Average Case Tractability of Non-homogeneous Tensor Product Problems

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Abstract

We study d-variate approximation problems in the average case setting with respect to a zero-mean Gaussian measure ν_d. Our interest is focused on measures having a structure of non-homogeneous tensor product, where covariance kernel of ν_d is a product of univariate kernels,

$$K_d(s,t) = \prod_{k=1}^{d} \mathcal{R}_k(s_k, t_k) \qquad \text{for} \quad s, t \in [0,1]^d.$$

We consider the normalized average error of algorithms that use finitely many evaluations of arbitrary linear functionals. The information complexity is defined as the minimal number $n^{\mathrm{avg}}(\varepsilon, d)$ of such evaluations for error in the d-variate case to be at most ε. The growth of $n^{\mathrm{avg}}(\varepsilon, d)$ as a function of ε^{-1} and d depends on the eigenvalues of the covariance operator of ν_d and determines whether a problem is tractable or not. Four types of tractability are studied and for each of them we find the necessary and sufficient conditions in terms of the eigenvalues of the integral operator with kernel $\mathcal{R}_k$.

We illustrate our results by considering approximation problems related to the product of Korobov kernels $\mathcal{R}_k$. Each $\mathcal{R}_k$ is characterized by a weight g_k and a smoothness r_k. We assume that weights are non-increasing and smoothness parameters are non-decreasing. Furthermore they may be related, for instance $g_k = g(r_k)$ for some non-increasing function g. In particular, we show that approximation problem is strongly polynomially tractable, i.e., $n^{\mathrm{avg}}(\varepsilon, d) \leq C \, \varepsilon^{-p}$ for all $d \in \mathbb{N}, \varepsilon \in (0,1]$, where C and p are independent of ε and d, iff

$$\liminf_{k \to \infty} \frac{\ln \frac{1}{g_k}}{\ln k} > 1.$$

For other types of tractability we also show necessary and sufficient conditions in terms of the sequences g_k and r_k.

1 Introduction

Multivariate problems occur in many applications. They are defined on classes of functions of d variables. Often the number of variables d is large. Examples include problems in computational finance, statistics and physics. These problems have been studied for different error criteria and in different settings including the worst and average case settings. The cost of an algorithm solving a problem depends on the accuracy ε and the number of variables d. A problem is intractable if the cost of any algorithm is an exponential function of ε^{-1} or d. Otherwise, the problem is tractable. Different types of tractable problems have been considered in the literature. In fact, tractability of multivariate problems has been recently a very active research area, see [9, 10, 11] and the references therein.

More precisely, the information complexity $n(\varepsilon, d)$ of a problem is the minimal number of information operations needed by an algorithm to solve the problem with accuracy ε. The allowed information operations consist of function evaluations, or, more generally, of evaluations of arbitrary continuous linear functionals. We have

- *weak* tractability if $n(\varepsilon, d)$ is *not* exponential in d and ε^{-1},

- *quasi-polynomial* tractability if $n(\varepsilon, d)$ is of order $\exp(\, t\, (1 + \ln d)(1 + \ln \varepsilon^{-1}))$,

- *polynomial* tractability if $n(\varepsilon, d)$ is of order $d^q\, \varepsilon^{-p}$,

- *strong* polynomial tractability if $n(\varepsilon, d)$ is of order ε^{-p}.

The bounds above hold for all d and all $\varepsilon \in (0, 1)$ with the parameters t, q, p and the pre-factors independent of d and ε^{-1}.

Strong polynomial tractability is the most challenging property. Then the information complexity is bounded independently of d. One may think that this property may hold only for trivial problems. Luckily, as we shall see, the opposite is sometimes true.

On the other hand, many multivariate problems are intractable. In particular, they suffer from the curse of dimensionality. One way to vanquish the curse is to shrink the class of functions by introducing the weights that monitor the influence of successive variables and groups of variables. For sufficiently fast decaying weights not only we vanquish the curse but obtain strong polynomial tractability; a survey of such results may be found again in [9, 10, 11].

The other way to vanquish the curse is by increasing the smoothness of functions with respect to the successive variables. This approach was taken recently in [14] for the worst case multivariate approximation in Korobov spaces. In this paper we extend the approach of [14] to the average case setting and, in a much broader context, to tensor product Gaussian random fields. In this case we denote $n(\varepsilon, d) = n^{\mathrm{avg}}(\varepsilon, d)$ and restrict ourselves to information

operations given by arbitrary continuous linear functionals since the use of function values leads to the same results due to [3] and Chapter 24 of [11].

More precisely, we consider non-homogeneous linear multivariate tensor product problems in the average case with the normalized error criterion. The normalized error is used to measure the error of an algorithm relative to the error of the zero algorithm. A precise problem statement is given in Section 2. The study of non-homogeneous case is necessary since homogeneous linear multivariate tensor product problems are intractable with this error criterion; see Chapter 6 in [9].

In Section 3 we recall spectral conditions for different types of tractability in the average case and prove some new conditions. The conditions are given in terms of the eigenvalues of the covariance operator of the corresponding Gaussian measure.

In Section 4 these conditions are applied to non-homogeneous tensor product approximation problems. We equip the space of continuous real functions defined on the d-dimensional unit cube $[0,1]^d$ with a zero-mean Gaussian measure with a covariance kernel of the form

$$K_d(s,t) = \prod_{k=1}^d \mathcal{R}_k(s_k, t_k), \qquad s, t \in [0,1]^d.$$

Then $n^{\mathrm{avg}}(\varepsilon, d)$ depends on spectral properties of the univariate integral operators with kernels $\mathcal{R}_k$. The main results of the paper , Theorems 6 – 8, present spectral conditions for polynomial, quasi-polynomial and weak tractability in this tensor product setting.

In Section 5 we illustrate these results for Korobov kernels,

$$\mathcal{R}_k(x,y) := 1 + 2\, g_k \sum_{j=1}^\infty j^{-2r_k} \cos(2\pi\, j(x-y)), \qquad x, y \in [0,1],$$

with varying smoothness parameters r_k such that

$$\tfrac{1}{2} < r_1 \le r_2 \le \cdots,$$

and weight parameters g_k such that

$$1 \ge g_1 \ge g_2 \ge \cdots > 0.$$

The sequences $\{r_k\}$ and $\{g_k\}$ may be related. We may have

$$g_k = g(r_k)$$

for some non-increasing function $g : [\tfrac{1}{2}, \infty) \to [0,1]$. The popular choice for Korobov space is to take $g(r) = (2\pi)^{-2r}$.

It turns out that:

- Weak tractability holds iff
$$\lim_{k \to \infty} g_k = 0.$$

- Quasi-polynomial tractability holds iff
$$\sup_{d \in \mathbb{N}} \frac{1}{\max(1, \ln d)} \sum_{k=1}^{d} g_k \max\left(1, \ln \frac{1}{g_k}\right) < \infty,$$

under the assumption that $\liminf_{k \to \infty} r_k / \ln k > 0$.

- Polynomial tractability is equivalent to strong polynomial tractability.

- Strong polynomial tractability holds iff
$$\rho_g := \liminf_{k \to \infty} \frac{\ln \frac{1}{g_k}}{\ln k} > 1.$$

If this holds then $n^{\mathrm{avg}}(\varepsilon, d) \le C \varepsilon^{-p}$ and the smallest p is
$$\max\left(\frac{2}{2r_1 - 1}, \frac{2}{\rho_g - 1}\right).$$

Other applications of our approach to tensor products problems are given in [7] for covariance kernels corresponding to Euler and Wiener integrated processes. We summarize the results of [7] in Section 6 and compare them to those of the Korobov case that we study here. By adjusting the weights g_k, the Korobov case behaves either like the Euler or Wiener case.

2 Problem setting

In this section we introduce multivariate problems in the average case setting. We define the information complexity and the different notions of tractability. More can be found in e.g., [9] and [16].

Let F_d be a Banach space of d-variate real functions defined on a Lebesgue measurable set $D_d \subset \mathbb{R}^d$. The space F_d is equipped with a zero-mean Gaussian measure μ_d defined on Borel sets of F_d. We denote by $C_{\mu_d} : F_d^* \to F_d$ the covariance operator of μ_d; e.g., see [9, Appendix B] for its definition. Let H_d be a Hilbert space with inner product and norm denoted by $\langle \cdot, \cdot \rangle_{H_d}$ and $\| \cdot \|_{H_d}$, respectively.

We want to approximate a continuous linear operator
$$S_d : \ F_d \ \to \ H_d.$$

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Let $\nu_d = \mu_d \, S_d^{-1}$ be the induced measure. Then ν_d is a zero-mean Gaussian measure on the Borel sets of H_d with covariance operator $C_{\nu_d} : H_d \to H_d$ given by

$$C_{\nu_d} = S_d \, C_{\mu_d} \, S_d^*$$

where $S_d^* : H_d \to F_d^*$ is the operator dual to S_d.

Then C_{ν_d} is self-adjoint, nonnegative definite, and has finite trace. Let $(\lambda_{d,j}, \eta_{d,j})_{j=1,2,\ldots}$ denote its eigenpairs

$$C_{\nu_d} \eta_{d,j} = \lambda_{d,j} \, \eta_{d,j} \quad \text{with} \quad \lambda_{d,1} \geq \lambda_{d,2} \geq \cdots .$$

Then

$$\mathrm{trace}(C_{\nu_d}) = \sum_{j=1}^{\infty} \lambda_{d,j} = \int_{H_d} \|g\|_{H_d}^2 \, \nu_d(\mathrm{d}g) = \int_{F_d} \|S_d f\|_{H_d}^2 \, \mu_d(\mathrm{d}f) < \infty.$$

We approximate $S_d f$ for $f \in F_d$ by algorithms A_n that use n function evaluations or n evaluations of arbitrary continuous linear functionals. It suffices to consider the case of arbitrary continuous functionals since it is known that the results are roughly the same for function values, see [3] and Chapter 24 of [11]. The average case error of A_n is defined as

$$e^{\mathrm{avg}}(A_n) = \left(\int_{F_d} \|S_d f - A_n(f)\|_{H_d}^2 \, \mu_d(\mathrm{d}f) \right)^{1/2}.$$

Without essential loss of generality, see e.g., [9] as well as [16], we can restrict ourselves in the average case setting to linear algorithms A_n of the form

$$A_n(f) = \sum_{j=1}^{n} L_j(f) \, g_j \quad \text{with} \quad L_j \in F_d^*, \quad g_j \in H_d.$$

For a given n, it is well known that the algorithm A_n that minimizes the average case error is of the form

$$A_n(f) = \sum_{k=1}^{n} \langle S_d f, \eta_{d,k} \rangle_{H_d} \, \eta_{d,k}, \tag{1}$$

and its average case error is

$$e^{\mathrm{avg}}(A_n) = \left(\sum_{j=n+1}^{\infty} \lambda_{d,j} \right)^{1/2}. \tag{2}$$

For $n = 0$ we have the zero algorithm $A_0 = 0$. Its average case error is called the initial error, and is given by the square-root of the trace of the operator C_{ν_d}, i.e., by (2) with $n = 0$.

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The average case information complexity $n^{\mathrm{avg}}(\varepsilon, d)$ is defined as the minimal n for which there is an algorithm whose average case error reduces the initial error by a factor ε,

$$n^{\mathrm{avg}}(\varepsilon, d) = \min\left\{ n \;\middle|\; \sum_{j=n+1}^{\infty} \lambda_{d,j} \leq \varepsilon^2 \sum_{j=1}^{\infty} \lambda_{d,j} \right\}. \tag{3}$$

We present the definitions of four types of tractability that will be studied in this paper. Let $S = \{S_d\}_{d=1,2,\dots}$ denote a sequence of multivariate problems. We say that

- S is *weakly tractable* iff

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln \max\left(1, n^{\mathrm{avg}}(\varepsilon, d)\right)}{\varepsilon^{-1} + d} = 0.$$

- S is *quasi-polynomially tractable* iff there are positive numbers C and t such that

$$n^{\mathrm{avg}}(\varepsilon, d) \leq C \exp\left(t\,(1 + \ln d)\,(1 + \ln \varepsilon^{-1}) \right) \qquad \text{for all} \quad d = 1, 2, \dots, \quad \varepsilon \in (0, 1).$$

The infimum of t satisfying the bound above is called the exponent of quasi-polynomial tractability and is denoted by $t^{\mathrm{qpol-avg}}$.

- S is *polynomially tractable* iff there are non-negative numbers C, q and p such that

$$n^{\mathrm{avg}}(\varepsilon, d) \leq C\, d^q\, \varepsilon^{-p} \qquad \text{for all} \quad d = 1, 2, \dots, \quad \varepsilon \in (0, 1).$$

- S is *strongly polynomially tractable* iff there are positive numbers C and p such that

$$n^{\mathrm{avg}}(\varepsilon, d) \leq C\, \varepsilon^{-p} \qquad \text{for all} \quad d = 1, 2, \dots, \quad \varepsilon \in (0, 1).$$

The infimum of p satisfying the last bound is called the exponent of strong polynomial tractability and is denoted by $p^{\mathrm{str-avg}}$.

Tractability can be fully characterized in terms of the eigenvalues $\lambda_{d,j}$. Necessary and sufficient conditions on weak, quasi-polynomial, polynomial and strong polynomial tractability can be found in Chapter 6 of [9] and Chapter 24 of [11]. In particular, S is polynomially tractable iff there exist $q \geq 0$ and $\tau \in (0, 1)$ such that

$$C := \sup_{d\in\mathbb{N}} \frac{\left(\sum_{j=1}^{\infty} \lambda_{d,j}^{\tau} \right)^{1/\tau}}{\sum_{j=1}^{\infty} \lambda_{d,j}} \, d^{-q} < \infty. \tag{4}$$

If so then

$$n^{\mathrm{avg}}(\varepsilon, d) \leq C^{\frac{\tau}{1-\tau}} \, d^{\frac{q\tau}{1-\tau}} \, \varepsilon^{\frac{-2\tau}{1-\tau}} \tag{5}$$

for all $d \in \mathbb{N}$ and $\varepsilon \in (0, 1)$.

Furthermore, S is strongly polynomially tractable iff (4) holds with $q = 0$. The exponent of strong polynomial tractability is

$$p^{\mathrm{str-avg}} = \inf\left\{ \frac{2\tau}{1-\tau} \;\middle|\; \tau \text{ satisfies (4) with } q = 0 \right\}. \tag{6}$$

3 General Bounds

We show bounds on $n^{\mathrm{avg}}(\varepsilon, d)$ which we will use to derive necessary and sufficient conditions for the four types of tractability. We first analyze an arbitrary problem $\{S_d\}$ and then restrict our attention to non-homogeneous tensor product problems.

We begin with a bound on $n^{\mathrm{avg}}(\varepsilon, d)$ which from a probabilistic point of view is an application of Chebyshev's inequality.

Lemma 1 *For any $\varepsilon \in (0, 1)$, $d \in \mathbb{N}$, $\tau \in (0, 1)$ and $z > 0$ we have*

$$n^{\mathrm{avg}}(\varepsilon, d) \leq \frac{\sum_{j=1}^{\infty} \lambda_{d,j}^z}{\left(\sum_{j=1}^{\infty} \lambda_{d,j}\right)^z} \left[\frac{\sum_{j=1}^{\infty} \lambda_{d,j}^\tau}{\left(\sum_{j=1}^{\infty} \lambda_{d,j}\right)^\tau} \right]^{z/(1-\tau)} \varepsilon^{-2z/(1-\tau)}. \tag{7}$$

Proof. Let $b := \left[\left(\sum_j \lambda_{d,j}\right) \varepsilon^2 / \left(\sum_j \lambda_{d,j}^\tau\right) \right]^{1/(1-\tau)}$. Then

$$\sum_{j:\lambda_{d,j}\leq b} \lambda_{d,j} \leq \sum_{j:\lambda_{d,j}\leq b} \lambda_{d,j}^\tau b^{1-\tau} \leq \sum_j \lambda_{d,j}^\tau b^{1-\tau} = \sum_j \lambda_{d,j} \, \varepsilon^2.$$

Hence,

$$\begin{aligned}
n^{\mathrm{avg}}(\varepsilon, d) &\leq \#\{j : \lambda_{d,j} > b\} = \#\{j : \lambda_{d,j}^z > b^z\} \leq \frac{\sum_{j:\,\lambda_{d,j}^z > b^z} \lambda_{d,j}^z}{b^z} \\
&\leq \frac{\sum_j \lambda_{d,j}^z}{b^z} = \frac{\sum_j \lambda_{d,j}^z}{\left(\sum_j \lambda_{d,j}\right)^z} \left[\frac{\left(\sum_j \lambda_{d,j}^\tau\right)}{\left(\sum_j \lambda_{d,j}\right)^\tau} \right]^{z/(1-\tau)} \varepsilon^{-2z/(1-\tau)},
\end{aligned}$$

as claimed. □

Note that (7) immediately proves sufficiency of polynomial tractability conditions in (4). Furthermore, if we set $z = \tau$ then we obtain the estimate (5) with the exponent of strong polynomial tractability at most $2\tau/(1 - \tau)$ for τ satisfying (4) with $q = 0$.

As we shall see now, the bound (7) is also useful when we consider quasi-polynomial tractability. In the rest of the paper we denote

$$\ln_+ d := \max(1, \ln d).$$

Theorem 2 *S is quasi-polynomially tractable iff there exists $\delta \in (0, 1)$ such that*

$$\sup_{d \in \mathbb{N}} \frac{\sum_{j=1}^{\infty} \lambda_{d,j}^{1 - \frac{\delta}{\ln_+ d}}}{\left(\sum_{j=1}^{\infty} \lambda_{d,j}\right)^{1 - \frac{\delta}{\ln_+ d}}} < \infty. \tag{8}$$

Proof. Sufficiency. Apply (7) with $\tau = 1 - \frac{\delta}{\ln_+ d} \in (0, 1)$ and $z = 1$. We obtain

$$n^{\mathrm{avg}}(\varepsilon, d) \leq \left[\frac{\left(\sum_j \lambda_{d,j}^{1 - \frac{\delta}{\ln_+ d}}\right) \varepsilon^{-2}}{\left(\sum_j \lambda_{d,j}\right)^{1 - \frac{\delta}{\ln_+ d}}} \right]^{\ln_+ d/\delta}$$

$$\leq M_\delta^{\ln_+ d/\delta} \varepsilon^{-2 \ln_+ d/\delta} = \exp\left(\frac{\ln M_\delta}{\delta} \ln_+ d + \frac{2}{\delta} \ln_+ d \ \ln \varepsilon^{-1} \right),$$

where M_δ is the supremum in (8). We can rewrite the last estimate as

$$n^{\mathrm{avg}}(\varepsilon, d) \leq \exp\left(t(1 + \ln d)(1 + \ln \varepsilon^{-1}) \right)$$

for $t = \delta^{-1} \max(2, \ln M_\delta)$. This means that S is quasi-polynomially tractable.

Necessity. Assume now that S is quasi-polynomially tractable, i.e.,

$$n^{\mathrm{avg}}(\varepsilon, d) \leq C \exp\left(t(\ln d + 1)(\ln \varepsilon^{-1} + 1) \right).$$

We verify (8) for some $\delta \in (0, 1)$. Note that (8) is invariant under multiplication of the eigenvalues by a positive number, and so is the value of $n^{\mathrm{avg}}(\varepsilon, d)$. That is why we may assume that $\sum_j \lambda_{d,j} = 1$. Quasi-polynomial tractability means that for all $\varepsilon > 0$ and $d \geq 1$ we have

$$\sum_{j \geq C \exp(t(\ln d + 1)(|\ln \varepsilon| + 1)) + 1} \lambda_{d,j} \leq \varepsilon^2.$$

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Let $\varepsilon := e\,(n/C)^{\frac{-1}{t(\ln d+1)}}$. Then

$$\sum_{j>n} \lambda_{d,j} \le e^2(n/C)^{\frac{-2}{t(\ln d+1)}} := e^2(n/C)^{-h} \tag{9}$$

with $h = 2/(t(1+\ln d))$.

To avoid too small eigenvalues, we introduce a regularization

$$\widehat{\lambda}_{d,j} := \max\{\lambda_{d,j}, h\,j^{-1-h}\}.$$

Note that (9) implies

$$\sum_{j>n} \widehat{\lambda}_{d,j} \le \sum_{j>n} \lambda_{d,j} + \sum_{j>n} hj^{-1-h} \le (e^2 C^h + 1)n^{-h}. \tag{10}$$

Let

$$N_m = \{j \in \mathbb{N} : \ 2^{m/h} \le j < 2^{(m+1)/h}\}, \qquad m = 0, 1, \dots.$$

Note that N_m depends on h. For any $\gamma \in (0,1)$ and any integer $m \ge 0$ we have

$$
\begin{aligned}
\sum_{j \in N_m} \lambda_{d,j}^{1-\gamma} \ &\le \ \sum_{j \in N_m} \widehat{\lambda}_{d,j}^{1-\gamma} \le \sum_{j \in N_m} \widehat{\lambda}_{d,j} \, [\min_{j \in N_m} \widehat{\lambda}_{d,j}]^{-\gamma} \\
&\le \ \sum_{j \ge 2^{m/h}} \widehat{\lambda}_{d,j} \, \left[h\,(2^{(m+1)/h})^{-1-h} \right]^{-\gamma} \\
\text{(by (10))} \ &\le \ (e^2 C^h + 1)\,2^{-m} \cdot h^{-\gamma} 2^{\frac{\gamma(m+1)(1+h)}{h}}.
\end{aligned}
$$

For a fixed $\delta \in (0,1)$, let $\gamma = \frac{\delta h}{1+h}$. We obtain

$$
\begin{aligned}
\sum_{j \in N_m} \lambda_{d,j}^{1-\gamma} \ &\le \ (e^2 C^h + 1)2^{-m} \cdot h^{-\gamma} 2^{\delta(m+1)} \\
&\le \ (e^2 C^h + 1)2^\delta \cdot h^{-\gamma} 2^{-(1-\delta)m} \le (e^2 C^h + 1)2^\delta \exp\left(|\ln h| \frac{\delta h}{h+1} \right)\, 2^{-(1-\delta)m}.
\end{aligned}
$$

Since

$$\sup_{0 < h \le \frac{2}{t}} |\ln h|h =: c(t) < \infty,$$

it follows that

$$\sum_j \lambda_{d,j}^{1-\gamma} = \sum_{m=0}^\infty \sum_{j \in N_m} \lambda_{d,j}^{1-\gamma} \le 2(e^2 C^h + 1)e^{c(t)} \sum_{m=0}^\infty 2^{-(1-\delta)m} =: c(t, \delta).$$

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Note that
$$\gamma = \frac{\delta h}{1+h} = \frac{2\delta}{t(1+\ln d)+2}.$$

Thus we have
$$\sup_{d \in \mathbb{N}} \sum_j \lambda_{d,j}^{1-\frac{2\delta}{t(1+\ln d)+2}} < \infty.$$

Let
$$\delta' := \min_{d \in \mathbb{N}} \frac{2\delta \ln_+ d}{t(1+\ln d)+2} \le \frac{2\delta}{t+2} < 1.$$

Then
$$1 - \frac{2\delta}{t(1+\ln d)+2} \le 1 - \frac{\delta'}{\ln_+ d} \qquad \text{for all} \quad d \in \mathbb{N},$$

and
$$\sup_{d \in \mathbb{N}} \sum_j \lambda_{d,j}^{1-\frac{\delta'}{\ln_+ d}} < \infty,$$

as required. This completes the proof. □

Theorem 2 does not address the exponent $t^{\mathrm{qpol-avg}}$ of quasi-polynomial tractability. There is, however, the bound on the exponent presented in the first part of the proof,
$$t^{\mathrm{qpol-avg}} \le \delta^{-1} \max(2, \ln M_\delta) \tag{11}$$

for all $\delta \in (0,1)$ satisfying (8).

The presence of M_δ may seem artificial. However, we now show that in general M_δ cannot be avoided in determining the exponent of quasi-polynomial tractability. Indeed, for $\delta \in (0,1)$, $M > 1$ and $d \ge 1$ let $N = N(d, M, \delta) := \lfloor M^{\ln_+ d/\delta} \rfloor$ and consider the following eigenvalues
$$\lambda_{d,j} := \begin{cases} 1 & \text{for} \quad j = 1, 2, \ldots, N, \\ 0 & \text{for} \quad j > N. \end{cases}$$

Then
$$M_\delta = \sup_{d \in \mathbb{N}} \frac{\sum_{j=1}^{\infty} \lambda_{d,j}^{1-\frac{\delta}{\ln_+ d}}}{\left(\sum_{j=1}^{\infty} \lambda_{d,j}\right)^{1-\frac{\delta}{\ln_+ d}}} = \sup_{d \in \mathbb{N}} N(d, M, \delta)^{\delta/\ln_+ d} = \lim_{d \to \infty} N(d, M, \delta)^{\delta/\ln_+ d} = M.$$

Hence quasi-polynomial tractability holds and for any $\varepsilon \in (0,1)$ we have
$$n^{\mathrm{avg}}(\varepsilon, d) = \lceil (1-\varepsilon^2)N \rceil \le C \exp\left(t(1+\ln d)(1+\ln \varepsilon^{-1})\right).$$

It follows that
$$t \geq \lim_{\varepsilon \to 1} \lim_{d \to \infty} \frac{\ln\left[(1 - \varepsilon^2)N(d, M, \delta)\right]}{\ln d} = \frac{\ln M}{\delta} = \frac{\ln M_\delta}{\delta}.$$
This justifies the presence of $\frac{\ln M_\delta}{\delta}$ in the bound (11) for exponent of quasi-polynomial tractability. However, we believe this bound is not always sharp.

We now show that the necessary condition on quasi-polynomial tractability can be simplified by eliminating the powers of $1 - \delta/\ln_+ d$. The following lemma will be a convenient tool for establishing this fact.

Lemma 3 *Let $\Lambda_d = \sum_{j=1}^{\infty} \lambda_{d,j}$. For any $\gamma > 0$ we have*

$$\Lambda_d^{-1} \sum_{j=1}^{\infty} \lambda_{d,j}^{1-\gamma} \geq \exp\left(-\gamma \sum_{j=1}^{\infty} \frac{\lambda_{d,j} \ln \lambda_{d,j}}{\Lambda_d}\right). \tag{12}$$

Proof. Jensen's inequality states that for a convex function $\phi(\cdot)$ defined on a convex set D, non-negative weights p_j satisfying $\sum_j p_j = 1$, and any set of arguments x_j from D we have

$$\sum_j p_j \phi(x_j) \geq \phi\left(\sum_j p_j x_j\right).$$

We apply Jensen's inequality with $p_j := \frac{\lambda_{d,j}}{\Lambda_d}$, $x_j := -\ln p_j$ and the function $\phi(x) = e^{\gamma x}$ for $x \in D := \mathbb{R}$. We obtain

$$\frac{\sum_j \lambda_{d,j}^{1-\gamma}}{\Lambda_d^{1-\gamma}} = \sum_j p_j^{1-\gamma} = \sum_j p_j \exp(-\gamma \ln p_j) = \sum_j p_j \phi(x_j)$$

$$\geq \phi\left(\sum_j p_j x_j\right) = \exp\left(\gamma \sum_j (-p_j \ln p_j)\right) = \exp\left(-\gamma \sum_j p_j \left(\ln \lambda_{d,j} - \ln \Lambda_d\right)\right)$$

$$= \Lambda_d^\gamma \exp\left(-\gamma \sum_j p_j \ln \lambda_{d,j}\right) = \Lambda_d^\gamma \exp\left(-\gamma \sum_j \frac{\lambda_{d,j} \ln \lambda_{d,j}}{\Lambda_d}\right).$$

This is equivalent to (12) and completes the proof. □

We will see in the next section that the right-hand side of (12) is convenient for tensor product problems. We are ready to simplify the necessary conditions for quasi-polynomial tractability.

Corollary 4 *If quasi-polynomial tractability holds then*

$$\sup_{d \in \mathbb{N}} \frac{1}{\ln_+ d} \sum_{j=1}^{\infty} \frac{\lambda_{d,j}}{\Lambda_d} \ln \left(\frac{\Lambda_d}{\lambda_{d,j}} \right) < \infty. \tag{13}$$

Proof. Quasi-polynomial tractability implies that (8) holds for some $\delta \in (0,1)$. Let $\gamma = \gamma(d) := \frac{\delta}{\ln_+ d}$. Using (12) we obtain

$$\frac{\sum_{j=1}^{\infty} \lambda_{d,j}^{1-\frac{\delta}{\ln_+ d}}}{\left(\sum_{j=1}^{\infty} \lambda_{d,j} \right)^{1-\frac{\delta}{\ln_+ d}}} = \frac{\sum_{j=1}^{\infty} \lambda_{d,j}^{1-\gamma}}{\Lambda_d^{1-\gamma}}$$

$$\geq \Lambda_d^{\gamma} \exp \left(-\gamma \sum_{j=1}^{\infty} \frac{\lambda_{d,j} \ln \lambda_{d,j}}{\Lambda_d} \right)$$

$$= \exp \left\{ \gamma \left(\ln \Lambda_d - \sum_{j=1}^{\infty} \frac{\lambda_{d,j} \ln \lambda_{d,j}}{\Lambda_d} \right) \right\}$$

$$= \exp \left\{ \gamma \sum_{j=1}^{\infty} \frac{\lambda_{d,j}}{\Lambda_d} \ln \left(\frac{\Lambda_d}{\lambda_{d,j}} \right) \right\}.$$

The claim (13) now follows from (8). $\qquad\square$

We will use later the following simple inequality that provides a sufficient condition for the curse of dimensionality. Recall that $\text{trace}(C_{\nu_d}) = \sum_{j=1}^{\infty} \lambda_{d,j}$ denotes the trace of the covariance operator.

Lemma 5 *For any $d \in \mathbb{N}$ and $\varepsilon > 0$ we have*

$$n^{\text{avg}}(\varepsilon, d) \geq (1 - \varepsilon^2) \frac{\text{trace}(C_{\nu_d})}{\lambda_{d,1}} = (1 - \varepsilon^2) \left(1 + \sum_{j=2}^{\infty} \frac{\lambda_{d,j}}{\lambda_{d,1}} \right).$$

In particular, if $\text{trace}(C_{\nu_d})/\lambda_{d,1} \geq (1+h)^d$ for some $h > 0$ and all $d \in \mathbb{N}$, then we have curse of dimensionality.

Proof. For $n = n^{\text{avg}}(\varepsilon, d)$ we have

$$\text{trace}(C_{\nu_d}) - n\lambda_{d,1} \leq \text{trace}(C_{\nu_d}) - \sum_{j=1}^{n} \lambda_{d,j} = \sum_{j=n+1}^{\infty} \lambda_{d,j} \leq \varepsilon^2 \, \text{trace}(C_{\nu_d}).$$

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Hence
$$n^{\mathrm{avg}}(\varepsilon, d) \geq (1 - \varepsilon^2)\,\mathrm{trace}(C_{\nu_d})/\lambda_{d,1},$$

as claimed. $\qquad\qquad\square$

4   Tensor Product Problems

In this section we assume that F_d, H_d and S_d are given by tensor products. That is,
$$F_d = F_1^{(1)} \otimes F_2^{(1)} \otimes \cdots \otimes F_d^{(1)} \quad \text{and} \quad H_d = H_1^{(1)} \otimes H_2^{(1)} \otimes \cdots \otimes H_d^{(1)}$$

for some Banach spaces $F_k^{(1)}$ of univariate real functions equipped with a zero-mean Gaussian measures $\mu_k^{(1)}$, and some Hilbert spaces $H_k^{(1)}$. (For Banach spaces the tensor product is defined as the *projective tensor product* [15, Ch. 2].) Here the superscript (1) reminds us that the objects are univariate. Furthermore we assume that
$$S_d = S_1^{(1)} \otimes S_2^{(1)} \otimes \cdots \otimes S_d^{(1)}$$

for continuous linear operators $S_k^{(1)} : F_k^{(1)} \to H_k^{(1)}$ and $k = 1, 2, \ldots, d$.

Let $\nu_k^{(1)} = \mu_k^{(1)}(S_k^{(1)})^{-1}$ and let $C_k^{(1)} : H_k^{(1)} \to H_k^{(1)}$ be the covariance operator of the measure $\nu_k^{(1)}$. The eigenpairs of $C_k^{(1)}$ are denoted by $(\lambda(k, j), \eta(k, j))$ and
$$\lambda(k, 1) \geq \lambda(k, 2) \geq \cdots \geq 0$$

as well as $\sum_{j=1}^{\infty} \lambda(k, j) < \infty$. To avoid the trivial case we assume that $\lambda(k, 1) > 0$ for all $k \in \mathbb{N}$.

The covariance operator C_{ν_d} is now the tensor product
$$C_{\nu_d} = C_1^{(1)} \otimes C_2^{(1)} \otimes \cdots \otimes C_d^{(1)}$$

and therefore the eigenvalues $\lambda_{d,j}$ and the eigenfunctions $\eta_{d,j}$ are given by corresponding products of the one-dimensional eigenvalues and eigenvectors $\lambda(k, j)$ and $\eta_{k,j}$, respectively. More precisely we have
$$\{\lambda_{d,j}\}_{j \in \mathbb{N}} = \left\{ \prod_{k=1}^{d} \lambda(k, j_k) \right\}_{j_1, j_2, \ldots j_d \in \mathbb{N}}.$$

Note that
$$\sum_{j \in \mathbb{N}} \lambda_{d,j}^{\tau} = \prod_{k=1}^{d} \sum_{j=1}^{\infty} \lambda(k, j)^{\tau} \qquad \text{for any } \tau > 0. \tag{14}$$

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We want to express necessary and sufficient conditions, for each of the four types of tractability, in terms of the eigenvalues $\lambda(k,j)$, $k,j \in \mathbb{N}$. The homogeneous case of the tensor product problem, i.e., when $F_k^{(1)} = F_1^{(1)}, H_k^{(1)} = H_1^{(1)}$ and $S_k^{(1)} = S_1^{(1)}$ which implies that

$$\lambda(k,j) = \lambda(1,j) \qquad \text{for all} \quad k,j = 1,2,\dots,$$

was studied in [9, Section 6.2] and in a recent paper [12]. In this section we mainly focus on a non-homogeneous case.

4.1 Polynomial Tractability

We know that $S = \{S_d\}$ is polynomially iff (4) holds for some $q \geq 0$. It is strongly polynomially tractable iff (4) holds with $q = 0$. We now simplify the condition (4) for tensor product problems.

Theorem 6 *Consider a tensor product problem $S = \{S_d\}$. Then*

- *S is strongly polynomially tractable iff there exists $\tau \in (0,1)$ such that*

$$\sum_{k=1}^{\infty} \sum_{j=2}^{\infty} \left(\frac{\lambda(k,j)}{\lambda(k,1)} \right)^{\tau} < \infty. \tag{15}$$

  If so the exponent of strong polynomial tractability is

$$p^{\text{str-avg}} = \inf \left\{ \frac{2\tau}{1-\tau} \;\middle|\; \tau \ \text{satisfies (15)} \right\}.$$

- *S is polynomially tractable iff there exists $\tau \in (0,1)$ such that*

$$Q_\tau := \sup_{d \in \mathbb{N}} \frac{1}{\ln_+ d} \sum_{k=1}^{d} \ln \left(1 + \sum_{j=2}^{\infty} \left(\frac{\lambda(k,j)}{\lambda(k,1)} \right)^{\tau} \right) < \infty. \tag{16}$$

  A simpler and stronger condition

$$\sup_{d \in \mathbb{N}} \frac{1}{\ln_+ d} \sum_{k=1}^{d} \sum_{j=2}^{\infty} \left(\frac{\lambda(k,j)}{\lambda(k,1)} \right)^{\tau} < \infty, \tag{17}$$

  is sufficient for polynomial tractability and necessary whenever

$$\sup_{k \in \mathbb{N}} \sum_{j=1}^{\infty} \left(\frac{\lambda(k,j)}{\lambda(k,1)} \right)^{\tau} < \infty. \tag{18}$$

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Proof. We prove the four conditions in the iff statements. Let

$$\widetilde{\lambda}(k,j) := \frac{\lambda(k,j)}{\lambda(k,1)} \quad \text{for all} \quad k,j \in \mathbb{N},$$

be the sequence of the normalized eigenvalues so that $1 = \widetilde{\lambda}(k,1) \geq \widetilde{\lambda}(k,j)$. We need to verify (4) which by (14) now asserts that for some $q \geq 0$ and $\tau \in (0,1)$ we have

$$C_{q,\tau} := \sup_{d \in \mathbb{N}} \frac{\left(\sum_{j=1}^{\infty} \lambda_{d,j}^{\tau}\right)^{1/\tau}}{\sum_{j=1}^{\infty} \lambda_{d,j}} d^{-q} = \sup_{d \in \mathbb{N}} \prod_{k=1}^{d} \frac{\left(\sum_{j=1}^{\infty} \widetilde{\lambda}(k,j)^{\tau}\right)^{1/\tau}}{\sum_{j=1}^{\infty} \widetilde{\lambda}(k,j)} d^{-q} < \infty.$$

For strong polynomial tractability $q = 0$, whereas for polynomial tractability $q \geq 0$.

1. *Sufficiency of* (15) *for strong polynomial tractability.* Note that

$$
\begin{aligned}
\prod_{k=1}^{d} \sum_{j=1}^{\infty} \widetilde{\lambda}(k,j)^{\tau} &= \prod_{k=1}^{d} \left(1 + \sum_{j=2}^{\infty} \widetilde{\lambda}(k,j)^{\tau}\right) \leq \prod_{k=1}^{d} \exp\left(\sum_{j=2}^{\infty} \widetilde{\lambda}(k,j)^{\tau}\right) \\
&= \exp\left(\sum_{k=1}^{d} \sum_{j=2}^{\infty} \widetilde{\lambda}(k,j)^{\tau}\right) \leq \exp\left(\sum_{k=1}^{\infty} \sum_{j=2}^{\infty} \widetilde{\lambda}(k,j)^{\tau}\right) < \infty \qquad (19)
\end{aligned}
$$

due to (15). On the other hand, $\prod_{k=1}^{d} \sum_{j=1}^{\infty} \widetilde{\lambda}(k,j) \geq 1$, hence $C_{0,\tau} < \infty$. This implies strong polynomial tractability.

2. *Necessity of* (15) *for strong polynomial tractability.* We now know that $C_{0,\tau} < \infty$ for some $\tau \in (0,1)$. This implies that

$$Q := \prod_{k=1}^{\infty} \left(1 + \sum_{j=2}^{\infty} \widetilde{\lambda}(k,j)^{\tau}\right) \leq C_{0,\tau}^{\tau} \prod_{k=1}^{\infty} \left(1 + \sum_{j=2}^{\infty} \widetilde{\lambda}(k,j)\right)^{\tau}.$$

Since $\widetilde{\lambda}(k,j) \leq 1$ and $\tau \in (0,1)$, we can estimate $\widetilde{\lambda}(k,j)$ by $\widetilde{\lambda}(k,j)^{\tau}$. This yields $Q \leq C_{0,\tau}^{\tau} Q^{\tau}$ or

$$\prod_{k=1}^{\infty} \left(1 + \sum_{j=2}^{\infty} \widetilde{\lambda}(k,j)^{\tau}\right) \leq C_{0,\tau}^{\tau/(1-\tau)} < \infty.$$

This is equivalent to

$$\sum_{k=1}^{\infty} \sum_{j=2}^{\infty} \widetilde{\lambda}(k,j)^{\tau} = \sum_{k=1}^{\infty} \sum_{j=2}^{\infty} \left(\frac{\lambda(k,j)}{\lambda(k,1)}\right)^{\tau} < \infty.$$

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Hence (15) holds, as claimed. The formula for the exponent of strong polynomial tractability follows from (6).

3. *Sufficiency of* (16) *for polynomial tractability.* By (16) we have

$$\prod_{k=1}^{d}\sum_{j=1}^{\infty}\widetilde{\lambda}(k,j)^{\tau} = \prod_{k=1}^{d}\left(1+\sum_{j=2}^{\infty}\widetilde{\lambda}(k,j)^{\tau}\right)$$

$$= \exp\left(\sum_{k=1}^{d}\ln\left(1+\sum_{j=2}^{\infty}\widetilde{\lambda}(k,j)^{\tau}\right)\right) \leq \max(e^{Q_{\tau}}, d^{Q_{\tau}}).$$

Using again the fact that $\prod_{k=1}^{d}\sum_{j=1}^{\infty}\widetilde{\lambda}(k,j) \geq 1$, we conclude that $C_{q,\tau} < \infty$ for $q = Q_{\tau}/\tau$, and obtain polynomial tractability. Since condition (17) is stronger than (16), it is also sufficient for polynomial tractability.

4. *Necessity of* (16) *for polynomial tractability.* We now know that $C_{q,\tau} < \infty$ for some $q \geq 0$ and $\tau \in (0,1)$. Proceeding as before we conclude that

$$\prod_{k=1}^{d}\left(1+\sum_{j=2}^{\infty}\widetilde{\lambda}(k,j)^{\tau}\right) \leq C_{q,\tau}^{\tau/(1-\tau)}\, d^{q\tau/(1-\tau)}.$$

Hence,

$$\sum_{k=1}^{d}\ln\left(1+\sum_{j=2}^{\infty}\widetilde{\lambda}(k,j)^{\tau}\right) \leq \frac{q\,\tau}{1-\tau}\,\ln_{+} d + \frac{\tau}{1-\tau}\ln C_{q,\tau},$$

and (16) follows.

It is easy to see that under the assumption (18), the conditions (16) and (17) are equivalent. Therefore, (17) is also necessary in this case. $\square$

We comment on the necessary condition for polynomial tractability. Typically, the coordinates in tensor product problems are ordered according to "decreasing importance". This means that the sequence $\sum_{j=2}^{\infty}\widetilde{\lambda}(k,j)^{\tau}$ is non-increasing in k. In this case (18) holds and the simple condition (17) is necessary and sufficient for polynomial tractability. However, in general, nothing prevents us from a strange ordering of important and unimportant coordinates so that the sequence of $\sum_{j=2}^{\infty}\widetilde{\lambda}(k,j)^{\tau}$ is not non-increasing in k. In this case the stronger condition (17) may fail as illustrated by the following example. Let

$$\lambda(k,j) = \widetilde{\lambda}(k,j) = \begin{cases} 1 & \text{for} \quad j=1, \\ 1 & \text{for} \quad j\in[2,k] \text{ and } k=2^{2^m} \text{ for non-negative integer } m, \\ 0 & \text{otherwise.} \end{cases}$$

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By counting the number of 1's in $\lambda(k,j)$ we easily conclude that

$$n^{\mathrm{avg}}(\varepsilon, d) \leq \prod_{m \in \mathbb{N}, \, 2^{2^m} \leq d} 2^{2^m} \leq 2^{2^{\ln_2(\ln_2(d^2))}} = d^2 \qquad \text{for all} \quad \varepsilon \in [0,1) \ \text{ and } \ d \in \mathbb{N}.$$

So polynomial tractability holds but condition (17) fails. Therefore, in general, it is not necessary for polynomial tractability.

4.2   Quasi-Polynomial Tractability

We now consider quasi-polynomial tractability of tensor products. First of all let us check how the right-hand side of Lemma 3 simplifies in this case. Let

$$\Lambda(k) := \sum_{j=1}^{\infty} \lambda(k,j) \quad \text{and by (14)} \quad \Lambda_d := \sum_{j=1}^{\infty} \lambda_{d,j} = \prod_{k=1}^{d} \Lambda(k).$$

For tensor products we have

$$
\begin{aligned}
\sum_{j=1}^{\infty} \lambda_{d,j} \ln \lambda_{d,j} &= \sum_{z=[z_1,z_2,\ldots,z_d] \in \mathbb{N}^d} \prod_{k=1}^{d} \lambda(k, z_k) \sum_{k=1}^{d} \ln \lambda(k, z_k) \\
&= \sum_{k=1}^{d} \sum_{z \in \mathbb{N}^d} \lambda(k, z_k) \ln \lambda(k, z_k) \prod_{\substack{1 \leq m \leq d \\ m \neq k}} \lambda(m, z_m) \\
&= \sum_{k=1}^{d} \left(\sum_{j=1}^{\infty} \lambda(k,j) \ln \lambda(k,j) \right) \prod_{\substack{1 \leq m \leq d \\ m \neq k}} \left(\sum_{j=1}^{\infty} \lambda(m,j) \right) \\
&= \sum_{k=1}^{d} \left(\sum_{j=1}^{\infty} \lambda(k,j) \ln \lambda(k,j) \right) \prod_{\substack{1 \leq m \leq d \\ m \neq k}} \Lambda(m) \\
&= \sum_{k=1}^{d} \left(\sum_{j=1}^{\infty} \lambda(k,j) \ln \lambda(k,j) \right) \frac{\Lambda_d}{\Lambda(k)}. \tag{20}
\end{aligned}
$$

Inequality (12) now becomes

$$\Lambda_d^{-1} \sum_{j=1}^{\infty} \lambda_{d,j}^{1-\gamma} \geq \exp\left(-\gamma \sum_{k=1}^{d} \frac{1}{\Lambda(k)} \sum_{j=1}^{\infty} \lambda(k,j) \ln \lambda(k,j) \right). \tag{21}$$

This inequality will be used in the following theorem which addresses quasi-polynomial tractability for tensor product problems.

Theorem 7 *Consider a tensor product problem $S = \{S_d\}$. Then*

- *S is quasi-polynomially tractable iff there exists $\delta \in (0,1)$ such that*

$$\sup_{d \in \mathbb{N}} \prod_{k=1}^{d} \frac{\sum_{j=1}^{\infty} \lambda(k,j)^{1-\frac{\delta}{\ln_+ d}}}{\left(\sum_{j=1}^{\infty} \lambda(k,j)\right)^{1-\frac{\delta}{\ln_+ d}}} < \infty. \tag{22}$$

- *If S is quasi-polynomially tractable then*

$$\sup_{d \in \mathbb{N}} \frac{1}{\ln_+ d} \sum_{k=1}^{d} \sum_{j=1}^{\infty} \frac{\lambda(k,j)}{\Lambda(k)} \ln\left(\frac{\Lambda(k)}{\lambda(k,j)}\right) < \infty. \tag{23}$$

- *If there exists $\delta > 0$ such that*

$$\sup_{d \in \mathbb{N}} \sum_{k=1}^{d} \ln\left(1 + \sum_{j=2}^{\infty} \left(\frac{\lambda(k,j)}{\lambda(k,1)}\right)^{1-\frac{\delta}{\ln d}}\right) < \infty, \tag{24}$$

or

$$\sup_{d \in \mathbb{N}} \sum_{k=1}^{d} \sum_{j=2}^{\infty} \left(\frac{\lambda(k,j)}{\lambda(k,1)}\right)^{1-\frac{\delta}{\ln d}} < \infty \tag{25}$$

then S is quasi-polynomially tractable.

Proof. In view of (14), criterion (22) is just the general criterion (8) in Theorem 2 specified for tensor products. The necessary condition in (23) is just a specification of the general necessary condition in (13) for tensor products. To see this, note that

$$\sum_{j=1}^{\infty} \frac{\lambda_{d,j}}{\Lambda_d} \ln\left(\frac{\Lambda_d}{\lambda_{d,j}}\right) = \ln \Lambda_d - \Lambda_d^{-1} \sum_{j=1}^{\infty} \lambda_{d,j} \ln \lambda_{d,j}$$

$$(\text{by } (20)) = \sum_{k=1}^{d} \ln \Lambda(k) - \sum_{k=1}^{d} \sum_{j=1}^{\infty} \frac{\lambda(k,j)}{\Lambda(k)} \ln \lambda(k,j)$$

$$= \sum_{k=1}^{d} \sum_{j=1}^{\infty} \frac{\lambda(k,j)}{\Lambda(k)} \ln\left(\frac{\Lambda(k)}{\lambda(k,j)}\right).$$

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To see that (24) is sufficient for quasi-polynomial tractability, observe that the fraction in (22) can be written with $\widetilde{\lambda}(k,j) = \lambda(k,j)/\lambda(k,1)$ as

$$\prod_{k=1}^{d} \frac{1 + \sum_{j=2}^{\infty} \widetilde{\lambda}(k,j)^{1-\frac{\delta}{\ln_+ d}}}{\left(1 + \sum_{j=2}^{\infty} \widetilde{\lambda}(k,j)\right)^{1-\frac{\delta}{\ln_+ d}}}.$$

Taking logarithms, we see that the numerator is bounded by (24) while the denominator is larger than 1. Hence (22) is bounded and we are done.

Since (25) is stronger than (24), it is also sufficient for quasi-polynomial tractability. □

4.3 Weak Tractability

We present a simple criterion of weak tractability for tensor products.

Theorem 8 *Consider a tensor product problem $S = \{S_d\}$. If for some $\tau \in (0,1)$*

$$\lim_{d \to \infty} d^{-1} \sum_{k=1}^{d} \sum_{j=2}^{\infty} \left(\frac{\lambda(k,j)}{\lambda(k,1)} \right)^{\tau} = 0 \tag{26}$$

then S is weakly tractable.

Proof. The idea is basically the same as in the proof of Theorem 6. Namely, we apply (7) with $z = 1$. As before, let $\widetilde{\lambda}(k,j) := \lambda(k,j)/\lambda(k,1)$. Then (7), by (14) can be rewritten as

$$n^{\mathrm{avg}}(\varepsilon, d) \le \prod_{k=1}^{d} \left[\frac{1 + \sum_{j=2}^{\infty} \widetilde{\lambda}(k,j)^{\tau}}{1 + \sum_{j=2}^{\infty} \widetilde{\lambda}(k,j)} \right]^{1/(1-\tau)} \varepsilon^{-2/(1-\tau)}.$$

Since the denominator above is larger than one, it may be dropped. Using (19) we have

$$n^{\mathrm{avg}}(\varepsilon, d) \le \left[\exp\left(\sum_{k=1}^{d} \sum_{j=2}^{\infty} \widetilde{\lambda}(k,j)^{\tau} \right) \varepsilon^{-2} \right]^{1/(1-\tau)} = \exp\left[(1-\tau)^{-1}(\theta_d d + 2\ln \varepsilon^{-1}) \right],$$

where

$$\theta_d := d^{-1} \sum_{k=1}^{d} \sum_{j=2}^{\infty} \widetilde{\lambda}(k,j)^{\tau} \to 0 \qquad \text{as} \quad d \to \infty \tag{27}$$

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due to (26). Equivalently,

$$\ln n^{\mathrm{avg}} a(\varepsilon, d) \leq (1 - \tau)^{-1} \left[\theta_d \, d + 2 \ln \varepsilon^{-1} \right].$$

By (27)

$$\lim_{d+\varepsilon^{-1} \to \infty} \frac{\theta_d \, d + 2 \ln \varepsilon^{-1}}{d + \varepsilon^{-1}} = 0,$$

and we obtain the weak tractability. $\qquad\square$

Note that (26) holds if

$$\lim_{k \to \infty} \sum_{j=2}^{\infty} \left(\frac{\lambda(k, j)}{\lambda(k, 1)} \right)^{\tau} = 0. \tag{28}$$

Hence (28) implies weak tractability. The last condition yields

$$\lim_{k \to \infty} \frac{\mathrm{trace}(C_k^{(1)})}{\lambda(k, 1)} = 1$$

so that the Gaussian measure is asymptotically concentrated on the one-dimensional subspace $\mathrm{span}(\eta(k, 1))$ of $H_k^{(1)}$.

5 Multivariate Approximation and Korobov Kernels

The non-homogeneous case offers the possibility of vanquishing the curse of dimensionality via variation of weights and smoothness parameters. We illustrate this by an example with Korobov kernels of decreasing weights g_k and increasing smoothness r_k. As we shall see, even strong polynomial tractability holds if the decay of g_k is sufficiently fast. Multivariate approximation for Korobov spaces in the worst case setting was recently studied in [14]. Here we present its average case analog.

In this section we consider a multivariate approximation problem defined over the space of continuous real functions equipped with a zero-mean Gaussian measure whose covariance is given as a Korobov kernel. More precisely, consider the approximation problem

$$\mathrm{APP} = \{\mathrm{APP}_d\}_{d \in \mathbb{N}} \quad \text{with} \quad \mathrm{APP}_d : C([0,1]^d) \to L_2([0,1]^d)$$

given by

$$\mathrm{APP}_d f = f \quad \text{for all} \quad f \in C([0,1]^d).$$

The space $C([0,1]^d)$ of continuous real functions is equipped with a zero-mean Gaussian measure μ_d whose covariance kernel

$$K_d(x,y) = \int_{C([0,1]^d)} f(x)f(y)\,\mu_d(df), \quad x,y \in [0,1]^d,$$

is given as follows. First of all we assume that K_d is of product form,

$$K_d(x,y) = \prod_{k=1}^{d} \mathcal{R}_k(x_k, y_k) \quad \text{for all} \quad x = [x_1, x_2, \ldots, x_d],\ y = [y_1, y_2, \ldots, y_d] \in [0,1]^d,$$

where $\mathcal{R}_k = \mathcal{R}_{r_k, g_k}$ are univariate Korobov kernels,

$$\mathcal{R}_{r,\beta}(x,y) := 1 + 2\beta \sum_{j=1}^{\infty} j^{-2r} \cos(2\pi j(x-y)) \quad \text{for all} \quad x, y \in [0,1].$$

Here $\beta \in (0,1]$ and r is a real number such that $r > \frac{1}{2}$. Note that for $y = x$ we have

$$\mathcal{R}_{r,\beta}(x,x) = 1 + 2\beta\,\zeta(2r),$$

where $\zeta(x) = \sum_{j=1}^{\infty} j^{-x}$ is the Riemann zeta function which is well-defined only for $x > 1$. That is why we have to consider $r > \frac{1}{2}$.

We assume that the sequence $\{r_k\}$ is non-decreasing,

$$\tfrac{1}{2} < r_1 \le r_2 \le \cdots \le r_d \le \cdots. \tag{29}$$

The weight sequence $\{g_k\}$ serves as a scaling and, as we shall see, tractability results will depend on the behavior of g_k at infinity. We assume that

$$1 \ge g_1 \ge g_2 \ge \cdots > 0. \tag{30}$$

As already mentioned, the sequences $\{r_k\}$ and $\{g_k\}$ may be related, $g_k = g(r_k)$ for some non-increasing function $g : [\frac{1}{2}, \infty) \to [0,1]$. The case which can be often found in the literature corresponds to $g_k = 1$ or $g_k = (2\pi)^{-2r_k}$. For $g_k = g(r_k)$ the behavior of g_k at infinity depends on the function g and the behavior of r_k at infinity. A summary of the properties of the Korobov kernels can be found in Appendix A of [9].

For a fixed d, the multivariate approximation problem under similar conditions was studied in [8, 13]. For varying d, the homogeneous case, i.e., $\mathcal{R}_k = \mathcal{R}$ for all k with $\mathcal{R}$ not necessarily equal to a Korobov kernel, was studied in [5, 6, 9]. In this case, we have the curse of dimensionality since $n^{\mathrm{avg}}(\varepsilon, d)$ depends exponentially on d.

The induced measure $\nu_d = \mu_d \mathrm{APP}_d^{-1}$ on $L_2([0,1]^d)$ is also a zero-mean Gaussian measure. It is known, see e.g., [9], that the eigenvalues of its covariance operator C_{ν_d} are given by

$$\lambda_{d,z} = \prod_{k=1}^{d} \lambda(k, z_k) \qquad \text{for all} \quad z = [z_1, z_2, \ldots, z_d] \in \mathbb{N}^d, \tag{31}$$

where $\lambda(k, 1) = 1$ and

$$\lambda(k, 2j) = \lambda(k, 2j+1) = \frac{g_k}{j^{2r_k}}, \qquad j \in \mathbb{N}. \tag{32}$$

Note that the trace of C_{ν_d} is

$$\mathrm{trace}(C_{\nu_d}) = \prod_{k=1}^{d} \left(1 + 2\, g_k\, \zeta(2r_k)\right).$$

We have the curse of dimensionality when

$$g_{\lim} := \lim_{k \to \infty} g_k > 0.$$

Indeed, in this case

$$\mathrm{trace}(C_{\nu_d}) \geq (1 + 2\, g_{\lim})^d,$$

and Lemma 5 yields the curse. Therefore $\lim_k g_k = 0$ is a necessary condition to vanquish the curse.

Theorem 9 *Consider the approximation problem* $\mathrm{APP} = \{\mathrm{APP}_d\}$ *in the average case with a zero-mean Gaussian measure whose covariance operator is given as the Korobov kernel with the weights* g_k *and smoothness* r_k *satisfying* (30) *and* (29), *respectively. Then*

- *APP is polynomially tractable iff*

$$\rho_g := \liminf_{k \to \infty} \frac{\ln \frac{1}{g_k}}{\ln k} > 1. \tag{33}$$

- *APP is strongly polynomially tractable iff it is polynomially tractable. If so, the exponent of strong polynomial tractability is*

$$p^{\mathrm{avg-str}} = \max\left(\frac{2}{2r_1 - 1}, \frac{2}{\rho_g - 1}\right).$$

- *If* APP *is quasi-polynomially tractable then*

$$\sup_{d \in \mathbb{N}} \frac{1}{\ln_+ d} \sum_{k=1}^{d} g_k \ln_+ \frac{1}{g_k} < \infty. \tag{34}$$

If (34) holds and

$$\liminf_{k \to \infty} \frac{r_k}{\ln k} > 0 \tag{35}$$

then APP *is quasi-polynomially tractable.*

- APP *is weakly tractable iff*

$$\lim_{k \to \infty} g_k = 0.$$

Proof: We will use Theorem 6 and proceed in a way similar to that of the proof of Theorem 1 in [14]. The main difference is that here $\tau \in (0, 1)$.

We first show that (33) implies strong polynomial tractability and then that polynomial tractability implies (33). Assume thus that (33) is satisfied. Then for some $\delta > 0$ and all large k we have

$$\frac{\ln \frac{1}{g_k}}{\ln k} \geq 1 + \delta.$$

Hence, there is a positive C such that for any $\tau \in (0, 1)$ we obtain

$$g_k^\tau \leq C^\tau \, k^{-\tau(1+\delta)} \qquad \text{for all} \qquad k \in \mathbb{N}.$$

If we choose $\tau \in (\frac{1}{1+\delta}, 1) \cap (\frac{1}{2r_1}, 1)$ then

$$\sum_{k=1}^{\infty} \sum_{j=2}^{\infty} \left(\frac{\lambda(k, j)}{\lambda(k, 1)} \right)^\tau = 2 \sum_{k=1}^{\infty} g_k^\tau \sum_{j=1}^{\infty} j^{-2\tau r_k} \leq 2 C^\tau \sup_k \zeta(2\tau r_k) \sum_{k=1}^{\infty} k^{-\tau(1+\delta)}$$

$$\leq 2 C^\tau \zeta(2\tau r_1) \, \zeta\left(\tau(1 + \delta) \right) < \infty,$$

and condition (15) of Theorem 6 yields strong polynomial tractability.

Assume now that polynomial tractability holds. Then for $\tau \in (\frac{1}{2r_1}, 1)$ we have

$$\sup_k \sum_{j=2}^{\infty} \left(\frac{\lambda(k, j)}{\lambda(k, 1)} \right)^\tau = 2 \sup_k [g_k^\tau \, \zeta(2\tau r_k)] = 2 g_1^\tau \, \zeta(2\tau r_1) < \infty.$$

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Therefore, condition (18) is verified, hence condition (17) is necessary for polynomial tractability. The latter condition for the Korobov case is

$$C := \sup_{d \in \mathbb{N}} \frac{2}{\ln_+ d} \sum_{k=1}^{d} g_k^\tau \, \zeta(2\tau r_k) \; < \infty,$$

for some $\tau \in (\frac{1}{2r_1}, 1)$. All terms of the last sum are larger or equal to g_d^τ and therefore for $d > 1$ we have $g_d^\tau \leq \frac{C \ln d}{2d}$. This is equivalent to

$$\frac{\ln \frac{1}{g_d}}{\ln d} \geq \frac{1}{\tau} \left(1 - \frac{\ln(C/2) + \ln \ln d}{\ln d} \right).$$

Hence,

$$\rho_g = \liminf_{d \to \infty} \frac{\ln \frac{1}{g_d}}{\ln d} \geq \frac{1}{\tau} > 1,$$

as required in (33).

We now turn to the exponent of strong polynomial tractability. We must have $\tau > \frac{1}{2r_1}$ and from the last displayed formula $\tau > \frac{1}{\rho_g}$. From Theorem 6 we obtain that

$$p^{\text{str-avg}} = \max \left(\frac{2}{2r_1 - 1}, \frac{2}{\rho_g - 1} \right).$$

This completes the proof of polynomial tractability.

Assume now that quasi-polynomial tractability holds. Then the necessary condition (23) is satisfied. Clearly, all terms appearing in this condition are positive. We simplify (23) by omitting all terms for $j \neq 2$, and obtain

$$\sup_{d \geq \mathbb{N}} \frac{1}{\ln_+ d} \sum_{k=1}^{d} \frac{\lambda(k,2)}{\Lambda(k)} \ln \left(\frac{\Lambda(k)}{\lambda(k,2)} \right) < \infty. \tag{36}$$

Recall that for the Korobov case, $\Lambda(k) = 1 + 2 \, g_k \, \zeta(2r_k)$ and $\lambda(k,2) = g_k$. Since $\Lambda(k) \geq 1$ and $\Lambda(k)/\lambda(k,2) \geq 3$ we obtain

$$\sup_{d \in \mathbb{N}} \frac{1}{\ln_+ d} \sum_{k=1}^{d} \frac{\lambda(k,2)}{\Lambda(k)} \ln_+ \left(\frac{1}{\lambda(k,2)} \right) < \infty.$$

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Furthermore, since $\{\Lambda(k)\}$ is non-increasing, we have

$$\sup_{d \in \mathbb{N}} \frac{1}{\ln_+ d} \sum_{k=1}^{d} \lambda(k,2) \ln_+ \left(\frac{1}{\lambda(k,2)} \right) < \infty.$$

This is equivalent to (34), and completes this part of the proof.

We now prove that (34) and (35) are sufficient for quasi-polynomial tractability. Theorem 7 states that APP is quasi-polynomially tractable iff there exists $\delta \in (0,1)$ such that (22) holds, i.e.,

$$\sup_{d \in \mathbb{N}} \prod_{k=1}^{d} \frac{\sum_{j=1}^{\infty} \lambda(k,j)^{\tau_d}}{\left(\sum_{j=1}^{\infty} \lambda(k,j) \right)^{\tau_d}} < \infty, \tag{37}$$

where $\tau_d = 1 - \frac{\delta}{\ln_+ d}$. Take any $\delta \in (0, \min(\frac{1}{2}, 1 - \frac{1}{2r_1}))$. Inequality $\delta < 1 - 1/(2r_1)$ ensures that all the sums above are finite because $2r_k \tau_d \geq 2r_1 \tau_1 > 1$.

We split the product in (37) into two products

$$\Pi_1(d) := \prod_{k=1}^{d} \left(\sum_{j=1}^{\infty} \lambda(k,j) \right)^{\frac{\delta}{\ln_+ d}}$$

and

$$\Pi_2(d) := \prod_{k=1}^{d} \frac{\sum_{j=1}^{\infty} \lambda(k,j)^{\tau_d}}{\sum_{j=1}^{\infty} \lambda(k,j)}.$$

In what follows we will write C for some positive number which is independent of d and k, and whose value may change for successive estimates.

For $\Pi_1(d)$ we use $(1+x)^t = \exp(t \ln(1+x)) \leq \exp(tx)$ and have

$$\begin{aligned}
\Pi_1(d) &= \prod_{k=1}^{d} \left(1 + \sum_{j=2}^{\infty} \lambda(k,j) \right)^{\frac{\delta}{\ln_+ d}} \leq \exp \left(\frac{\delta}{\ln_+ d} \sum_{k=1}^{d} \sum_{j=2}^{\infty} \lambda(k,j) \right) \\
&\leq \exp \left(\frac{C}{\ln_+ d} \sum_{k=1}^{d} g_k \, \zeta(2r_k) \right) \leq \exp \left(\frac{C \, \zeta(2r_1)}{\ln_+ d} \sum_{k=1}^{d} g_k \right).
\end{aligned}$$

Clearly, (34) implies that $\sup_{d \in \mathbb{N}} \Pi_1(d) < \infty$.

We now turn to the product $\Pi_2(d)$. We estimate each of its factors by

$$\frac{\sum_{j=1}^{\infty} \lambda(k,j)^{\tau_d}}{\sum_{k=1}^{\infty} \lambda(k,j)} \leq \frac{1 + 2\lambda(k,2)^{\tau_d}}{1 + 2\lambda(k,2)} + \sum_{j=4}^{\infty} \lambda(k,j)^{\tau_d}. \tag{38}$$

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Note that if $|\ln\lambda(2,k)| \le 3\ln_+ d$, then

$$\frac{1+2\lambda(k,2)^{\tau_d}}{1+2\lambda(k,2)} = \frac{1+2\lambda(k,2)\exp\left(\frac{-\delta\,\ln\lambda(k,2)}{\ln_+ d}\right)}{1+2\lambda(k,2)}$$

$$\le \frac{1+2\lambda(k,2)\left(1+\frac{C|\ln\lambda(k,2)|}{\ln_+ d}\right)}{1+2\lambda(k,2)} \le 1+\frac{C\lambda(k,2)|\ln\lambda(k,2)|}{\ln_+ d},$$

while if $|\ln\lambda(k,2)| \ge 3\ln_+ d$ then $\delta < \frac{1}{2}$ implies

$$\frac{1+2\lambda(k,2)^{\tau_d}}{1+\lambda(k,2)} \le 1+2\lambda(k,2)^{\tau_d} \le 1+2\lambda(k,2)^{1/2} \le 1+2d^{-3/2}.$$

Thus, in any case

$$\frac{1+2\lambda(k,2)^{\tau_d}}{1+2\lambda(k,2)} \le 1+2d^{-3/2}+\frac{C\lambda(k,2)|\ln\lambda(k,2)|}{\ln_+ d}. \tag{39}$$

It remains to evaluate the sum in (38). An easy and elementary calculation shows that (34) implies $\lambda(k,2) = g_k \le \frac{C}{k}$. On the other hand, (35) yields $r_k \ge h\ln k - C$ for all $k \in N$ with appropriate $h,C > 0$. We obtain now

$$\sum_{j=4}^{\infty}\lambda(k,j)^{\tau_d} \le C\,\lambda(k,4)^{\tau_d} = C\,\lambda(k,2)^{\tau_d}4^{-r_k\tau_d}$$

$$\le C\cdot(C/k)^{1-\delta/\ln_+ d}\,2^{-(h\ln k-C)} \le Ck^{-(1+u)}, \tag{40}$$

where $u = h\ln 2 > 0$. Combining (38), (39)) and (40), and using again $1+x \le \exp(x)$ we easily check that

$$\frac{\sum_{j=1}^{\infty}\lambda(k,j)^{\tau_d}}{\sum_{j=1}^{\infty}\lambda(k,j)} \le \exp\left(2d^{-3/2}+\frac{C\,\lambda(k,2)|\ln\lambda(k,2)|}{\ln_+ d}+Ck^{-(1+u)}\right).$$

Then it follows that

$$\Pi_2(d) \le \exp\left(\sum_{k=1}^{d}\left(2d^{-3/2}+\frac{C\,\lambda(k,2)|\ln\lambda(k,2)|}{\ln_+ d}+Ck^{-(1+u)}\right)\right)$$

$$\le \exp\left(\sum_{k=1}^{d}\left(2d^{-3/2}+\frac{C\,g_k\,\ln_+\frac{1}{g_k}}{\ln_+ d}+Ck^{-(1+u)}\right)\right),$$

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and (34) implies that $\sup_{d \in \mathbb{N}} \Pi_2(d) < \infty$. Therefore,

$$\sup_{d \in \mathbb{N}} \Pi_1(d)\,\Pi_2(d) \le \sup_{d \in \mathbb{N}} \Pi_1(d)\ \sup_{d \in \mathbb{N}} \Pi_2(d) < \infty.$$

Hence, (37) holds so that the quasi-polynomial tractability is proved.

We now consider weak tractability. Sufficiency. Let $\lim_k g_k = 0$. Then for an arbitrarily small positive δ there exists $k(\delta)$ such that $g_k \le \delta$ for all $k \ge k(\delta)$. We check the assumption (26) of Theorem 8. For $\tau \in (1/(2r_1), 1)$ and $d > k(\delta)$ we have

$$a_d := \frac{1}{d} \sum_{k=1}^{d} \sum_{j=2}^{\infty} \left(\frac{\lambda(k,j)}{\lambda(k,1)} \right)^{\tau} = \frac{2}{d} \sum_{k=1}^{d} g_k^{\tau}\, \zeta(2r_k \tau)$$

$$\le \frac{2\zeta(2r_1\tau)\, k(\delta)}{d} + \frac{(d - k(\delta))\, \delta}{d}.$$

Hence,

$$\limsup_{d \to \infty} a_d \le \delta.$$

For δ tending to zero, we conclude that $\limsup_d a_d = \lim_d a_d = 0$, and obtain weak tractability due to Theorem 8.

Necessity. We have already showed that $\lim_k g_k = 0$ is a necessary condition for weak tractability. This completes the proof. $\qquad\square$

We do not know if (35) is needed for quasi-polynomial tractability. However, for $g_k = g(r_k)$ with $g(r) = \vartheta^r$ and $\vartheta \in (0,1)$, or $g(r) = r^{-s}$ and $s > 0$, this condition clearly follows from (34) since the latter implies that $g_k \le \frac{C}{k}$. For such weights and smoothness parameters, (34) is a necessary and sufficient condition for quasi-polynomial tractability.

We illustrate Theorem 9 for special weights.

- Let $g_k = v^{r_k}$ with $v \in (0,1)$.

    - Strong polynomial tractability holds   iff $\rho_r := \liminf_{k \to \infty} \frac{r_k}{\ln k} > \frac{1}{\ln v^{-1}}$.
      If so the exponent is $p^{\text{avg-str}} = \max \left(\frac{2}{2r_1 - 1}, \frac{2}{\rho_r[\ln v^{-1}] - 1} \right)$.

    - Quasi-polynomial tractability holds   iff $\sum_{k=1}^{d} v^{r_k} \max(1, r_k) = \mathcal{O}(\ln d)$.

    - Weak tractability holds   iff $\lim_{k \to \infty} r_k = \infty$.

- Let $g_k = r_k^{-s}$ for $s > 0$.

- Strong polynomial tractability holds   iff $\rho_r := \liminf_{k \to \infty} \frac{\ln r_k}{\ln k} > \frac{1}{s}$.

  If so the exponent is $p^{\mathrm{avg-str}} = \max\left(\frac{2}{2r_1 - 1}, \frac{2}{\rho_r s - 1}\right)$.

- Quasi-polynomial tractability holds   iff $\sum_{k=1}^{d} r_k^{-s} \max(1, \ln r_k) = \mathcal{O}(\ln d)$.

- Weak tractability holds   iff $\lim_{k \to \infty} r_k = \infty$.

It is also important to notice that Theorem 9 holds for constant smoothness parameters $r_k \equiv r > \frac{1}{2}$ if g_k are *not* related to r_k and satisfy the conditions presented in Theorem 9. This corresponds to appropriately decaying product weights, the case that was also studied in [9] p. 276.

6   Comparison of Korobov, Euler, and Wiener Kernels

Another application of our general results is given in [7], where tensor products of multi-parametric Wiener and Euler integrated processes are considered. We briefly summarize the results of [7] to compare them to the results of the previous section.

Let $W(t), t \in [0,1]$, be a standard Wiener process, i.e. a Gaussian random process with zero mean and covariance $K_{1,0}^{\mathrm{E}}(s,t) = K_{1,0}^{\mathrm{W}}(s,t) := \min(s,t)$. Consider two sequences of integrated random processes $X_r^{\mathrm{W}}, X_r^{\mathrm{E}}$ on $[0,1]$ defined inductively on r by $X_0^{\mathrm{W}} = X_0^{\mathrm{E}} = W$, and for $r = 0, 1, 2, \ldots$

$$
\begin{aligned}
X_{r+1}^{\mathrm{W}}(t) &= \int_0^t X_r^{\mathrm{W}}(s)\mathrm{d}s, \\
X_{r+1}^{\mathrm{E}}(t) &= \int_{1-t}^1 X_r^{\mathrm{E}}(s)\mathrm{d}s.
\end{aligned}
$$

$\{X_r^{\mathrm{W}}\}$ is called the univariate integrated Wiener process, while $\{X_r^{\mathrm{E}}\}$ is called the univariate integrated Euler process.

Clearly, X_r^{W} and X_r^{E} have the same smoothness properties but they satisfy different boundary conditions.

The covariance kernel of X_r^{W} is given by

$$
K_{1,r}^{\mathrm{W}}(x,y) = \int_0^{\min(x,y)} \frac{(x-u)^r}{r!} \frac{(y-u)^r}{r!} \, \mathrm{d}u
$$

and is called the Wiener kernel, while the covariance kernel of X_r^{E} is given by

$$
K_{1,r}^{\mathrm{E}}(x,y) = \int_{[0,1]^r} \min(x,s_1) \min(s_1,s_2) \cdots \min(s_r,y) \, \mathrm{d}s_1 \, \mathrm{d}s_2 \cdots \mathrm{d}s_r
$$

and is called the Euler kernel. The last kernel can be expressed in terms of Euler polynomials, hence the name of the process and its kernel.

The corresponding tensor product kernels on $[0, 1]^d$ are given by

$$K_d^{\mathrm{W}}(s, t) = \prod_{k=1}^{d} K_{1, r_k}^{\mathrm{W}}(s_k, t_k), \quad \text{and} \quad K_d^{\mathrm{E}}(s, t) = \prod_{k=1}^{d} K_{1, r_k}^{\mathrm{E}}(s_k, t_k).$$

As for the Korobov case, the sequence $\{r_k\}$ with integers

$$r_1 \leq r_2 \leq \cdots \leq r_d \leq \cdots ,$$

describes the increasing smoothness of a process with respect to the successive coordinates.

We now compare tractability results for processes described by the Euler, Korobov and Wiener kernels from [7] and from Theorem 9. Some results are the same:

- strong polynomial tractability and polynomial tractability are equivalent,

- there is a lim-inf-type criterion for polynomial tractability,

- there is a narrow zone where quasi-polynomial tractability holds while polynomial tractability fails,

- weak tractability is equivalent to a convergence without rate, $\lim_k r_k = \infty$ for both integrated processes, or to $\lim_k g_k = 0$ for Korobov case,

- if weak tractability fails then the curse of dimensionality appears.

The conditions on strong polynomial tractability for Euler and Wiener integrated processes are different. Namely, strong polynomial tractability holds iff

$$\rho_E := \liminf_{d \to \infty} \frac{r_d}{\ln d} > \frac{1}{2 \ln 3} \qquad \text{for Euler integrated process,}$$

$$\rho_W := \liminf_{d \to \infty} \frac{r_d}{d^s} > 0 \quad \text{for some } s > \tfrac{1}{2} \qquad \text{for Wiener integrated process.}$$

For the Korobov case, strong polynomial tractability depends on $\{g_d\}$ and holds iff

$$\rho_K := \liminf_{d \to \infty} \frac{\ln \frac{1}{g_d}}{\ln d} > 1,$$

For $g_d = 9^{-r_d}$, we see that $\rho_K = (2 \ln 3) \rho_E$ and conditions for strong polynomial tractability for the Euler and Korobov cases are equivalent.

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For $g_d = d^{-r_d/d^s}$, we see that $\rho_W = \rho_K$. Hence, strong polynomial tractability holds for the Wiener and Korobov cases if $\rho_W > 1$, whereas it holds only for the Wiener case when $\rho_W \in (0, 1]$.

Without going to technical details, we may say that all depends on the two largest eigenvalues for the univariate cases. These eigenvalues are quite different for the Euler and Wiener cases, whereas for the Korobov case they depend on the weights g_k. By adjusting these weights, the Korobov case behaves either like the Euler or Wiener case.

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