Scattering and Sparse Partitions, and their Applications

Arnold Filtser

Columbia University

January 17, Simons A&G collaboration
Universal Steiner tree

\[ G = (V, E, w) \] weighted graph,
Universal Steiner tree

\[ G = (V, E, w) \text{ weighted graph,} \]
Universal Steiner tree

\[ G = (V, E, w) \] weighted graph,

\[ \text{cost} = w(T) \]

\[ \text{opt} = \text{Minimal Steiner tree} \]

Theorem ([Jia, Lin, Noubir, Rajaraman, Sundaram 05])

Suppose \( G \) admits \((\sigma, \tau)\)-sparse partition scheme, \( \Rightarrow \) solution to the UST problem with stretch \( O(\tau \sigma^2 \log \tau n) \).
Universal Steiner tree

\[ G = (V, E, w) \text{ weighted graph, } \quad \text{cost} = w(T) \]
Universal Steiner tree

\[ G = (V, E, w) \] weighted graph, \( \text{cost} = w(T) \) \( \text{opt} = \text{Minimal Steiner tree} \)
Universal Steiner tree

$G = (V, E, w)$ weighted graph, $\text{cost} = w(T)$ $\text{opt} = \text{Minimal Steiner tree}$
Universal Steiner tree

\[ G = (V, E, w) \text{ weighted graph, } \quad \text{cost} = w(T) \quad \text{opt} = \text{Minimal Steiner tree} \]
Universal Steiner tree

$G = (V, E, w)$ weighted graph, $\text{cost} = w(T)$, $\text{opt} = \text{Minimal Steiner tree}$
Universal Steiner tree

\[ G = (V, E, w) \] weighted graph, \[ \text{cost} = w(T) \] \[ \text{opt} = \text{Minimal Steiner tree} \]
Universal Steiner tree

\[ G = (V, E, w) \] weighted graph, \quad \text{cost} = w(T) \quad \text{opt} = \text{Minimal Steiner tree}
Universal Steiner tree

\[ G = (V, E, w) \] weighted graph, \( \text{cost} = w(T) \) \( \text{opt} = \text{Minimal Steiner tree} \)
Universal Steiner tree

\[ G = (V, E, w) \] weighted graph, \quad \text{cost} = w(T) \quad \text{opt} = \text{Minimal Steiner tree}
Universal Steiner tree

$G = (V, E, w)$ weighted graph, \( \text{cost} = w(T) \) \( \text{opt} = \text{Minimal Steiner tree} \)
Universal Steiner tree

\[ G = (V, E, w) \] weighted graph, \( \text{cost} = w(T) \) \( \text{opt} = \text{Minimal Steiner tree} \)

\[ \text{stretch}(T) = \max_{K \subseteq V} \frac{T(K)}{\text{opt}(K)} \]
Universal Steiner tree

\[ G = (V, E, w) \] weighted graph, \quad \text{cost} = w(T) \quad \text{opt} = \text{Minimal Steiner tree}

\[ \text{stretch}(T) = \max_{K \subseteq V} \frac{T(K)}{\text{opt}(K)} \]

\textbf{Theorem ([Jia, Lin, Noubir, Rajaraman, Sundaram 05])}

\textit{Suppose } G \textit{ admits } (\sigma, \tau)\textit{-sparse partition scheme,}

\[ \Rightarrow \text{ solution to the UST problem with stretch } O(\tau \sigma^2 \log \tau n). \]
Steiner Point removal problem

$G = (V, E, w)$ - a weighted graph.
$K \subseteq V$ - a terminal set of size $k$. 

Construct a new graph $M = (K, E', w_M)$ such that:

- $M$ has small distortion:
  $$\forall t, t' \in K, d_G(t, t') \leq d_M(t, t') \leq \alpha \cdot d_G(t, t').$$
- $M$ is a graph minor of $G$. 

The distortion is:
$$d_M(t, t') = \frac{d_G(t, t')}{2} = 2$$

Theorem ([Fil19], improving [Kamma, Krauthgamer, Nguyen 15], [Cheung 18])

Given $G$ with $k$ terminals, there is a solution to the SPR problem with distortion $O(\log k)$.

The only known lower bound is $8$.

What about special graph families?

Theorem ([Fil20])

Suppose that every induced subgraph $G[A]$ of $G$ admits $(\sigma, \tau)$-scattering partition scheme, $\Rightarrow$ solution to the SPR problem with distortion $O(\tau^3 \sigma^3)$. 

Arnold Filtser
Scattering and Sparse Partitions, and their Applications
Steiner Point removal problem

\[ G = (V, E, w) \] - a weighted graph.
\[ K \subseteq V \] - a terminal set of size \( k \).

Construct a new graph \( M = (K, E', w_M) \) such that:

\[ \forall t, t' \in K, d_G(t, t') \leq d_M(t, t') \leq \alpha \cdot d_G(t, t') \]

\( M \) is a graph minor of \( G \).

The distortion is:
\[ d_M(t, t') = d_G(t, t') = 2 \]

Theorem (\([\text{Fil} 19]\) improving \([\text{Kamma, Krauthgamer, Nguyen 15}], [\text{Cheung 18}]\))

Given \( G \) with \( k \) terminals, there is a solution to the SPR problem with distortion \( O(\log k) \).

The only known lower bound is 8.

What about special graph families?

Theorem (\([\text{Fil} 20]\))

Suppose that every induced subgraph \( G[A] \) of \( G \) admits \((\sigma, \tau)\)-scattering partition scheme, \( \Rightarrow \) solution to the SPR problem with distortion \( O(\tau^3 \sigma^3) \).
Steiner Point removal problem

\[ G = (V, E, w) - \text{a weighted graph.} \]

\[ K \subseteq V - \text{a terminal set of size } k. \]

Construct a new graph \( M = (K, E', w_M) \) such that:

- \( M \) has small distortion:

\[ \forall t, t' \in K, \quad d_G(t, t') \leq d_M(t, t') \leq \alpha \cdot d_G(t, t'). \]
Steiner Point removal problem

\( G = (V, E, w) \) - a **weighted** graph.

\( K \subseteq V \) - a **terminal** set of size \( k \).

Construct a new graph \( M = (K, E', w_M) \) such that:

- \( M \) has small **distortion**:

\[
\forall t, t' \in K, \quad d_G(t, t') \leq d_M(t, t') \leq \alpha \cdot d_G(t, t').
\]

- \( M \) is a graph **minor** of \( G \).
Steiner Point removal problem

$G = (V, E, w)$ - a weighted graph.
$K \subseteq V$ - a terminal set of size $k$.

Construct a new graph $M = (K, E', w_M)$ such that:
- $M$ has small distortion:
  \[ \forall t, t' \in K, \quad d_G(t, t') \leq d_M(t, t') \leq \alpha \cdot d_G(t, t') \]
- $M$ is a graph minor of $G$.

The distortion is: $\frac{d_M(t, t')}{d_G(t, t')} = \frac{4}{2} = 2$
Steiner Point removal problem

\( G = (V, E, w) \) - a weighted graph.
\( K \subseteq V \) - a terminal set of size \( k \).

Construct a new graph \( M = (K, E', w_M) \) such that:

- \( M \) has small distortion:
  \[
  \forall t, t' \in K, \quad d_G(t, t') \leq d_M(t, t') \leq \alpha \cdot d_G(t, t').
  \]

- \( M \) is a graph minor of \( G \).

**Theorem ([Fil 19] (improving [Kamma, Krauthgamer, Nguyen 15], [Cheung 18]) )**

*Given \( G \) with \( k \) terminals, there is a solution to the \textit{SPR} problem with distortion \( O(\log k) \).*
Steiner Point removal problem

\( G = (V, E, w) \) - a \textbf{weighted} graph.
\( K \subseteq V \) - a \textbf{terminal} set of size \( k \).

Construct a new graph \( M = (K, E', w_M) \) such that:

- \( M \) has small \textbf{distortion}:
  \[
  \forall t, t' \in K, \quad d_G(t, t') \leq d_M(t, t') \leq \alpha \cdot d_G(t, t').
  \]

- \( M \) is a graph \textbf{minor} of \( G \).

\textbf{Theorem ([Fil 19] (improving [KKN 15] [Che 18]))}

\textit{Given} \( G \) \textit{with} \( k \) \textit{terminals}, \textit{there is a solution to the SPR problem}
\textit{with distortion} \( O(\log k) \).

The only known lower bound is \( 8 \) [Chan, Xia, Konjevod, Richa 06].
Steiner Point removal problem

\[ G = (V, E, w) \] - a weighted graph.

\[ K \subseteq V \] - a terminal set of size \( k \).

Construct a new graph \( M = (K, E', w_M) \) such that:

- \( M \) has small distortion:
  \[ \forall t, t' \in K, \quad d_G(t, t') \leq d_M(t, t') \leq \alpha \cdot d_G(t, t') \] .

- \( M \) is a graph minor of \( G \).

**Theorem ([Fil 19] (improving [KKN 15] [Che 18]))**

Given \( G \) with \( k \) terminals, there is a solution to the \( \text{SPR} \) problem with distortion \( O(\log k) \).

The only known lower bound is 8 [CXKR 06].

What about special graph families?
Steiner Point removal problem

- \( M \) has small \textbf{distortion}: \( \forall t, t' \in K, \quad d_G(t, t') \leq d_M(t, t') \leq \alpha \cdot d_G(t, t) \).
- \( M \) is a graph \textbf{minor} of \( G \).

**Theorem (\cite{Fil19} (improving \cite{KKN15} \cite{Che18}))**

\( G \) with \textbf{k terminals}, there is a solution to the \textbf{SPR problem} with distortion \( O(\log k) \).

The only known lower bound is 8 \cite{CXKR06}.

What about special graph families?

**Theorem (\cite{Fil20})**

\( G[A] \) of \( G \) admits \((\sigma, \tau)\)-\textbf{scattering partition scheme}, \( \Rightarrow \) solution to the \textbf{SPR problem} with distortion \( O(\tau^3 \sigma^3) \).
Sparse partitions

\( \mathcal{P} \) is a \((\sigma, \tau, \Delta)\)-sparse partition if:

- The diameter of each cluster \( \leq \Delta \).
- Every ball of radius \( \leq \sigma \) intersects at most \( \tau \) clusters.

Theorem ([JLNRS 05])

Suppose \( G \) admits \((\sigma, \tau)\)-sparse partition scheme, \( \Rightarrow \) solution to the UST problem with stretch \( O(\tau \sigma^2 \log \tau n) \).
Sparse partitions

\( \mathcal{P} \) is a \((\sigma, \tau, \Delta)\)-sparse partition if:

- The diameter of each cluster \( \leq \Delta \).
- Every ball of radius \( \leq \sigma \) intersects at most \( \tau \) clusters.

Theorem ([JLNRS 05])

Suppose \( G \) admits \((\sigma, \tau)\)-sparse partition scheme, \( \Rightarrow \) solution to the UST problem with stretch \( O(\tau \sigma^2 \log \tau n) \).
Sparse partitions

\( \mathcal{P} \) is a \((\sigma, \tau, \Delta)\)-sparse partition if:

- The diameter of each cluster \( \leq \Delta \).

Theorem ([JLNRS 05])

Suppose \( G \) admits a \((\sigma, \tau)\)-sparse partition scheme, \( \Rightarrow \) solution to the UST problem with stretch \( O(\tau \sigma^2 \log \tau n) \).
Sparse partitions

\( \mathcal{P} \) is a \((\sigma, \tau, \Delta)\)-sparse partition if:

- The diameter of each cluster \(\leq \Delta\).
- Every ball of radius \(\leq \frac{\Delta}{\sigma}\) intersects at most \(\tau\) clusters.

Theorem ([JLNRS 05])

Suppose \(G\) admits \((\sigma, \tau)\)-sparse partition scheme, \(\Rightarrow\) solution to the UST problem with stretch \(O(\tau \sigma^2 \log \tau n)\).
Sparse partitions

\( \mathcal{P} \) is a \((\sigma, \tau, \Delta)\)-sparse partition if:

- The diameter of each cluster \( \leq \Delta \).
- Every ball of radius \( \leq \frac{\Delta}{\sigma} \) intersects at most \( \tau \) clusters.

Theorem ([JLNRS 05])

Suppose \( G \) admits \((\sigma, \tau)\)-sparse partition scheme, \( \Rightarrow \) solution to the UST problem with stretch \( O(\tau \sigma^2 \log \tau n) \).
Sparse partitions

\( \mathcal{P} \) is a \((\sigma, \tau, \Delta)\)-sparse partition if:

- The diameter of each cluster \( \leq \Delta \).
- Every ball of radius \( \leq \frac{\Delta}{\sigma} \) intersects at most \( \tau \) clusters.

Theorem ([JLNRS 05])

Suppose \( G \) admits \((\sigma, \tau)\)-sparse partition scheme, \( \Rightarrow \) solution to the UST problem with stretch \( O(\tau \sigma^2 \log \tau n) \).
Sparse partitions

\( \mathcal{P} \) is a \((\sigma, \tau, \Delta)\)-sparse partition if:

- The diameter of each cluster \( \leq \Delta \).
- Every ball of radius \( \leq \frac{\Delta}{\sigma} \) intersects at most \( \tau \) clusters.

\((\sigma, \tau)\)-sparse partition scheme: \( \forall \Delta > 0 \ \exists (\sigma, \tau, \Delta)\)-sparse partition.
Sparse partitions

\( \mathcal{P} \) is a \((\sigma, \tau, \Delta)\)-sparse partition if:

- The diameter of each cluster \( \leq \Delta \).
- Every ball of radius \( \leq \frac{\Delta}{\sigma} \) intersects at most \( \tau \) clusters.

\((\sigma, \tau)\)-sparse partition scheme: \( \forall \Delta > 0 \ \exists (\sigma, \tau, \Delta)\)-sparse partition.

Theorem ([JLNRS 05])

Suppose \( G \) admits \((\sigma, \tau)\)-sparse partition scheme,

\[ \Rightarrow \text{solution to the UST problem with stretch } O(\tau \sigma^2 \log \tau n). \]
Strong Vs. Weak Diameter

Given a subset $A \subseteq V$, 

**Weak Diameter** of $A := \max_{v,u \in A} d_G(v,u)$.
Strong Vs. Weak Diameter

Given a subset $A \subseteq V$,

**Weak Diameter** of $A := \max_{v, u \in A} d_G(v, u)$.

**Strong Diameter** of $A := \max_{v, u \in A} d_{G[A]}(v, u)$.

(Induced subgraph)
Strong Vs. Weak Diameter

Given a subset $A \subseteq V$,

**Weak Diameter** of $A := \max_{v,u \in A} d_G(v, u)$.

**Strong Diameter** of $A := \max_{v,u \in A} d_G[A](v, u)$.

(induced subgraph)
Strong Vs. Weak Diameter

Given a subset $A \subseteq V$,

**Weak Diameter of $A$** := $\max_{v, u \in A} d_G(v, u)$.

**Strong Diameter of $A$** := $\max_{v, u \in A} d_{G[A]}(v, u)$.

$(\text{induced subgraph})$

$$d_G(u, v) = 2$$
Strong vs. Weak Diameter

Given a subset $A \subseteq V$,

**Weak Diameter of** $A := \max_{v, u \in A} d_G(v, u)$.

**Strong Diameter of** $A := \max_{v, u \in A} d_G[A](v, u)$.

(induced subgraph)

\[
d_G(u, v) = 2 \\
d_G[A](u, v) = 6
\]
**Strong Vs. Weak Diameter**

Given a subset $A \subseteq V$,

**Weak Diameter** of $A := \max_{v,u \in A} d_G(v, u)$.

**Strong Diameter** of $A := \max_{v,u \in A} d_{G[A]}(v, u)$.

(Induced subgraph)

\[
\begin{align*}
d_G(u, v) &= 2 \\
d_{G[A]}(u, v) &= 6
\end{align*}
\]

Weak diameter of $A = 4$.

Strong diameter of $A = 6$. 

Arnold Filtser
Scattering and Sparse Partitions, and their Applications 5 / 15
Strong Vs. Weak Diameter

Given a subset $A \subseteq V$,

**Weak Diameter** of $A := \max_{v,u \in A} d_G(v, u)$.

**Strong Diameter** of $A := \max_{v,u \in A} d_{G[A]}(v, u)$.

Theorem ([JLNRS 05])

Suppose $G$ admits $(\sigma, \tau)$–weak sparse partition scheme, \implies solution to the UST problem with stretch $O(\tau \sigma^2 \log \tau n)$.

\[d_G(u, v) = 2\]
\[d_{G[A]}(u, v) = \infty\]
Strong Vs. Weak Diameter

Given a subset $A \subseteq V$,

**Weak Diameter of** $A := \max_{v, u \in A} d_G(v, u)$.

**Strong Diameter of** $A := \max_{v, u \in A} d_{G[A]}(v, u)$.

Theorem ([JLNRS 05])

Suppose $G$ admits $(\sigma, \tau)$–weak sparse partition scheme, then a solution to the UST problem with stretch $O(\tau \sigma^2 \log \tau n)$. 

\[ d_G(u, v) = 2 \]
\[ d_{G[A]}(u, v) = \infty \]

Weak diameter of $A = 4$.
Strong diameter of $A = \infty$. 

(induced subgraph)
Strong Vs. Weak Diameter

\( \mathcal{P} \) is a \((\sigma, \tau, \Delta)\)-\textbf{strong/weak} sparse partition if:

- The \textbf{strong/weak} diameter of each cluster \( \leq \Delta \).
- Every ball of radius \( \leq \frac{\Delta}{\sigma} \) intersects at most \( \tau \) clusters.

\((\sigma, \tau)\)-\textbf{strong/weak} sparse partition scheme: \( \exists (\sigma, \tau, \Delta)\)-\textbf{strong/weak} sparse partition for all \( \Delta > 0 \).
Strong Vs. Weak Diameter

\( \mathcal{P} \) is a \((\sigma, \tau, \Delta)\)-strong/weak sparse partition if:

- The strong/weak diameter of each cluster \( \leq \Delta \).

- Every ball of radius \( \leq \frac{\Delta}{\sigma} \) intersects at most \( \tau \) clusters.

\((\sigma, \tau)\)-strong/weak sparse partition scheme: \( \exists (\sigma, \tau, \Delta)\)-strong/weak sparse partition for all \( \Delta > 0 \).

**Theorem ([JLNRS 05])**

Suppose \( G \) admits \((\sigma, \tau)\)-weak sparse partition scheme,

\[ \Rightarrow \text{solution to the UST problem with stretch } O(\tau \sigma^2 \log \tau n). \]
(σ, τ)-strong/weak sparse partition scheme: ∃ (σ, τ, Δ)-strong/weak sparse partition for all Δ > 0.

**Theorem ([JLNRS 05])**

Suppose G admits (σ, τ)-weak sparse partition scheme,

⇒ solution to the UST problem with stretch $O(\tau \sigma^2 \log_\tau n)$.

[JLNRS 05] produces a non-subgraph solution to the UST problem.
Strong Vs. Weak Diameter

\((\sigma, \tau)\)-strong/weak sparse partition scheme: \(\exists (\sigma, \tau, \Delta)\)-strong/weak sparse partition for all \(\Delta > 0\).

**Theorem ([JLNRS 05])**

Suppose \(G\) admits \((\sigma, \tau)\)-weak sparse partition scheme,\n\[\Rightarrow\] solution to the UST problem with stretch \(O(\tau \sigma^2 \log \tau n)\).

[JLNRS 05] produces a non-subgraph solution to the UST problem.

[BDRRS 12]: subgraph solution using hierarchy of strong sparse partitions.
Scattering partitions

$\mathcal{P}$ is a $(\sigma, \tau, \Delta)$-scattering partition if:

- Each cluster is connected.
- The weak-diameter of each cluster $\leq \Delta$.
- Every shortest path of length $\leq \sigma$ intersects at most $\tau$ clusters.

Theorem ([Fil20])

Suppose that every induced subgraph $G[A]$ of $G$ admits a $(\sigma, \tau)$-scattering partition scheme, $\Rightarrow$ solution to the SPR problem with distortion $O(\tau^3 \sigma^3)$. 
Scattering partitions

\( \mathcal{P} \) is a \((\sigma, \tau, \Delta)\)-scattering partition if:

- Each cluster is connected.
- The weak-diameter of each cluster \( \leq \Delta \).
- Every shortest path of length \( \leq \Delta \) intersects at most \( \tau \) clusters.

Theorem ([Fil20])

Suppose that every induced subgraph \( G[A] \) of \( G \) admits \((\sigma, \tau)\)-scattering partition scheme, \( \Rightarrow \) solution to the SPR problem with distortion \( O(\tau^3 \sigma^3) \).
Scattering partitions

\( \mathcal{P} \) is a \((\sigma, \tau, \Delta)\)-scattering partition if:

- Each cluster is connected.
- The weak-diameter of each cluster \( \leq \Delta \).

Theorem (\([\text{Fil}]_{20}\))

Suppose that every induced subgraph \( G[A] \) of \( G \) admits \((\sigma, \tau)\)-scattering partition scheme, \( \Rightarrow \) solution to the SPR problem with distortion \( O(\tau^3 \sigma^3) \).
Scattering partitions

$\mathcal{P}$ is a $(\sigma, \tau, \Delta)$-scattering partition if:

- Each cluster is connected.
- The weak-diameter of each cluster $\leq \Delta$.
- Every shortest path of length $\leq \frac{\Delta}{\sigma}$ intersects at most $\tau$ clusters.
**Scattering partitions**

$\mathcal{P}$ is a $(\sigma, \tau, \Delta)$-scattering partition if:

- Each cluster is connected.
- The weak-diameter of each cluster $\leq \Delta$.
- Every shortest path of length $\leq \Delta/\sigma$ intersects at most $\tau$ clusters.

**Theorem** ([Fil20])

Suppose that every induced subgraph $G[A]$ of $G$ admits $(\sigma, \tau)$-scattering partition scheme, $\Rightarrow$ solution to the SPR problem with distortion $O(\tau^3 \sigma^3)$. 
Scattering partitions

\( \mathcal{P} \) is a \((\sigma, \tau, \Delta)\)-scattering partition if:

- Each cluster is connected.
- The weak-diameter of each cluster \( \leq \Delta \).
- Every shortest path of length \( \leq \frac{\Delta}{\sigma} \) intersects at most \( \tau \) clusters.
Scattering partitions

\( \mathcal{P} \) is a \((\sigma, \tau, \Delta)\)-scattering partition if:

- Each cluster is connected.
- The weak-diameter of each cluster \(\leq \Delta\).
- Every shortest path of length \(\leq \frac{\Delta}{\sigma}\) intersects at most \(\tau\) clusters.

\((\sigma, \tau)\)-scattering partition scheme: \( \forall \Delta > 0 \ \exists \ (\sigma, \tau, \Delta)\)-scattering partition.
Scattering partitions

$\mathcal{P}$ is a $(\sigma, \tau, \Delta)$-scattering partition if:

- Each cluster is connected.
- The weak-diameter of each cluster $\leq \Delta$.
- Every shortest path of length $\leq \Delta \sigma$ intersects at most $\tau$ clusters.

$(\sigma, \tau)$-scattering partition scheme: $\forall \Delta > 0 \ \exists (\sigma, \tau, \Delta)$-scattering partition.

Theorem ([Fil 20])

Suppose that every induced subgraph $G[A]$ of $G$ admits $(\sigma, \tau)$-scattering partition scheme,

$\Rightarrow$ solution to the SPR problem with distortion $O(\tau^3 \sigma^3)$. 
Observations

$$(\sigma, \tau, \Delta)-\text{strong sparse} \quad \Rightarrow \quad (\sigma, \tau, \Delta)-\text{weak sparse}$$.

- Each cluster **strong** diameter $\leq \Delta$.
- Every ball of radius $\leq \frac{\Delta}{\sigma}$ intersects at most $\tau$ clusters.

- Each cluster **weak** diameter $\leq \Delta$.
- Every ball of radius $\leq \frac{\Delta}{\sigma}$ intersects at most $\tau$ clusters.
Observations

$$(\sigma, \tau, \Delta)$$-strong sparse \implies $$(\sigma, \tau, \Delta)$$-scattering.
Observations

\[(\sigma, \tau, \Delta)-\text{strong sparse} \quad \Rightarrow \quad (\sigma, \tau, \Delta)-\text{scattering}.\]
Scattering
Weak

Strong

Trees?
Theorem ([Fil 20])

Suppose all \( n \)-vertex trees admit a \((\sigma, \tau)\)-strong sparse partition scheme.

Then \( \tau \geq \frac{1}{3} \cdot n^{\frac{2}{\sigma+1}} \).
Corollary

\[ \forall n > 1, \text{ there are trees } T_1, T_2 \text{ such that,} \]

- \( T_1 \) do not admit \( \left( \frac{\log n}{\log \log n}, \log n \right) \)-strong sparse partition scheme.
- \( T_2 \) do not admit \( \left( \sqrt{\log n}, 2\sqrt{\log n} \right) \)-strong sparse partition scheme.
Theorem ([Fil 20])

Every tree is $(2, 3)$-scatterable.
Theorem ([Fil 20])

Every tree admits a \((4, 3)\)-weak sparse partition scheme.
Scattering

Weak

Strong

Trees
Doubling Metrics

Metric space has **doubling dimension** $d$ if every radius $r$ ball can be **covered** by $2^d$ balls of radius $\frac{r}{2}$. 
Doubling Metrics

Metric space has **doubling dimension** $d$ if every radius $r$ ball can be covered by $2^d$ balls of radius $\frac{r}{2}$.

Example: Every $d$-dimensional Euclidean space has doubling dimension $O(d)$. 
Doubling Metrics

Metric space has **doubling dimension** $d$ if every radius $r$ ball can be covered by $2^d$ balls of radius $\frac{r}{2}$.

**Packing Property**

$N \subseteq X$ set s.t. $x, y \in N$ it holds that $d(x, y) \geq \delta$. Then $\forall x, R,$

$$|B(x, R) \cap N| \leq \left(\frac{R}{\delta}\right)^{O(d)}.$$
Doubling Metrics

Metric space has **doubling dimension** $d$ if every radius $r$ ball can be **covered** by $2^d$ balls of radius $\frac{r}{2}$.

**Packing Property**

$N \subseteq X$ set s.t. $x, y \in N$ it holds that $d(x, y) \geq \delta$. **Then** $\forall x, R,$

$$|B(x, R) \cap N| \leq \left(\frac{R}{\delta}\right)^{O(d)}.$$

The graph $G = (V, E, w)$ has doubling dimension $O(d)$, **if** $(V, d_G)$ (the shortest path metric) has doubling dimension $O(d)$. 
Theorem ([JLNRS 05]) Every graph with doubling dimension $d$ admits a $(1, 2^{O(d)})$-weak sparse partition scheme.
Theorem ([JLNRS 05])

Every graph with **doubling dimension** $d$ admits a $(1, 2^{O(d)})$-weak sparse partition scheme.
Theorem ([JLNRS 05])

Every graph with **doubling dimension** $d$ admits a 

$(1, 2^{O(d)})$-**weak** sparse partition scheme.
Theorem ([Fil 20])

Every graph with **doubling dimension** $d$ admits a

$(O(d), \tilde{O}(d))$-**strong** sparse partition scheme.
Theorem ([Fil 20])

Every graph with \textbf{doubling dimension} \( d \) admits a \( (O(d), \tilde{O}(d)) \)-\textbf{strong} sparse partition scheme.
MPX [Miller, Peng, Xu 2013]
\[ \delta t_1 = 4 \]
\[ \delta t_2 = 2 \]
\[ \delta t_3 = 1 \]
\[ \delta t_4 = 4 \]
\[ \delta_{t_1} = 4 \quad \delta_{t_2} = 2 \quad \delta_{t_3} = 1 \quad \delta_{t_4} = 4 \]
\[\delta_{t_1} = 4\]
\[\delta_{t_2} = 2\]
\[\delta_{t_3} = 1\]
\[\delta_{t_4} = 4\]
\( \delta t_1 = 4 \)

\( \delta t_2 = 2 \)

\( \delta t_3 = 1 \)

\( \delta t_4 = 4 \)
$\delta t_1 = 4$
$\delta t_2 = 2$
$\delta t_3 = 1$
$\delta t_4 = 4$
δt_4 = 4
δt_3 = 1
δt_2 = 2
δt_1 = 4
Inherently connected!
Inherently \textbf{connected}!

Formally, for $v \in V$ set $f_v(t) = \delta_t - d_G(v, t)$.

$v \textbf{ joins}$ the cluster $C_t$ of the center $t \textbf{ maximizing } f_v$. 

Partition Algorithm

Algorithm: 1. Let $N$ be a $\Delta$-net.
Partition Algorithm

Algorithm: 1. Let $N$ be a $\Delta$-net.

Definition ($\Delta$-net)

Set $N$ s.t.:
- $\forall u, v \in N, d_G(u, v) > \Delta$.
- $\forall v \in V$ there is a net point $u \in N$ s.t. $d_G(u, v) \leq \Delta$. 
Partition Algorithm

Algorithm: 1. Let $N$ be a $\Delta$-net.
            2. For each center $t \in N$ sample $\delta_t \sim \text{BExp}(\Delta/d, 4\Delta)$. 

Partition Algorithm

**Algorithm:**
1. Let $N$ be a $\Delta$-net.
2. For each center $t \in N$ sample $\delta_t \sim \text{BExp}(\Delta/d, 4\Delta)$.

**Definition (Betailed exponential distribution $\text{BExp}(\lambda, \lambda_T)$)**

$X = \min\{X', \lambda_T\}$ where $X' \sim \text{Exp}(\lambda)$. 
Partition Algorithm

Algorithm: 1. Let $N$ be a $\Delta$-net.
2. For each center $t \in N$ sample $\delta_t \sim B\text{Exp}(\Delta/d, 4\Delta)$.
3. Run [MPX 13] ($v$ goes to $\text{arg max} f_v(t) = \delta_t - d_G(v, t)$).

Definition (Betailed exponential distribution $B\text{Exp}(\lambda, \lambda_T)$)

$X = \min\{X', \lambda_T\}$ where $X' \sim \text{Exp}(\lambda)$. 
Partition Algorithm

Algorithm:
1. Let $N$ be a $\Delta$-net.
2. For each center $t \in N$ sample $\delta_t \sim \text{BExp} \left( \frac{\Delta}{d}, 4\Delta \right)$.
3. Run [MPX 13] ($v$ goes to $\text{arg max} \ f_v(t) = \delta_t - d_G(v, t)$).

$$B = B_G(v, \frac{\Delta}{d})$$

$B_G(v, \Delta/d)$
Partition Algorithm

**Algorithm:**
1. Let $N$ be a $\Delta$-net.
2. For each center $t \in N$ sample $\delta_t \sim \text{BExp}(\Delta/d, 4\Delta)$.
3. Run [MPX 13] ($v$ goes to arg max $f_v(t) = \delta_t - d_G(v, t)$).

Set $N_v = N \cap B_G(v, 6\Delta)$. By packing argument: $|N_v| = 2^{O(d)}$. 
Partition Algorithm

Algorithm:
1. Let $N$ be a $\Delta$-net.
2. For each center $t \in N$ sample $\delta_t \sim \text{BExp}(\Delta/d, 4\Delta)$.
3. Run [MPX 13] ($v$ goes to $\arg \max f_v(t) = \delta_t - d_G(v, t)$).

Set $N_v = N \cap B_G(v, 6\Delta)$. By packing argument: $|N_v| = 2^{O(d)}$. 
Partition Algorithm

Algorithm:
1. Let $N$ be a $\Delta$-net.
2. For each center $t \in N$ sample $\delta_t \sim \text{BExp}(\Delta/d, 4\Delta)$.
3. Run [MPX 13] ($v$ goes to arg max $f_v(t) = \delta_t - d_G(v, t)$).

Consider $u \in B$, for all $t \in N$, $|f_v(t) - f_u(t)| \leq d_G(u, v) \leq \frac{\Delta}{d}$
Partition Algorithm

**Algorithm:**
1. Let $N$ be a $\Delta$-net.
2. For each center $t \in N$ sample $\delta_t \sim \text{BExp}(\Delta/d, 4\Delta)$.
3. Run [MPX 13] ($v$ goes to $\arg\max f_v(t) = \delta_t - d_G(v, t)$).

Consider $u \in B$, for all $t \in N$, $|f_v(t) - f_u(t)| \leq d_G(u, v) \leq \frac{\Delta}{d}$

$B$ can intersects center $t'$ only if $f_v(t') \geq f_v(t_1) - \frac{2\Delta}{d}$. 
Partition Algorithm

**Algorithm:**

1. Let $N$ be a $\Delta$-net.
2. For each center $t \in N$ sample $\delta_t \sim \text{BExp}(\Delta/d, 4\Delta)$.
3. Run [MPX 13] ($v$ goes to arg max $f_v(t) = \delta_t - d_G(v, t)$).

For how many $t \in N_v$, $f_v(t) \in [f_v(t_1) - \frac{2\Delta}{d}, f_v(t_1)]$?
Let $\delta'_t \sim \text{Exp}(\lambda)$, $\delta_t = \min\{\delta'_t, \lambda_T\}$ (note $\delta_t \sim \text{BExp}(\lambda, \lambda_T)$) \hspace{1cm} (\lambda = \Delta/d, \lambda_T = 4\Delta.)
Let $\delta'_t \sim \text{Exp}(\lambda)$, $\delta_t = \min\{\delta'_t, \lambda_T\}$ (note $\delta_t \sim \text{BExp}(\lambda, \lambda_T)$ $(\lambda = \Delta/d, \lambda_T = 4\Delta)$.)

Recall $f_v(t) = \delta_t - d_G(v, t)$. Set $f'_v(t) = \delta'_t - d_G(v, t)$.
Let \( \delta'_t \sim \text{Exp}(\lambda) \), \( \delta_t = \min\{\delta'_t, \lambda_T\} \) (note \( \delta_t \sim \text{BExp}(\lambda, \lambda_T) \)) \( (\lambda = \Delta/d, \lambda_T = 4\Delta.) \)

Recall \( f_v(t) = \delta_t - d_G(v, t) \). Set \( f'_v(t) = \delta'_t - d_G(v, t) \).
Let $\delta'_t \sim \text{Exp}(\lambda)$, $\delta_t = \min\{\delta'_t, \lambda_T\}$ (note $\delta_t \sim \text{BExp}(\lambda, \lambda_T)$, $(\lambda = \Delta/d, \lambda_T = 4\Delta$).

Recall $f_v(t) = \delta_t - d_G(v, t)$. Set $f'_v(t) = \delta'_t - d_G(v, t)$. 

$$[f_v(t_1) - \frac{2\Delta}{d}, f_v(t_1)]$$
Let $\delta_t' \sim \text{Exp}(\lambda)$, $\delta_t = \min\{\delta_t', \lambda_T\}$ (note $\delta_t \sim \text{BExp}(\lambda, \lambda_T)$) \(\lambda = \frac{\Delta}{d}, \lambda_T = 4\Delta\). Recalling $f_v(t) = \delta_t - d_G(v, t)$. Set $f_v'(t) = \delta_t' - d_G(v, t)$.

Fix $s = \Theta(d)$ and $t'_s$. 

$B = B_G(v, \frac{\Delta}{d})$
Let $\delta_t' \sim \text{Exp}(\lambda)$, $\delta_t = \min\{\delta_t', \lambda_T\}$ (note $\delta_t \sim \text{BExp}(\lambda, \lambda_T)$, $\lambda = \Delta/d$, $\lambda_T = 4\Delta$.)

Recall $f_v(t) = \delta_t - d_G(v, t)$. Set $f'_v(t) = \delta_t' - d_G(v, t)$.

Fix $s = \Theta(d)$ and $t'_s$. By Memorylessness

$$\Pr \left[ f'_v(t) > f'_v(t'_s) + \frac{2\Delta}{d} \mid f'_v(t) \geq f'_v(t'_s) \right] \geq \Pr \left[ \delta'_t > \frac{2\Delta}{d} \right] = e^{-\frac{2\Delta}{d}/\lambda} = \Omega(1).$$
Let $\delta_t' \sim \text{Exp}(\lambda)$, $\delta_t = \min\{\delta_t', \lambda_T\}$ (note $\delta_t \sim \text{BExp}(\lambda, \lambda_T)$, $(\lambda = \Delta/d, \lambda_T = 4\Delta$).

Recall $f_v(t) = \delta_t - d_G(v, t)$. Set $f'_v(t) = \delta_t' - d_G(v, t)$.

Fix $s = \Theta(d)$ and $t'_s$. By Memorylessness

$$\Pr \left[ f'_v(t) > f'_v(t'_s) + \frac{2\Delta}{d} \mid f'_v(t) \geq f'_v(t'_s) \right] \geq \Pr \left[ \delta'_t > \frac{2\Delta}{d} \right] = e^{-\frac{2\Delta}{d}/\lambda} = \Omega(1).$$

Using Chernoff, for $\tilde{s} = \frac{1}{\Omega(1)} s = \Omega(d)$, $f'_v(t'_s) > f'_v(t'_s) + \frac{2\Delta}{d}$. 

\[ [f'_v(t_1) - \frac{2\Delta}{d}, f_v(t_1)] \]

\[ \cdots \]

\[ f_v(t_3) \quad f_v(t_2) \quad f_v(t_1) \]

\[ \cdots \]

\[ f'_v(t'_s) \quad f'_v(t'_1) \quad f'_v(t'_s) \quad f'_v(t'_3) \quad f'_v(t'_2) \quad f'_v(t'_1) \]
Let $\delta'_t \sim \text{Exp}(\lambda)$, $\delta_t = \min\{\delta'_t, \lambda_T\}$ (note $\delta_t \sim B\text{Exp}(\lambda, \lambda_T)$ ($\lambda = \Delta/d$, $\lambda_T = 4\Delta$.)

Recall $f_v(t) = \delta_t - d_G(v, t)$. Set $f'_v(t) = \delta'_t - d_G(v, t)$.

Fix $s = \Theta(d)$ and $t'_s$. By Memorylessness

$$
\Pr \left[ f'_v(t) > f'_v(t'_s) + \frac{2\Delta}{d} \mid f'_v(t) \geq f'_v(t'_s) \right] \geq \Pr \left[ \delta'_t > \frac{2\Delta}{d} \right] = e^{-\frac{2\Delta}{d}/\lambda} = \Omega(1).
$$

Using Chernoff, for $\bar{s} = \frac{1}{\Omega(1)} s = \Omega(d)$, $f'_v(t'_s) > f'_v(t'_s) + \frac{2\Delta}{d}$. 

Let $\delta'_t \sim \text{Exp}(\lambda)$, $\delta_t = \min\{\delta'_t, \lambda_T\}$ (note $\delta_t \sim \text{BExp}(\lambda, \lambda_T)$ \quad ($\lambda = \Delta/d$, $\lambda_T = 4\Delta$.)

Recall $f_v(t) = \delta_t - d_G(v, t)$. Set $f'_v(t) = \delta'_t - d_G(v, t)$.

Fix $s = \Theta(d)$ and $t'_s$. By Memorylessness

$$\Pr\left[f'_v(t) > f'_v(t'_s) + \frac{2\Delta}{d} \mid f'_v(t) \geq f'_v(t'_s)\right] \geq \Pr\left[\delta'_t > \frac{2\Delta}{d}\right] = e^{-\frac{2\Delta}{d}/\lambda} = \Omega(1) \quad .$$

Using Chernoff, for $\tilde{s} = \frac{1}{\Omega(1)} s = \Omega(d)$, $f'_v(t'_s) > f'_v(t'_s) + \frac{2\Delta}{d}$.

$t \in N_v$ is betailed with probability $\Pr[\delta_t = \lambda_T] = \Pr[\delta'_t \geq \lambda_T] = e^{-\frac{\lambda_T}{\lambda}} = e^{-4d}$.
Let $\delta'_t \sim \text{Exp}(\lambda)$, $\delta_t = \min\{\delta'_t, \lambda T\}$ (note $\delta_t \sim B\text{Exp}(\lambda, \lambda T)$ \hspace{1cm} ($\lambda = \Delta/d$, $\lambda T = 4\Delta$.)

Recall $f_v(t) = \delta_t - d_G(v, t)$. Set $f'_v(t) = \delta'_t - d_G(v, t)$.

Fix $s = \Theta(d)$ and $t'_s$. By Memorylessness

$$
\Pr \left[ f'_v(t) > f'_v(t'_s) + \frac{2\Delta}{d} \mid f'_v(t) \geq f'_v(t'_s) \right] \geq \Pr \left[ \delta'_t > \frac{2\Delta}{d} \right] = e^{-\frac{2\Delta}{d}/\lambda} = \Omega(1).$$

Using Chernoff, for $\tilde{s} = \frac{1}{\Omega(1)} s = \Omega(d)$, $f'_v(t'_s) > f'_v(t'_s) + \frac{2\Delta}{d}$.

$t \in N_v$ is betailed with probability $\Pr[\delta_t = \lambda T] = \Pr[\delta'_t \geq \lambda T] = e^{-\frac{\lambda T}{\lambda}} = e^{-4d}$.

The probability that at least $\tilde{s}$ centers in $N_v$ are betailed bounded by

$$|N_v|^{\tilde{s}} \cdot \left( e^{-4d} \right)^{\tilde{s}} = 2^{O(d \tilde{s})} \cdot \left( e^{-4d \tilde{s}} \right) = e^{-\Omega(d \tilde{s})}$$
Let $\delta'_t \sim \text{Exp}(\lambda)$, $\delta_t = \min\{\delta'_t, \lambda_T\}$ (note $\delta_t \sim \text{BExp}(\lambda, \lambda_T)$) ($\lambda = \Delta/d$, $\lambda_T = 4\Delta$.)

Recall $f_v(t) = \delta_t - d_G(v, t)$. Set $f'_v(t) = \delta'_t - d_G(v, t)$.

Fix $s = \Theta(d)$ and $t'_s$. By Memorylessness

$$\Pr \left[ f'_v(t) > f'_v(t'_s) + \frac{2\Delta}{d} \mid f'_v(t) \geq f'_v(t'_s) \right] \geq \Pr \left[ \delta'_t > \frac{2\Delta}{d} \right] = e^{-\frac{2\Delta}{d}/\lambda} = \Omega(1).$$

Using Chernoff, for $\tilde{s} = \frac{1}{\Omega(1)}s = \Omega(d)$, $f'_v(t'_s) > f'_v(t'_s) + \frac{2\Delta}{d}$.

$t \in N_v$ is **betailed** with probability $\Pr[\delta_t = \lambda_T] = \Pr[\delta'_t \geq \lambda_T] = e^{-\frac{\lambda_T}{\lambda}} = e^{-4d}$.

The probability that at least $\tilde{s}$ centers in $N_v$ are bailed bounded by

$$|N_v|^{\tilde{s}} \cdot (e^{-4d})^{\tilde{s}} = 2^{O(d\tilde{s})} \cdot (e^{-4d\tilde{s}}) = e^{-\Omega(d\tilde{s})}$$
Let $\delta'_t \sim \text{Exp}(\lambda)$, $\delta_t = \min\{\delta'_t, \lambda_T\}$ (note $\delta_t \sim \text{BExp}(\lambda, \lambda_T)$) ($\lambda = \Delta/d$, $\lambda_T = 4\Delta$.)

Recall $f_v(t) = \delta_t - d_G(v, t)$. Set $f'_v(t) = \delta'_t - d_G(v, t)$.

Fix $s = \Theta(d)$ and $t'_s$.

There is index $m \leq \bar{s} \leq s$, s.t. $f'_v(t_m) > f'_v(t'_s) + \frac{2\Delta}{d}$ and $t'_m$ is not betailed.
Let $\delta'_t \sim \text{Exp}(\lambda)$, $\delta_t = \min\{\delta'_t, \lambda_T\}$ (note $\delta_t \sim \text{BExp}(\lambda, \lambda_T)$) \hspace{1cm} (\lambda = \Delta/d, \lambda_T = 4\Delta.

Recall $f_v(t) = \delta_t - d_G(v, t)$. Set $f'_v(t) = \delta'_t - d_G(v, t)$.

Fix $s = \Theta(d)$ and $t'_s$.

There is index $m \leq \tilde{s} \leq s$, s.t. $f'_v(t_m) > f'_v(t'_s) + \frac{2\Delta}{d}$ and $t'_m$ is not betailed.

For every $t \not\in \{t'_1, t'_2, \ldots, t'_s\}$, it holds that

$$f'_v(t) \leq f'_v(t'_s)$$
Let $\delta'_t \sim \text{Exp}(\lambda)$, $\delta_t = \min\{\delta'_t, \lambda_T\}$ (note $\delta_t \sim \text{BExp}(\lambda, \lambda_T)$) (note $\lambda = \Delta/d$, $\lambda_T = 4\Delta$.)

Recall $f_v(t) = \delta_t - d_G(v, t)$. Set $f'_v(t) = \delta'_t - d_G(v, t)$.

Fix $s = \Theta(d)$ and $t'_s$.

There is index $m \leq \tilde{s} \leq s$, s.t. $f'_v(t_m) > f'_v(t'_s) + \frac{2\Delta}{d}$ and $t'_m$ is not betailed.

For every $t \notin \{t'_1, t'_2, \ldots, t'_s\}$, it holds that

$$f_v(t) \leq f'_v(t) \leq f'_v(t'_s)$$
Let $\delta'_t \sim \text{Exp}(\lambda)$, $\delta_t = \min\{\delta'_t, \lambda_T\}$ (note $\delta_t \sim \text{BExp}(\lambda, \lambda_T)$) ($\lambda = \Delta/d$, $\lambda_T = 4\Delta$.)

Recall $f_v(t) = \delta_t - d_G(v, t)$. Set $f'_v(t) = \delta'_t - d_G(v, t)$.

Fix $s = \Theta(d)$ and $t'_s$.

There is index $m \leq \tilde{s} \leq s$, s.t. $f'_v(t_m) > f'_v(t'_s) + \frac{2\Delta}{d}$ and $t'_m$ is not bailed.

For every $t \not\in \{t'_1, t'_2, \ldots, t'_s\}$, it holds that

$$f_v(t) \leq f'_v(t) \leq f'_v(t'_s) < f'_v(t'_m) - \frac{2\Delta}{d}$$
Let $\delta'_t \sim \text{Exp}(\lambda)$, $\delta_t = \min\{\delta'_t, \lambda T\}$ (note $\delta_t \sim \text{BExp}(\lambda, \lambda T)$) $(\lambda = \Delta/d$, $\lambda T = 4\Delta)$.

Recall $f_v(t) = \delta_t - d_G(v, t)$. Set $f'_v(t) = \delta'_t - d_G(v, t)$.

Fix $s = \Theta(d)$ and $t'_s$.

There is index $m \leq \bar{s} \leq s$, s.t. $f'(t_m) > f'(t'_s) + \frac{2\Delta}{d}$ and $t'_m$ is not betailed.

For every $t \notin \{t'_1, t'_2, \ldots, t'_s\}$, it holds that

$$f_v(t) \leq f'_v(t) \leq f'_v(t'_s) < f'_v(t'_m) - \frac{2\Delta}{d} = f_v(t'_m) - \frac{2\Delta}{d}.$$
Let $\delta'_t \sim \text{Exp}(\lambda)$, $\delta_t = \min\{\delta'_t, \lambda_T\}$ (note $\delta_t \sim \text{BExp}(\lambda, \lambda_T)$) \quad \text{(Note $\lambda = \Delta/d$, $\lambda_T = 4\Delta$.)}

Recall $f_v(t) = \delta_t - d_G(v, t)$. Set $f'_v(t) = \delta'_t - d_G(v, t)$.

Fix $s = \Theta(d)$ and $t'_s$.

There is index $m \leq \tilde{s} \leq s$, s.t. $f'_v(t_m) > f'_v(t'_s) + \frac{2\Delta}{d}$ and $t'_m$ is not betailed.

For every $t \not\in \{t'_1, t'_2, \ldots, t'_s\}$, it holds that

$$f_v(t) \leq f'_v(t) \leq f'_v(t'_s) < f'_v(t'_m) - \frac{2\Delta}{d} = f_v(t'_m) - \frac{2\Delta}{d} \leq f_v(t_1) - \frac{2\Delta}{d}. $$
Let $\delta_t \sim \text{Exp}(\lambda)$, $\delta_t = \min\{\delta_t', \lambda T\}$ (note $\delta_t \sim \text{BExp}(\lambda, \lambda T)$) and $\lambda = \Delta / d$, $\lambda T = 4\Delta$.

Recall $f_v(t) = \delta_t - d_G(v, t)$. Set $f'_v(t) = \delta'_t - d_G(v, t)$.

Fix $s = \Theta(d)$ and $t'_s$.

There is index $m \leq \tilde{s} \leq s$, s.t. $f'_v(t_m) > f'_v(t'_s) + \frac{2\Delta}{d}$ and $t'_m$ is not betailed.

For every $t \notin \{t'_1, t'_2, \ldots, t'_s\}$, it holds that

$$f_v(t) \leq f'_v(t) \leq f'_v(t'_s) < f'_v(t'_m) - \frac{2\Delta}{d} = f_v(t'_m) - \frac{2\Delta}{d} \leq f_v(t_1) - \frac{2\Delta}{d}.$$ 

**Corollary**

W.h.p. $B = B_G(v, \frac{\Delta}{d})$ intersects at most $s = O(d)$ clusters.
Fix $s = \Theta(d)$ and $t'_s$.

There is index $m \leq \tilde{s} \leq s$, s.t. $f'_v(t_m) > f'_v(t'_s) + \frac{2\Delta}{d}$ and $t'_m$ is not betailed.

For every $t \notin \{t'_1, t'_2, \ldots, t'_s\}$, it holds that $f_v(t) < f_v(t_1) - \frac{2\Delta}{d}$

**Corollary**

\textit{W.h.p.} $B = B_G(v, \frac{A}{d})$ intersects at most $s = O(d)$ clusters.

Using the Lovász Local Lemma, we conclude
Fix $s = \Theta(d)$ and $t'_s$.

There is index $m \leq \bar{s} \leq s$, s.t. $f'_v(t_m) > f'_v(t'_s) + \frac{2\Delta}{d}$ and $t'_m$ is not betailed.

For every $t \not\in \{t'_1, t'_2, \ldots, t'_s\}$, it holds that $f_v(t) < f_v(t_1) - \frac{2\Delta}{d}$

**Corollary**

W.h.p. $B = B_G(v, \frac{\Delta}{d})$ intersects at most $s = O(d)$ clusters.

Using the Lovász Local Lemma, we conclude

**Theorem ([Fil 20])**

Every graph with **doubling dimension** $d$ admits a

$(O(d), \tilde{O}(d))$-**strong** sparse partition scheme.
Theorem ([Fil 20])

Every graph with **doubling dimension** $d$ admits a $(O(d), \tilde{O}(d))$-**strong** sparse partition scheme.
Theorem ([Fil 20])

Every graph with pathwidth $\rho$ admits a $(O(\rho), O(\rho^2))$-strong sparse partition scheme, and a $(8, 5\rho)$-weak sparse partition scheme.
Theorem ([Fil 20])

Every cactus graph admits a (4, 5)-scattering partition scheme, and a (O(1), O(1))-weak sparse partition scheme.
Theorem ([Fil 20])

Every **chordal** graph admits a \((2, 3)\)-**scattering** partition scheme,  
and a \((24, 3)\)-**weak** sparse partition scheme.
Theorem ([Fil 20])

Suppose that the space $(\mathbb{R}^d, \| \cdot \|_2)$ admits a $(\sigma, \tau)$-weak sparse partition scheme.

Then $\tau \geq (1 + \frac{1}{2\sigma})^d$ (alternatively $\sigma > \frac{d}{4\ln \tau}$).
Theorem ([Fil 20])

The space \((\mathbb{R}^d, \| \cdot \|_2)\) admits a \((1, 2d)\)-scattering partition scheme.
Theorem ([Fil 20])

The space \((\mathbb{R}^d, \| \cdot \|_2)\) admits a \((1, 2d)\)-scattering partition scheme.

(For weak: \(\tau \geq (1 + \frac{1}{2^\sigma})^d \Rightarrow no (O(1), 2^{\Omega(d)})\)-weak partition scheme).
Theorem ([Fil 20])

Every $K_{r,r}$-free graph admits an $(O(r^2), 2^r)$-weak sparse partition scheme.
Theorem ([Fil 20])

Every $K_{r,r}$-free graph admits an $(O(r^2), 2^r)$-weak sparse partition scheme.

What about scattering?
Conjecture

Planar graphs are \((O(1), O(1))\)-scattering.
**Conjecture**

*Planar graphs are* \((O(1), O(1))\)-scattering.

Will imply a solution for the *SPR* problem with *distortion* \(O(1)\) for *planar* graphs!
Consider a **general** weighted $n$ vertex graph $G$:
- [JLNRS 05]: $G$ admits ($O(\log n), O(\log n)$)-**weak** sparse partition scheme.
Consider a **general** weighted $n$ vertex graph $G$:

- **[JLNRS 05]**: $G$ admits $(O(\log n), O(\log n))$-**weak** sparse partition scheme.
- **[KKN 14]** (implicitly): $G$ admits $(O(\log n), O(\log n))$-**scattering** partition scheme.
Consider a general weighted $n$ vertex graph $G$:

- [JLNRS 05]: $G$ admits $(O(\log n), O(\log n))$-weak sparse partition scheme.
- [KKN 14] (implicitly): $G$ admits $(O(\log n), O(\log n))$-scattering partition scheme.
- [Fil 20]: $G$ admits $(O(\log n), O(\log n))$-strong sparse partition scheme.
Consider a **general** weighted $n$ vertex graph $G$:

- **[JLNRS 05]**: $G$ admits $(O(\log n), O(\log n))$-weak sparse partition scheme.
- **[KKN 14]** (implicitly): $G$ admits $(O(\log n), O(\log n))$-scattering partition scheme.
- **[Fil 20]**: $G$ admits $(O(\log n), O(\log n))$-strong sparse partition scheme.
- **[Fil 20]**: $\exists G$ which **do not** admit $(O(\frac{\log n}{\log \log n}), O(\log n))$-weak sparse partition scheme.
Conjecture

Every $n$ vertex graph admits $(O(1), O(\log n))$-scattering partition scheme. Furthermore, this is tight.
Theorem ([JLNRS 05])

Suppose $G$ admits $(\sigma, \tau)$-weak sparse partition scheme,
$\Rightarrow$ solution to the UST problem with stretch $O(\tau \sigma^2 \log \tau n)$.

Theorem ([Fil 20])

Suppose that every induced subgraph $G[A]$ of $G$ admits $(\sigma, \tau)$-scattering partition scheme,
$\Rightarrow$ solution to the SPR problem with distortion $O(\tau^3 \sigma^3)$. 
**Conjecture**

Planar graphs are \((O(1), O(1))\)-scattering.

**Conjecture**

General \(n\) vertex graph are \((O(1), O(\log n))\)-scattering. Furthermore, this is tight.
Conjecture
Planar graphs are $(O(1), O(1))$-scattering.

Conjecture
General $n$ vertex graph are $(O(1), O(\log n))$-scattering. Furthermore, this is tight.

Thank you for listening!