Tractability of tensor product problems in the average case setting

Anargyros Papageorgiou and Iasonas Petras

June 1, 2010

Abstract

It has been an open problem to derive a necessary and sufficient condition for a linear tensor product problem $S = \{S_d\}$ in the average case setting to be weakly tractable but not polynomially tractable. As a result of the tensor product structure, the eigenvalues of the covariance operator of the induced measure in the one dimensional problem characterize the complexity of approximating S_d, $d \geq 1$, with accuracy ε. If $\sum_{j=1}^{\infty} \lambda_j < 1$ and $\lambda_2 > 0$, we know that S is not polynomially tractable iff $\limsup_{j\to\infty} \lambda_j j^p = \infty$ for all $p > 1$. Thus we settle the open problem by showing that S is weakly tractable iff $\sum_{j>n} \lambda_j = o\left(\ln^{-2} n\right)$. In particular, assume that

$$\ell = \lim_{j\to\infty} \lambda_j j \ln^3(j+1),$$

exists. Then S is weakly tractable iff $\ell = 0$.

1 Introduction

The complexity of multivariate problems is often studied with respect to the required accuracy ε while assuming the number of variables d is fixed. Treating d as a constant disregards important aspects of the complexity. There are many problems where the cost of any algorithm solving them to within ε grows exponentially with d.

Henryk Woźniakowski initiated research studying the complexity of multivariate problems as a function of the accuracy ε and the dimension d about fifteen years ago. His work and that of his colleagues has produced numerous results, many of them quite recent, which are included in the recently published book *Tractability of Multivariate Problems, Volume 1: Linear Information*, by Erich Novak and himself. A second volume *Tractability of Multivariate Problems, Volume 2: Standard Information for Functionals*, by the same authors, is expected to be published by the European Mathematical Society in 2010. The two books contain 91 open problems. In this paper we solve Open Problem 28, which can be found in the first volume.

The problem we study in this paper concerns linear tensor product problems in the average case setting. A linear problem $S = \{S_d\}$ is obtained through a sequence of continuous linear operators S_d each defined on a space of functions of $d \geq 1$ variables [2]. In the worst case setting, the tensor product structure is introduced by setting

$$S_d = S_1^{\otimes d},$$

where S_1 is defined on a space of univariate functions. This construction is generalized in the average case setting. In fact, fewer assumptions are necessary. For $S_d : F_d \to G_d$, only the target space G_d needs to be a tensor product space $G_d = G_1^{\otimes d}$, where G_1 is a Hilbert space. The space F_d is equipped with a Gaussian measure that is *derived* from a given Gaussian measure on F_1. We will go over the details later in this paper.

We are interested in algorithms approximating the operator S_d using n evaluations of arbitrary linear functionals and we consider their average error. The information complexity (complexity for brevity) is the minimal number of evaluations needed to approximate S_d to within accuracy ε. In order to underline the dependency on ε and d, we denote the complexity by $n(\varepsilon, d)$.

The problem S is *polynomially* tractable iff $n(\varepsilon, d)$ grows as a polynomial in d and ε^{-1}. In particular, when $n(\varepsilon, d)$ is bounded by a quantity independent of d and polynomial in ε^{-1} the problem S is *strongly polynomially* tractable.

The problem S is *weakly tractable* iff

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n(\varepsilon, d)}{\varepsilon^{-1} + d} = 0,$$

otherwise the problem is intractable. Hence, a problem is weakly tractable if its complexity is not exponential in both ε^{-1} and d. Note that weakly tractable problems have complexity that is subexponential even though it may grow faster than any polynomial in ε^{-1} or d.

The complexity of linear tensor product problems in the average case setting is characterized by the eigenvalues of the covariance operator of the induced measure on the space G_1. These eigenvalues, due to the tensor product structure, determine the rate of decay of the eigenvalues of the covariance operator in the d-dimensional problem and, through them, they determine the (average) error of optimal algorithms, as we will see soon.

We know that if the sum of all the eigenvalues, in the one-dimensional problem, is greater than or equal to 1 then S is intractable. When the opposite is true, the notions of polynomial and strong polynomial tractability are equivalent. When the eigenvalues in the one dimensional problem satisfy

$$\lambda_j = O(j^{-p}) \quad \text{with} \quad p > 1,$$

the problem is polynomially tractable. See [2, Ch. 6.2] for all the details. It has been an open question to characterize the eigenvalues for which S is weakly tractable but not polynomially tractable. The precise statement of this question is Open Problem 28 in [2], and we solve it in this paper.

In particular, consider the linear tensor product problem $S = \{S_d\}$ in the average case setting with $\sum_{j=1}^{\infty} \lambda_j < 1$, $\lambda_2 > 0$, for the absolute error criterion. We show that:

- S is weakly tractable iff

$$t_n = \sum_{j>n} \lambda_j = o\left(\frac{1}{\ln^2(n+1)}\right).$$

- In particular, suppose that

$$\ell = \lim_{j\to\infty} \lambda_j j \ln^3(j+1)$$

  exists. Then S is weakly tractable iff $\ell = 0$.

In this paper we deal only with the absolute error criterion since linear tensor product problems are intractable in the average case setting with the normalized error criterion for $\lambda_2 > 0$. M. A. Lifshits and E. V. Tulyakova in [1, Sec. 3] derive the complexity with the normalized error criterion when $d \to \infty$ and ε is fixed. It appears that their approach can be used to derive the complexity for the absolute error criterion as well. However, we do not pursue this since we are mainly interested in determining when weak tractability holds.

2 Linear Tensor Product Problems

In this section we briefly introduce linear tensor product problems in the average case setting. The material is from [2, Ch.6] and we include it here for the convenience of the reader.

Let

$$S_d : F_d \to G_d,$$

be a continuous linear operator mapping a separable Banach space F_d to a separable Hilbert space G_d, $d \geq 1$. We assume that the space G_d is the tensor product of d copies of a separable Hilbert space G, i.e., $G_d = \otimes_{i=1}^d G$. Thus G_d is spanned by $\otimes_{i=1}^d g_i$, $g_i \in G$, and has an inner product defined by

$$\langle \otimes_{i=1}^d g_i, \otimes_{i=1}^d h_i \rangle_{G_d} = \prod_{i=1}^d \langle g_i, h_i \rangle_G \quad \text{for} \quad g_i, h_i \in G.$$

Hence,

$$S_d f = \sum_{j\in\mathbb{N}^d} \langle S_d f, \eta_{d,j} \rangle_{G_d} \eta_{d,j} \quad \text{for } f \in F_d,$$

where

$$\eta_{d,j} = \otimes_{k=1}^d \eta_{j_k} \quad j = [j_1, j_2, \ldots, j_d] \in \mathbb{N}^d, \tag{1}$$

and $\{\eta_i\}_{i\in\mathbb{N}}$ is an orthonormal system in G.

Consider a zero-mean Gaussian measure μ_d on F_d with $\int_{F_d} \|S_d f\|_{G_d}^2 \, \mu_d(df) < \infty$. Let $\nu_d = \mu_d S_d^{-1}$ be the induced measure on G_d, which is also a zero-mean Gaussian measure. Let C_{ν_d} denote the covariance operator of ν_d and let $(\lambda_{d,j}, \eta_{d,j})$, $j \in \mathbb{N}^d$, be its eigenvalues and the corresponding eigenvectors.

We also assume that the eigenvalues satisfy the conditions below, in order to preserve the tensor product structure of G_d and its orthonormal system $\{\eta_{d,j}\}_{j\in\mathbb{N}^d}$. For $d=1$, $\lambda_{1,j}=\lambda_j$, with $\lambda_1 \geq \lambda_2 \geq \ldots \geq 0$ and

$$\sum_{j=1}^{d} \lambda_j = \operatorname{trace}(C_{\nu_1}) < \infty.$$

For $d \geq 1$, we assume

$$\lambda_{d,j} = \Pi_{k=1}^{d} \lambda_{j_k} \ \text{ for all } j = [j_1, j_2, \ldots, j_d] \in \mathbb{N}^d, \tag{2}$$

and

$$\operatorname{trace}(C_{\nu_d}) = \sum_{j\in\mathbb{N}^d} \lambda_{d,j} = \left(\sum_{i=1}^{\infty} \lambda_j \right)^d .$$

A *linear tensor product problem* in the *average case setting* is a multivariate problem $S=\{S_d\}$ with the eigenpairs of the covariance operator C_{ν_d} satisfying the conditions (1, 2).

For notational convenience, let us now reindex the eigenvalues and eigenvectors to obtain $\{\lambda_{d,j}\}_{j\in\mathbb{N}^d} = \{\lambda_{d,i}\}_{i\in\mathbb{N}}$ and $\{\eta_{d,j}\}_{j\in\mathbb{N}^d} = \{\eta_{d,i}\}_{i\in\mathbb{N}}$, respectively. Also assume the eigenvalues are ordered, $\lambda_{d,1} \geq \lambda_{d,2} \geq \cdots \geq 0$.

Suppose that we can use arbitrary linear functionals on F_d as information operations, i.e., we can use functionals from the class Λ^{all}, as denoted in [2, 3]. Then it is known, see e.g. [3], that the algorithm

$$A_{d,n}(f) = \sum_{i=1}^{n} \langle S_d f, \eta_{d,i} \rangle_{G_d} \eta_{d,i} \tag{3}$$

minimizes the average error

$$e(A_{d,n}) = \left(\int_{F_1} \|Sf - A_{d,n}(f)\|_{G_d}^2 \, \mu(df) \right)^{1/2},$$

among all possible algorithms using at most n information operations. It is also known that the error of this optimal algorithm is obtained from the truncated trace of C_{ν_d} and

$$e(A_{d,n}) = \left(\sum_{i=n+1}^{\infty} \lambda_{d,i} \right)^{1/2} . \tag{4}$$

The information complexity of the problem S_d for accuracy ε with the absolute error criterion is the minimal number of information operations needed to guarantee that the average case error is at most ε, and is given by

$$n(\varepsilon, d) = \min \left\{ n : \sum_{i=n+1}^{\infty} \lambda_{d,i} \leq \varepsilon^2 \right\} .$$

The problem S is polynomially tractable iff there exist constants $c, p_2 \geq 0$, $p_1 > 0$ such that

$$n(\varepsilon, d) \leq c\, d^{p_2} \varepsilon^{-p_1} \qquad \forall\, d \geq 1,\ \varepsilon \in (0,1).$$

When $p_2 = 0$ the problem is strongly polynomially tractable.

As we have already mentioned, the problem S is weakly tractable iff

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n(\varepsilon, d)}{\varepsilon^{-1} + d} = 0.$$

The reader is referred to [2] for more details.

3 Weak Tractability

Linear tensor product problems in the average case setting are discussed in [2, Ch. 6]. We briefly review some of the results, which motivate Open Problem 28 in the book [2]. Details can be found in [2, Th. 6.5, Th. 6.6].

Recall that we deal only with the absolute error criterion, since linear tensor product problems are intractable in the average case setting with the normalized error criterion for $\lambda_2 > 0$.

If the one-dimensional eigenvalues satisfy $\sum_{j=1}^{\infty} \lambda_j \geq 1$ then the linear tensor product problem $S = \{S_d\}$ is intractable. From this point on we consider the case

$$\sum_{j=1}^{\infty} \lambda_j < 1, \qquad \lambda_2 > 0.$$

Then the following conditions are equivalent:

1. S is polynomially tractable.
2. S is strongly polynomially tractable.
3. There exists a $\tau \in (0, 1)$ such that $\sum_{j=1}^{\infty} \lambda_j^{\tau} \leq 1$.

Moreover, if $\lambda_j = O(j^{-p})$ with $p > 1$, the following conditions are equivalent:

1. S is weakly tractable.
2. S is polynomially tractable.
3. S is strongly polynomially tractable.
4. $\sum_{j=1}^{\infty} \lambda_j < 1$.

Combining the above with

$$\sum_{j=1}^{\infty} \lambda_j^{\tau} \leq 1 \;\text{ for }\; \tau \in (0, 1) \;\text{ iff }\; \sum_{j=1}^{\infty} \lambda_j < 1 \;\text{ and }\; \lambda_j = \mathcal{O}(j^{-p}) \;\text{ for }\; p > 1,$$

(see, [2, p. 258] for the proof) we conclude that one possibility for having a weakly tractable problem which is not polynomially tractable is if

$$\lambda_j = O\left(\frac{1}{j \ln^q (j+1)}\right) \quad \text{for } q > 1.$$

This observation led to the following open problem in [2]:

Open Problem 28

Consider the linear tensor product problem in the average case setting $S = \{S_d\}$ with $\sum_{j=1}^{\infty} \lambda_j < 1$ and $\lambda_2 > 0$. Study this problem for the absolute error criterion and for the class Λ^{all}. Verify whether there are eigenvalues λ_j for which we have weak tractability but not polynomial tractability. If so, characterize all such $\{\lambda_j\}$. In particular, characterize the numbers q for which we have weak tractability for

$$\lambda_j = \Theta\left(\frac{1}{j \ln^q (j+1)}\right). \quad \square$$

We are interested in estimating the information complexity $n(\varepsilon, d)$. Let $a = \sum_{j=1}^{\infty} \lambda_j$. The error of the zero algorithm that does not use any information at all is $a^{d/2}$. Hence, the interesting case is when the required accuracy satisfies $\varepsilon^2 < a^d$. Let us start with the relatively easier task of characterizing the numbers q and then we will deal with the general case.

Lemma 1. *Consider the eigenvectors of C_{ν_d} given by*

$$\eta_{d,j} = \eta_{j_1} \otimes \cdots \otimes \eta_{j_d},$$

where $j = [j_1, j_2, \ldots, j_d]$, for $j_k = 1, \ldots, m$, and $k = 1, \ldots, d$. The average error of the algorithm

$$\phi_{d,m^d}(f) = \sum_{j_1,\ldots,j_d=1}^{m} \langle S_d(f), \eta_{d,j} \rangle \eta_{d,j}$$

is bounded from above as follows

$$e^2(\phi_{d,m^d}) \leq d\, a^{d-1}\, t_m,$$

where $t_m = \sum_{j=m+1}^{\infty} \lambda_j$.

Proof. The error of ϕ_{d,m^d} satisfies

$$\begin{aligned}
e^2(\phi_{d,m^d}) &= \sum_{j_1,\dots j_d\geq 1} \lambda_{j_1}\dots\lambda_{i_d} - \sum_{j_1,\dots j_d=1}^{m} \lambda_{j_1}\dots\lambda_{i_d} \\
&= \sum_{j_1>m,j_2,\dots,j_d\geq 1} \lambda_{j_1}\dots\lambda_{j_d} + \sum_{j_1\leq m,j_2,\dots,j_d\geq 1} \lambda_{j_1}\dots\lambda_{j_d} - \sum_{j_1,\dots j_d=1}^{m} \lambda_{j_1}\dots\lambda_{j_d} \\
&= t_m a^{d-1} + \sum_{j_1\leq m,j_2,\dots,j_d\geq 1} \lambda_{j_1}\dots\lambda_{j_d} - \sum_{j_1,\dots j_d=1}^{m} \lambda_{j_1}\dots\lambda_{j_d} \\
&\leq 2t_m a^{d-1} + \sum_{j_1,j_2\leq m,j_3,\dots,j_d\geq 1} \lambda_{j_1}\dots\lambda_{j_d} - \sum_{j_1,\dots j_d=1}^{m} \lambda_{j_1}\dots\lambda_{j_d} \\
&\vdots \\
&\leq d a^{d-1} t_m.
\end{aligned}$$

We remark that the algorithm ϕ_{d,m^d} minimizes the average error among all algorithms that use the the same information as ϕ_{d,m^d} although this information is not optimal, in general. The reason is that the eigenvectors $\eta_{d,j}$ do not correspond to the m^d largest eigenvalues. Hence, if m is large enough and ϕ_{d,m^d} satisfies the accuracy demand ε then m^d is an upper bound of $n(\varepsilon, d)$. $\square$

Theorem 1. *Consider the linear tensor product problem $S = \{S_d\}$ in the average case setting with $\sum_{j=1}^{\infty} \lambda_j < 1$, $\lambda_2 > 0$, for the absolute error criterion and the class of Λ^{all}.*

- *S is weakly tractable iff*

$$t_n = \sum_{j>n} \lambda_j = o\left(\frac{1}{\ln^2(n+1)}\right).$$

- *In particular, suppose that*

$$\ell = \lim_{j\to\infty} \lambda_j j \ln^3(j+1)$$

  exists. Then S is weakly tractable iff $\ell = 0$.

Proof. We begin by showing that

$$t_n = \sum_{j>n} \lambda_j = o\left(\frac{1}{\ln^2(n+1)}\right)$$

is a sufficient condition for weak tractability. Let ε^{-1} and/or d to be sufficiently large. The error of the algorithm ϕ_{d,m^d} of Lemma 1 satisfies

$$e^2(\phi_{d,m^d}) \leq d a^{d-1} t_m = d a^{d-1} \frac{s_m}{\ln^2(m+1)},$$

where $s_m = o(1)$.

Let $m = m(\varepsilon, d)$ be the smallest integer such that

$$e^2(\phi_{d,m^d}) \le da^{d-1}\frac{s_m}{\ln^2 m} \le \varepsilon^2 < a^d.$$

Then $m \to \infty$ as $\varepsilon \to 0$ and/or $d \to \infty$. Clearly $n(d,\varepsilon) \le m^d$ and

$$\ln m \ge (da^{d-1}s_m)^{1/2}\varepsilon^{-1}.$$

By definition of $m(\varepsilon, d)$, there exists a constant c such that

$$\ln m(\varepsilon, d) \le c\,(da^{d-1}s_{m(\varepsilon,d)-1})^{1/2}\varepsilon^{-1}.$$

Hence,

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n(\varepsilon,d)}{\varepsilon^{-1}+d} \le \lim_{\varepsilon^{-1}+d\to\infty} \frac{d\ln m(\varepsilon,d)}{\varepsilon^{-1}+d} = \lim_{\varepsilon^{-1}+d\to\infty} \frac{c\,d\left[a^{d-1}s_{m(\varepsilon,d)-1}\right]^{1/2}\varepsilon^{-1}}{\varepsilon^{-1}+d} = 0.$$

On the other hand, it is relatively easy to show that

$$t_n = \sum_{j>n}\lambda_j = o\left(\frac{1}{\ln^2(n+1)}\right)$$

is a necessary condition for weak tractability. One can use the same proof as the one used in [2, p. 178] for the worst case. For completeness, we include it here. Assume S is weakly tractable, i.e.,

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n(\varepsilon,d)}{\varepsilon^{-1}+d} = 0.$$

Setting $d = 1$, we get $\frac{1}{\varepsilon^{-1}+1} = o(\ln^{-1} n(\varepsilon,1))$ as $\varepsilon \to 0$. Equivalently, $\varepsilon = o(\ln^{-1} n(\varepsilon,1))$. Also

$$\varepsilon^2 \ge e^2(A_{1,n(\varepsilon,1)}) = t_{n(\varepsilon,1)}.$$

So

$$t_n = o(\ln^{-2}(n+1)).$$

This completes the proof of the first part of the theorem.

For the second part of the theorem it is easy to see that $\ell = 0$ is a necessary condition for weak tractability. Indeed, let $d = 1$ and ε be sufficiently small. Assume there exists a constant c such that $\ell = \lim_{j\to\infty}\lambda_j j\ln^3(j+1) \ge c > 0$. Then t_n is bounded from below as follows

$$t_n = e^2(A_{1,n}) \ge c\sum_{j>n}\frac{1}{j\ln^3(j+1)} = \Omega\left(\frac{1}{\ln^2(n+2)}\right)$$

and we have a contradiction.

We now show that the condition $\ell = 0$ is sufficient for weak tractability. Let

$$\lambda_j = \frac{g(j)}{j\ln^3(j+1)}$$

and since $\ell = 0$ we have $g(j) = o(1)$. Let ε^{-1} and/or d be sufficiently large. We have

$$t_n = \sum_{j>n} \frac{g(j)}{j \ln^3(j+1)} \leq \frac{s_n}{\ln^2(n+1)},$$

where $s_n = \sup_{j>n} g(j) = o(1)$. Hence $t_n = o(\ln^{-2}(n+1))$, and the first part of the theorem yields that S is weakly tractable. $\square$

Remark 1. In the second part of Theorem 1 we assumed that the limit of $\lambda_j j \ln^3(j+1)$ exists as $j \to \infty$ and we showed a necessary and sufficient condition for weak tractability. If, on the other hand, this limit does not exist the problem may still be weakly tractable.

Indeed, the condition $t_n = \sum_{j>n} \lambda_j = o(\ln^{-2}(n+1))$ implies that $n\lambda_{2n} = o(\ln^{-2}(n+1))$. Therefore,

$$\lambda_n = o\left(\frac{1}{n \ln^2(n+1)}\right)$$

is a necessary condition for weak tractability. Moreover, proceeding in a way similar to that in the proof of Theorem 1, we can show a second necessary condition, namely

$$\liminf_{n\to\infty} \lambda_n n \ln^3(n+1) = 0.$$

It is interesting to observe that as long as the slower converging subsequence of eigenvalues does not contribute excessively to t_n the problem can be weakly tractable. We illustrate this by an example.

Let k_0 be a sufficiently large integer. For $k = k_0, k_0+1, \ldots$, let $j = \lceil e^{k^2} \rceil$, and

$$\lambda_i = \frac{1}{j \ln^{3+\gamma}(j+1)} \qquad i = j+1, \ldots, \lceil j + j\ln(j+1) \rceil,$$

and $\gamma \in (1/2, 1)$. So we have a segment of $\lceil j \ln(j+1) \rceil$ eigenvalues that are equal and the first eigenvalue in the segment, λ_j, goes to zero faster than the last $\lambda_{\lceil j + j\ln(j+1)\rceil}$. Furthermore, since k_0 is large enough the segments are disjoint. We define the remaining eigenvalues by

$$\lambda_j = \frac{1}{j \ln^{3+\gamma}(j+1)}.$$

Hence, $\lambda_n = o(n^{-1} \ln^{-2}(n+1))$ and $\liminf_{n\to\infty} \lambda_n n \ln^3(n+1) = 0$. However, $\limsup_{n\to\infty} \lambda_n n \ln^3(n+1) = \infty$ since $\gamma < 1$. Thus the limit ℓ of Theorem 1 does not exist. Nevertheless, S is weakly tractable.

Indeed,

$$\sum_{i=j+1}^{\lceil j + j\ln(j+1)\rceil} \lambda_i \leq c' \frac{1}{\ln^{2+\gamma}(j+1)},$$

where c' is an absolute constant. The contribution of all such segments starting at $j = \lceil e^{k^2} \rceil$, $k \in \mathbb{N}$, to t_n is

$$c' \sum_{j=\lceil e^{k^2}\rceil > n,\ k\in\mathbb{N}} \frac{1}{\ln^{2+\gamma}(j+1)} \leq c' \frac{1}{\ln^{2+\gamma}(n+1)} + c'' \int_{x^2 > \ln n} \frac{dx}{x^{2(2+\gamma)}} = o\left(\frac{1}{\ln^2(n+1)}\right),$$

where c'' is an absolute constant and the last equality holds since $\gamma > 1/2$. It is easy to see that the contribution to t_n of the remaining eigenvalues is also $o(\ln^{-2}(n+1))$. Since $t_n = o(\ln^{-2}(n+1))$ the problem S is weakly tractable, as claimed.

Finally, it is relatively easy to see that a problem can be weakly tractable even though it is not polynomially tractable. We state this fact in the following corollaries.

Corollary 1. *Consider the linear tensor product problem $S = \{S_d\}$ in the average case setting with $\sum_{j=1}^{\infty} \lambda_j < 1$ for the absolute error criterion and the class of Λ^{all}. If $\lambda_j = \Theta\left(\frac{1}{j \ln^q(j+1)}\right)$, then the problem is weakly tractable if and only if $q > 3$.*

Proof. This directly follows from Theorem 1. □

Corollary 2. *Consider the linear tensor product problem $S = \{S_d\}$ in the average case setting with $\sum_{j=1}^{\infty} \lambda_j < 1$, $\lambda_2 > 0$, for the absolute error criterion and the class of Λ^{all}. Then S is weakly tractable but not polynomially tractable iff*

$$t_n = \sum_{j>n} \lambda_j = o\left(\frac{1}{\ln^2(n+1)}\right).$$

and

$$\limsup_{j\to\infty} \lambda_j j^p = \infty \qquad \text{for all } p > 1.$$

Proof. This directly follows from Theorem 1 and [2, Th. 6.7]. □

Acknowledgements

We are grateful Erich Novak and to Henryk Woźniakowski for insightful remarks and suggestions that significantly improved this paper. This research was supported in part by NSF.

References

[1] M. A. Lifshits and E. V. Tulyakova, *Curse of dimensionality in approximation of random fields*, Prob. Math. Stat., 26(1), 2006, 97–112.

[2] E. Novak and H. Woźniakowski, *Tractability of Multivariate Problems*, Volume 1, Linear Information, European Math. Soc., Zürich, 2008.

[3] J. F. Traub, G. W. Wasilkowski and H. Woźniakowski, *Information-Based Complexity*, Academic Press, New York, 1988.

Authors' addresses

Anargyros Papageorgiou
ap@cs.columbia.edu
Department of Computer Science, Columbia University,
New York, NY 10027, USA.

Iasonas Petras
ipetras@cs.columbia.edu
Department of Computer Science, Columbia University,
New York, NY 10027, USA.