th>Stretch</th>
<th>Label</th>
<th>Table</th>
<th>Decision time</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 3.68k = O(k) )</td>
<td>( O(k \cdot \log n) )</td>
<td>( O(k \cdot n^{\frac{1}{k}}) )</td>
<td>( O(1) ) (initial: ( O(k) ))</td>
</tr>
</tbody>
</table>
Theorem (Ramsey based Compact Routing Scheme)

For any $n$-vertex graph, there is a routing scheme with:

<table>
<thead>
<tr>
<th>Stretch</th>
<th>Label</th>
<th>Table</th>
<th>Decision time</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O(k \cdot \log \log n)$</td>
<td>$O(\log n)$</td>
<td>$O(k \cdot n^{\frac{1}{k}})$</td>
<td>$O(1)$</td>
</tr>
</tbody>
</table>

Theorem (Thorup, Zwick 01, Chechik 13)

For any $n$-vertex graph, there is a routing scheme with:

<table>
<thead>
<tr>
<th>Stretch</th>
<th>Label</th>
<th>Table</th>
<th>Decision time</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3.68k = O(k)$</td>
<td>$O(k \cdot \log n)$</td>
<td>$O(k \cdot n^{\frac{1}{k}})$</td>
<td>$O(1)$ (initial: $O(k)$)</td>
</tr>
</tbody>
</table>

By choosing $k = \log n$, we get:

<table>
<thead>
<tr>
<th></th>
<th>Stretch</th>
<th>Label</th>
<th>Table</th>
<th>D. time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Here</td>
<td>$O(\log n \cdot \log \log n)$</td>
<td>$O(\log n)$</td>
<td>$O(\log n)$</td>
<td>$O(1)$</td>
</tr>
<tr>
<td>[TZ01]</td>
<td>$O(\log n)$</td>
<td>$O(\log^2 n)$</td>
<td>$O(\log n)$</td>
<td>$O(1)$ ($O(\log n)$)</td>
</tr>
</tbody>
</table>
Theorem (Main Result)

For every $n$-vertex weighted graph $G = (V, E, w)$ and $k \geq 1$, there exists a subset $M$ of size $n^{1-1/k}$ and spanning tree $T$ of $G$ with distortion $O(k \cdot \log \log n)$ w.r.t $M \times V$. 

Theorem (Main Result)

For every $n$-vertex weighted graph $G = (V, E, w)$ and $k \geq 1$, there exists a subset $M$ of size $n^{1-1/k}$ and spanning tree $T$ of $G$ with distortion $O(k \cdot \log \log n)$ w.r.t $M \times V$.

- Framework: Petal decomposition.
Technical Ideas

Theorem (Main Result)

For every \( n \)-vertex weighted graph \( G = (V, E, w) \) and \( k \geq 1 \), there exists a subset \( M \) of size \( n^{1 - 1/k} \) and spanning tree \( T \) of \( G \) with distortion \( O(k \cdot \log \log n) \) w.r.t \( M \times V \).

- Framework: Petal decomposition.
- Hierarchically padded decompositions.
Theorem (Main Result) 

For every $n$-vertex weighted graph $G = (V, E, w)$ and $k \geq 1$, there exists a subset $M$ of size $n^{1-1/k}$ and spanning tree $T$ of $G$ with distortion $O(k \cdot \log \log n)$ w.r.t $M \times V$.

- Framework: Petal decomposition.
- Hierarchically padded decompositions.
- Region growing.
Petal Decomposition
Petal Decomposition

\[ X_1 \]

\[ y_1 = t_0 \]
Petal Decomposition
Petal Decomposition
Petal Decomposition
Petal Decomposition
Petal Decomposition
Petal Decomposition

\[ y_1 = t_0 \]

\[ X_0 \]

\[ X_1 \]

\[ X_2 \]

\[ X_3 \]

\[ X_4 \]

\[ X_5 \]

\[ X_6 \]

\[ X_7 \]

\[ X_8 \]
Petal Decomposition
Petal Decomposition

\[ y_1 = t_0 \]

\[ y_2, y_3, y_4, y_5, y_6, y_7, y_8 \]

\[ x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \]

\[ X_0, X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8 \]
Petal Decomposition
Petal Decomposition

- Each cluster $X$ (petal) has a center vertex $x$. 

![Diagram of Petal Decomposition]
Petal Decomposition

- Each cluster $X$ (petal) has a center vertex $x$.
- The radius $\Delta$ defined w.r.t the center.
Petal Decomposition

- Each cluster $X$ (petal) has a center vertex $x$.
- The radius $\Delta$ defined w.r.t the center.
- The radius decrease by $\frac{3}{4}$ factor in each hierarchy step.
Petal Decomposition

- Each cluster $X$ (petal) has a center vertex $x$.
- The radius $\Delta$ defined w.r.t the center.
- The radius decrease by $\frac{3}{4}$ factor in each hierarchy step.
- The radius of $T$ is at most 4 times larger than in $G$. 
Petal Decomposition

- Each cluster $X$ (petal) has a center vertex $x$.
- The radius $\Delta$ defined w.r.t to the center.
- The radius decreases by $\frac{3}{4}$ factor in each hierarchical step.
- The radius of $T$ is at most 4 times larger than in $G$.

**Corollary**

Suppose $v, u$ were separated while being in cluster of radius $\Delta$. Then $d_T(v, u) \leq 8 \cdot \Delta$. 
Petal Growth

Degree of freedom:
parameter $R \in [\text{lo}, \text{hi}]$ ($\text{hi} - \text{lo} = \frac{\Delta}{8}$).
Petal Growth

Degree of freedom: parameter $R \in [lo, hi]$ ($hi - lo = \frac{A}{8}$).

$W_r$ denotes the petal (cluster) created for $R = r$.

Monotonicity: $r' \leq r \Rightarrow W_{r'} \subseteq W_r$. 

Arnold Filtser

Ramsey Spanning Trees and their Applications
Petal Growth

Degree of freedom:

Parameter $R \in [\text{lo}, \text{hi}]$ ($\text{hi} - \text{lo} = \frac{\Delta}{8}$).

$W_r$ denotes the petal (cluster) created for $R = r$.

Monotonicity: $r' \leq r \Rightarrow W_{r'} \subseteq W_r$.

Set $\delta = \Delta / (k \cdot \log \log n)$.
Petal Growth

Degree of freedom: parameter $R \in [\text{lo}, \text{hi}]$ (hi – lo = $\frac{\Delta}{8}$).

$W_r$ denotes the petal (cluster) created for $R = r$.

Monotonicity: $r' \leq r \Rightarrow W_{r'} \subseteq W_r$.

Set $\delta = \Delta / (k \cdot \log \log n)$.

Vertex $v$ s.t. $B(v, \delta) \subseteq W_r$ is padded.
Petal Growth

Degree of freedom:
parameter \( R \in [lo, hi] \) (hi − lo = \( \frac{\Delta}{8} \)).

\( W_r \) denotes the **petal** (cluster) created for \( R = r \).

Monotonicity: \( r' \leq r \Rightarrow W_{r'} \subseteq W_r \).

Set \( \delta = \frac{\Delta}{(k \cdot \log \log n)} \).

Vertex \( v \) s.t. \( B(v, \delta) \subseteq W_r \) is **padded**.

All vertices out of \( W_{r+\delta} \setminus W_{r-\delta} \) (restricted area) are padded.
Degree of freedom:
parameter $R \in [lo, hi]$ ($hi - lo = \frac{\Delta}{8}$).

$W_r$ denotes the **petal** (cluster) created for $R = r$.

Monotonicity: $r' \leq r \Rightarrow W_{r'} \subseteq W_r$.

Set $\delta = \Delta / (k \cdot \log \log n)$.

Vertex $v$ s.t. $B(v, \delta) \subseteq W_r$ is **padded**.

All vertices out of $W_{r+\delta} \setminus W_{r-\delta}$ (restricted area) are padded.

Padded vertices suffer distortion **at most** $\Delta / \delta = O(k \cdot \log \log n)$!
Petal Growth

Degree of freedom: parameter $R \in [\text{lo}, \text{hi}]$ ($\text{hi} - \text{lo} = \frac{\Delta}{8}$).

$W_r$ denotes the petal (cluster) created for $R = r$.

Monotonicity: $r' \leq r \Rightarrow W_{r'} \subseteq W_r$.

Set $\delta = \Delta / (k \cdot \log \log n)$.

Vertex $v$ s.t. $B(v, \delta) \subseteq W_r$ is padded.

All vertices out of $W_{r+\delta} \setminus W_{r-\delta}$ (restricted area) are padded.

Padded vertices suffer distortion at most $\Delta / \delta = O(k \cdot \log \log n)$!

Goal: find $r$, with many padded vertices! (sparse restricted area).
Petal Decomposition
Petal Decomposition

A vertex which is **padded in all the levels** will have small distortion w.r.t all other vertices.
Petal Decomposition

A vertex which is **padded in all the levels** will have small distortion w.r.t all other vertices.

**Goal**: choose parameters \( r \in [lo, hi] \) s.t. at least \( n^{1 - \frac{1}{k}} \) vertices will be **padded in all levels**.
Petal Decomposition

A vertex which is **padded in all the levels** will have small distortion w.r.t all other vertices.

**Goal:** choose parameters \((r \in [lo, hi])\) s.t. at least \(n^{1 - \frac{1}{k}}\) vertices will be **padded in all levels**.

A vertex is called **active** if it is **padded** in all levels up till now.
Region Growing

For petal $W_r$:

Active $x \in W_{r-\delta}$ remains active.

Active $x \in W_{r+\delta} \setminus W_{r-\delta}$ ceases to be active.
Region Growing

For petal $W_r$:

Active $x \in W_{r-\delta}$ remains active.

Active $x \in W_{r+\delta} \setminus W_{r-\delta}$ ceases to be active.
Region Growing

For petal $W_r$:

Active $x \in W_{r-\delta}$ remains active.

Active $x \in W_{r+\delta} \setminus W_{r-\delta}$ ceases to be active.
Region Growing

For petal $W_r$:

Active $x \in W_{r-\delta}$ remains active.

Active $x \in W_{r+\delta} \setminus W_{r-\delta}$ ceases to be active.
Region Growing

For petal $W_r$:

Active $x \in W_{r-\delta}$ remains active.

Active $x \in W_{r+\delta} \setminus W_{r-\delta}$ ceases to be active.

Intuition

There is $r \in [lo, hi]$ such that $W_{r-\delta}$ is large enough compared to $W_{r+\delta}$.
Intuition

There is $r \in [lo, hi]$ such that $W_{r-\delta}$ is large enough compared to $W_{r+\delta}$.
Intuition

There is $r \in [lo, hi]$ such that $W_{r-\delta}$ is large enough compared to $W_{r+\delta}$.
**Intuition**

*There is* \( r \in [lo, hi] \) *such that* \( W_{r-\delta} \) *is large enough compared to* \( W_{r+\delta} \).

**Corollary**

*At least* \( n^{1-1/k} \) *vertices remain active at the end of the process.*
Intuition

There is $r \in [lo, hi]$ such that $W_{r-\delta}$ is large enough compared to $W_{r+\delta}$.

Corollary

At least $n^{1-1/k}$ vertices remain active at the end of the process.

Theorem (Main Result)

For every $n$-vertex weighted graph $G = (V, E, w)$ and $k \geq 1$, there exists a subset $M$ of size $n^{1-1/k}$ and spanning tree $T$ of $G$ with distortion $O(k \cdot \log \log n)$ w.r.t $M \times V.$
Open Questions

1. **Remove** the log log $n$ factor.

Conjecture

For every $n$-vertex weighted graph $G = (V, E, w)$ and $k \geq 1$, there exists a subset $M$ of size $n^{1-1/k}$ and spanning tree $T$ of $G$ with distortion $O(k \cdot \log \log n)$ w.r.t $M \times V$. 
Open Questions

1. **Remove** the log log $n$ factor.

Conjecture

For every $n$-vertex weighted graph $G = (V, E, w)$ and $k \geq 1$, there exists a subset $M$ of size $n^{1-1/k}$ and spanning tree $T$ of $G$ with distortion $O(k \cdot \log \log n)$ w.r.t $M \times V$.

2. **Improve construction** for deterministic distance oracle.

<table>
<thead>
<tr>
<th>Distance Oracle</th>
<th>Distortion</th>
<th>Size</th>
<th>Query</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>This paper</strong> + C14</td>
<td>$2k - 1$</td>
<td>$O(k \cdot n^{1+1/k})$</td>
<td>$O(1)$</td>
</tr>
<tr>
<td>C15 (Randomized)</td>
<td>$2k - 1$</td>
<td>$O(n^{1+1/k})$</td>
<td>$O(1)$</td>
</tr>
</tbody>
</table>
Open Questions

1. **Remove** the log log $n$ factor.

Conjecture

For every $n$-vertex **weighted graph** $G = (V, E, w)$ and $k \geq 1$, there exists a **subset** $M$ of size $n^{1-1/k}$ and **spanning tree** $T$ of $G$ with **distortion** $O(k \cdot \text{log log } n)$ w.r.t $M \times V$.

2. Improve construction for deterministic distance oracle.

<table>
<thead>
<tr>
<th>Distance Oracle</th>
<th>Distortion</th>
<th>Size</th>
<th>Query</th>
</tr>
</thead>
<tbody>
<tr>
<td>This paper $+$ C14</td>
<td>$2k - 1$</td>
<td>$O(k \cdot n^{1+1/k})$</td>
<td>$O(1)$</td>
</tr>
<tr>
<td>C15 (Randomized)</td>
<td>$2k - 1$</td>
<td>$O(n^{1+1/k})$</td>
<td>$O(1)$</td>
</tr>
</tbody>
</table>

3. Find more **applications** to Ramsey spanning trees!