On Notions of Distortion and an Almost Minimum Spanning Tree with Constant Average Distortion

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Warning

The talk is about an improved version of the paper.
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For details: www.cs.bgu.ac.il/~arnoldf/
Embedding

Let $\mathcal{X} = (X, d_X)$, $\mathcal{Y} = (Y, d_Y)$ be metric spaces. A function $f : \mathcal{X} \rightarrow \mathcal{Y}$ is called an embedding if for every $x, y \in X$,

$$d_X(x, y) \leq d_Y(f(x), f(y)) \leq t \cdot d_X(x, y)$$

where $t$ is a constant called the distortion of the embedding $f$. The average distortion of $f$ is defined as

$$\frac{1}{|X|^2} \sum_{u, v \in X} d_Y(f(u), f(v)) d_X(u, v).$$

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Embedding

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\(f : (X, d_X) \rightarrow (Y, d_Y)\) is called an embedding.

Distortion

\(f\) has distortion \(t\) if for every \(x, y \in X\),

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Average distortion

\[
\frac{1}{\binom{|X|}{2}} \cdot \sum_{v, u \in X} \frac{d_Y(f(v), f(u))}{d_X(v, u)}
\]
Graph spanner

Given a weighted graph $G = (V, E, w)$, a subgraph $H = (V, E_H, w)$ of $G$ is a spanner of $G$ with distortion $t$ if

$$\forall u, v \in V, \quad d_H(u, v) \leq t \cdot d_G(u, v)$$
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$$\forall u, v \in V, \quad d_H(u, v) \leq t \cdot d_G(u, v)$$

The lightness of a $H$ is

$$\Psi(H) = \frac{\sum_{e \in E_H} w(e)}{w(MST)}.$$
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Lightness vs Average Distortion in Trees

$G = (V, E, w)$ is a weighted graph.
Lightness vs Average Distortion in Trees

\[ G = (V, E, w) \] is a weighted graph.

The **MST** has **lightness** 1!
Lightness vs Average Distortion in Trees

$G = (V, E, w)$ is a weighted graph.

The MST has **lightness** 1!

But unbounded **average distortion**...
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The MST has **lightness** 1!
But unbounded **average distortion**...

**Theorem (Abraham, Bartal and Neiman 2006)**

*Every weighted graph contains a spanning tree with $O(1)$ average distortion.*
$G = (V, E, w)$ is a weighted graph.

The MST has **lightness** 1!
But unbounded **average distortion**...

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**Theorem (Abraham, Bartal and Neiman 2006)**

Every weighted graph contains a **spanning tree** with $O(1)$ **average distortion**.
But unbounded **lightness**...
Main result

Theorem (Constant lightness and average distortion (This work))

For any parameter $0 < \rho < 1$, every weighted graph contains a spanning tree with $O\left(\frac{1}{\rho}\right)$ average distortion.
Main result

Theorem (Constant lightness and average distortion (This work))

For any parameter $0 < \rho < 1$, every weighted graph contains a spanning tree with

- $1 + \rho$ lightness.
**Main result**

**Theorem (Constant lightness and average distortion (This work))**

For any parameter $0 < \rho < 1$, every weighted graph contains a spanning tree with

- $1 + \rho$ *lightness*.
- $O(1/\rho)$ *average distortion*. 

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On Notions of Distortion

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Main result

**Theorem (Constant lightness and average distortion (This work))**

For any parameter $0 < \rho < 1$, every weighted graph contains a spanning tree with

- $1 + \rho$ lightness.
- $O(1/\rho)$ average distortion.

Tight!
Prioritized Distortion

\[ f : (X, d_X) \to (Y, d_Y). \text{ The Distortion of } f \text{ is } \max_{x,y} \frac{d_Y(f(x),f(y))}{d_X(x,y)}. \]
Prioritized Distortion

\( f : (X, d_X) \rightarrow (Y, d_Y) \). The **Distortion** of \( f \) is \( \max_{x,y} \frac{d_Y(f(x), f(y))}{d_X(x, y)} \).

Given a monotone increasing function \( \alpha : \mathbb{N} \rightarrow \mathbb{R}_+ \)
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Given a monotone increasing function \( \alpha : \mathbb{N} \rightarrow \mathbb{R}_+ \)
Priority \( \pi = (x_1, \ldots, x_n) \).
Prioritized Distortion

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**Priority Distortion**

\( f : X \rightarrow Y \) has **priority distortion** \( \alpha \) w.r.t. \( \pi \) if

\[ \forall x_j, y \in X \quad d_X(x_j, y) \leq d_Y(f(x_j), f(y)) \leq \alpha(j) \cdot d_X(x_j, y) \]
Prioritized Distortion

\[ f : (X, d_X) \rightarrow (Y, d_Y). \text{ The Distortion of } f \text{ is } \max_{x,y} \frac{d_Y(f(x), f(y))}{d_X(x, y)}. \]

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Priority Distortion

\[ f : X \rightarrow Y \text{ has priority distortion } \alpha \text{ w.r.t. } \pi \text{ if } \]
\[ \forall x_j, y \in X \quad d_X(x_j, y) \leq d_Y(f(x_j), f(y)) \leq \alpha(j) \cdot d_X(x_j, y) \]

Theorem (Prioritized Spanner (This work))

Given a graph \( G = (V, E) \), parameter \( 0 < \rho < 1 \) and any priority ranking \( \pi \) of \( V \), there exists a spanner \( H \) with lightness \( 1 + \rho \)
and prioritized distortion \( \tilde{O}(\log j) / \rho \).
Scaling Distortion

Embedding $f : X \to Y$ has scaling distortion $\beta : (0, 1) \to \mathbb{R}_+$ if

$\forall \epsilon \in (0, 1)$ at least $(1 - \epsilon)$-fraction of the pairs suffer distortion at most $\beta(\epsilon)$.
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Theorem (Abraham, Bartal and Neiman 2006)

Every weighted graph contains a spanning tree with scaling distortion $O(1/\sqrt{\epsilon}).$
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Scaling Distortion implies constant average distortion

If $f$ has scaling distortion $O\left(\frac{1}{\epsilon^{1-\delta}}\right)$ for $\delta > 0$ then
$$\text{Average Distortion } = O(1).$$
Theorem (Priority implies scaling)

Given a metric space \((X, d_X)\), there exists a priority ranking \(\pi = (x_1, \ldots, x_n)\) s.t. every embedding with priority distortion \(\alpha\) w.r.t \(\pi\) into \((Y, d_Y)\) has scaling distortion \(O(\alpha (4/\epsilon))\).
Theorem (Priority implies scaling)

Given a metric space \((X, d_X)\), there exists a priority ranking \(\pi = (x_1, \ldots, x_n)\) such that every embedding with priority distortion \(\alpha\) w.r.t \(\pi\) into \((Y, d_Y)\) has scaling distortion \(O(\alpha^{4/\epsilon})\).
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has scaling distortion \(O(\alpha(4/\epsilon))\).

Scaling also implies priority!
Theorem (Prioritized Spanner)

Spanner with **lightness** $1 + \rho$ and **prioritized distortion** $\tilde{O}(\log j) / \rho$. 
Scaling Light Spanner

Theorem (Prioritized Spanner)

Spanner with lightness $1 + \rho$ and prioritized distortion $\tilde{O}(\log j) / \rho$.

Theorem (Priority implies Scaling)

Priority distortion $\alpha$ w.r.t $\pi$ implies scaling distortion $O(\alpha(4/\epsilon))$. 
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Theorem (Scaling Spanner)

Spanner with **lightness** $1 + \rho$ and **scaling distortion** $\tilde{O}(\log (1/\epsilon)) / \rho$. 
Theorem (Scaling Spanner)

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Theorem (Scaling Spanner)

Spanner with lightness $1 + \rho$ and scaling distortion $\tilde{O} \left( \log \left( \frac{1}{\epsilon} \right) \right) / \rho$.

Theorem (Abraham, Bartal and Neiman 2006)

Any graph contains a spanning tree with scaling distortion $O \left( \frac{1}{\sqrt{\epsilon}} \right)$. 
Theorem (Scaling Spanner)

Spanner with lightness $1 + \rho$ and scaling distortion $\tilde{O}(\log (1/\epsilon))/\rho$.

Theorem (Abraham, Bartal and Neiman 2006)

Any weighted graph contains a spanning tree with scaling distortion $O(1/\sqrt{\epsilon})$.

Lemma (Scaling embeddings Composition)

If $f : (X, d_X) \rightarrow (Y, d_Y)$ (respectively, $g : (Y, d_Y) \rightarrow (Z, d_Z)$) has scaling distortion $\alpha$ (resp., $\beta$). Then $f \circ g$ has scaling distortion $\alpha(\epsilon/2) \cdot \beta(\epsilon/2)$. 
Theorem (Scaling Spanner)

Spanner with lightness $1 + \rho$ and scaling distortion $\tilde{O}\left(\log\left(\frac{1}{\epsilon}\right)/\rho\right)$.

Theorem (Abraham, Bartal and Neiman 2006)

Any w.graph contains a spanning tree with scaling distortion $O(1/\sqrt{\epsilon})$.

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If $f : (X, d_X) \rightarrow (Y, d_Y)$ (respectively, $g : (Y, d_Y) \rightarrow (Z, d_Z)$) has scaling distortion $\alpha$ (resp., $\beta$). Then $f \circ g$ has scaling distortion $\alpha(\epsilon/2) \cdot \beta(\epsilon/2)$.

Theorem

Spanning tree with lightness $1 + \rho$ and scaling distortion $\tilde{O}(\sqrt{1/\epsilon})/\rho$. 
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Corollary (Main result)

Spanning tree with lightness $1 + \rho$ and average distortion $O(1/\rho)$. 

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Light Tree with Constant Average Distortion

**Theorem**

Spanning tree with lightness $1 + \rho$ and scaling distortion $\tilde{O}(\sqrt{1/\epsilon})/\rho$.

**Corollary (Main result)**

Scaling distortion $O(\frac{1}{\epsilon^{1-\delta}})$ implies $O(1)$ average distortion.
Theorem

Spanning tree with lightness $1 + \rho$ and scaling distortion $\tilde{O}(\sqrt{1/\epsilon})/\rho$.

Scaling distortion $O(\frac{1}{\epsilon^{1-\delta}})$ implies $O(1)$ average distortion.

Corollary (Main result)

Spanning tree with lightness $1 + \rho$ and average distortion $O(1/\rho)$. 
BRACE YOURSELVES

TECHNICAL DETAILS ARE COMING
Theorem (Priority implies scaling)

Given a metric space \((X, d_X)\), there exists a priority ranking \(\pi = (x_1, \ldots, x_n)\) s.t. every embedding with priority distortion \(\alpha\) w.r.t \(\pi\) into \((Y, d_Y)\) has scaling distortion \(O(\alpha(4/\epsilon))\).
Given $x \in X$ and $\epsilon \in (0, 1)$, $R(x, \epsilon)$ is the minimal radius $r$ such that

$$|B_X (x, r)| \geq \epsilon \cdot n$$
Given \( x \in X \) and \( \epsilon \in (0, 1) \), \( R(x, \epsilon) \) is the \textbf{minimal} radius \( r \) such that

\[
|B_X (x, r)| \geq \epsilon \cdot n
\]
**ε-Density Net** is a subset $N \subseteq X$ such that:

- $\forall x \in X$ there exists $y \in N$ such that $d_X(x, y) \leq 2R(x, \epsilon)$.
- $|N| \leq 1/\epsilon$. 

**Theorem (H. Chan, M. Dinitz and A. Gupta 2006)** For every metric space and $\epsilon \in (0, 1)$ there exists an $\epsilon$-density-net.
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\textbf{Theorem (H.Chan, M.Dinitz and A.Gupta 2006)}

For every metric space and \(\varepsilon \in (0, 1)\) there exists an \(\varepsilon\)-density-net.
Priority implies Scaling - Proof

For $1 \leq i \leq \lceil \log n \rceil$ set $\epsilon_i = 2^{-i}$. Let $N_i$ be an $\epsilon_i$-density net.
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Permutation selection:

\[
\begin{array}{cccccccccccc}
    & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \cdots & x_{14} & x_{2^1-1} & \cdots & x_{2^5-2} & x_{2^5-1} & \cdots \\
N_1 & N_2 & N_3 & N_4 & & & & & & & & & & \\
\end{array}
\]
Theorem (Prioritized Spanner)

Given a graph \( G = (V, E) \), parameter \( 0 < \rho < 1 \) and any priority ranking \( \pi \) of \( V \), there exists a spanner \( H \) with lightness \( 1 + \rho \) and prioritized distortion \( \tilde{O}(\log j) / \rho \).
Prioritized Light Spanner

**Theorem (Prioritized Spanner)**

Given a graph $G = (V, E)$, parameter $0 < \rho < 1$ and any priority ranking $\pi$ of $V$, there exists a spanner $H$ with lightness $1 + \rho$ and prioritized distortion $\tilde{O}(\log j) / \rho$.

**Lemma (Terminal light spanner)**

Given a graph $G = (V, E)$, a subset $K \subseteq V$ of terminals of size $k$, and a parameter $0 < \delta < 1$, there exists a spanner $H$ that:

1. Contains the MST of $G$.
2. Has lightness $1 + \delta$.
3. Every pair in $K \times V$ has distortion $O(\log k \delta)$.
Theorem (Prioritized Spanner)

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On Notions of Distortion

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1) Contains the MST of $G$.
2) Has lightness $1 + \delta$.
3) Every pair in $K \times V$ has distortion $O\left(\frac{\log k}{\delta}\right)$. 
Theorem (Chechik and Wulff-Nilsen (SODA 16), following Chandra et.al and Elkin et.al.)

For every weighted $n$-vertex graph $G$ and parameters $t > 1, \epsilon > 0$ there exist a $(2t - 1)(1 + \epsilon)$ spanner of lightness $O_\epsilon(n^{1/t})$. 
Theorem (Chechik and Wulff-Nilsen (SODA 16), following Chandra et.al and Elkin et.al.)

For every weighted $n$-vertex graph $G$ and parameters $t > 1$, $\epsilon > 0$ there exist a $(2t - 1)(1 + \epsilon)$ spanner of lightness $O_\epsilon(n^{1/t})$.

For $t = \log n$ and $\epsilon = 1$, they get $O(\log n)$-spanner with lightness $O(1)$. 
Theorem (Light spanners reduction)

Suppose that for every $n$ vertex graph $G$ there is a spanner $H$ that:

1. Has lightness $\ell$.
2. Has distortion $t$.

Then, for every $n$ vertex graph $G$ and parameter $0 < \delta < 1$, there is a spanner $H$ that:

1. Has lightness $1 + \delta \ell$.
2. Has distortion $t/\delta$.
3. Contains the MST.
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General reduction to light spanners

Theorem (Light spanners reduction)

Suppose that for every $n$ vertex graph $G$ there is a spanner $H$ that:

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General reduction to light spanners

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Then, for every $n$ vertex graph $G$ and parameter $0 < \delta < 1$, there is a spanner $H$ that:

1. Has lightness $1 + \delta \ell$. $1 + \delta$
2. Has distortion $t/\delta$. \(O(\log n)/\delta\)
3. Contains the MST.
Theorem (Prioritized Spanner)

Spanner with lightness $1 + \rho$ and prioritized distortion $\tilde{O}(\log j) / \rho$. 

Is it possible to get prioritized distortion $O(\log j) / \rho$?

Efficient implementation

While the current implementation is polynomial, it is still far from practical.
Theorem (Prioritized Spanner)

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Open Problems

**Theorem (Prioritized Spanner)**

*Spanner with lightness* $1 + \rho$ *and prioritized distortion* $\tilde{O}(\log j)/\rho$.

Is it possible to get prioritized distortion $O(\log j)/\rho$?

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While the current implementation is polynomial, it is still far from practical.