Sufficient Conditions for Fast Quasi-Monte Carlo Convergence

A. Papageorgiou
Department of Computer Science
Columbia University
New York, NY 10027

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Abstract

We study the approximation of d-dimensional integrals. We present sufficient conditions for fast quasi-Monte Carlo convergence. They apply to isotropic and non-isotropic problems and, in particular, to a number of problems in computational finance. We show that the convergence rate of quasi-Monte Carlo is of order $n^{-1+p\{\log n\}^{-1/2}}$ with $p \geq 0$. This is a worst case result. Compared to the expected rate $n^{-1/2}$ of Monte Carlo it shows the superiority of quasi-Monte Carlo.

1 Introduction

Many applications, for instance, in finance and in physics, require the calculation of high dimensional integrals. The Monte Carlo method is frequently used to approximate them. Its expected error is of order $n^{-1/2}$ and for functions with uniformly bounded variances the expected error is independent of the dimension. Since its convergence is not fast, a large number of evaluations may be necessary.

Quasi-Monte Carlo methods use deterministic samples at points that belong to low discrepancy sequences and approximate the integrals by the arithmetic average of n function evaluations. According to the Koksma-Hlawka inequality their worst case error is of order $\log^d n/n$, where d denotes the dimension [7]. A similar bound was shown for the average error of high dimensional integration by Woźniakowski [18].

Since $\log^d n/n$ becomes huge when n is fixed and d is large, as sometimes happens in practice, traditionally, there has been a certain degree of concern about quasi-Monte Carlo. However, tests by Paskov and Traub in [12, 13] and quite a few others, e.g., [1, 2, 3, 5, 8, 10], have shown that quasi-Monte Carlo methods are very effective for very high dimensional integrals in computational finance. In these tests the observed error is about n^{-1}. A survey of the state of the art may be found in the monograph by Traub and Werschulz [17].

Sloan and Woźniakowski [14] were the first to derive a class of functions for which the worst case error of quasi-Monte Carlo is of order $n^{-1/p}$, $p \in [1, 2]$, and does not depend on d. Without going into the details of their paper, we mention that they considered non-isotropic integration problems by assuming that the behavior of the functions in the successive dimensions is moderated by different weights.

Quasi-Monte Carlo methods have also been found effective for isotropic problems, where all integration variables are equally important. Papageorgiou and Traub [11] tested quasi-Monte Carlo using an isotropic integral from physics and found that it converged faster than Monte Carlo. Later, Papageorgiou [9] showed that the quasi-Monte Carlo error is of order $\sqrt{\log n}/n$ for a certain class of isotropic integrals, which includes the example of [11].

Explaining the surprisingly good performance of quasi-Monte Carlo for other high dimensional integrals and, in particular, for integrals arising in finance, is a challenging problem.

In this paper we present sufficient conditions for fast quasi-Monte Carlo convergence. They can be used in the study of isotropic and non-isotropic problems, and, specifically, in the study of a number of problems in finance.

In particular, we study quasi-Monte Carlo in the worst case for a number of different function classes. We show that its convergence is

$$O(n^{-1+p\{\log n\}^{-1/2}}) = O(n^{-1+o(1)}),$$

where $p \geq 0$ is a constant that depends on the class definition, and the factor in the big O notation is independent of d.

Since this is a worst case result, and since this convergence is faster than the expected rate $n^{-1/2}$ of Monte Carlo it shows the superiority of quasi-Monte Carlo for the integration of functions from our classes.

Finally, we consider some applications in finance. We show how they can be analyzed using the conditions of our framework, and the corresponding quasi-Monte Carlo convergence rates.

2 Problem Definition

We consider the approximation of a weighted high-dimensional integral of the form

$$I_{d,g}(f) = \int_{\mathbb{R}^d} f(g(x))\phi_d(x)\,dx, \tag{1}$$

where d is the dimension, $\phi_d(x) = (2\pi)^{-d/2}e^{-\|x\|^2/2}$ is a Gaussian weight. We assume that $g\colon \mathbb{R}^d \to \mathbb{R}$ is a given continuous function, and $f : \mathbb{R} \to \mathbb{R}$ such that $I_{d,g}(f) < \infty$.

In a recent paper [9] we studied the approximation of the integral

$$\int_{\mathbb{R}^d} f(\|x\|)e^{-\|x\|^2}\,dx,$$

for a function f such that $\operatorname{ess\,sup}\{|f'(r)| : \ r \in \mathbb{R}\} \leq M$, $M > 0$, where $\|\cdot\|$ denotes the Euclidean norm. We found that the quasi-Monte Carlo error is $O(\sqrt{\log n}/n)$. The last integral corresponds to $g(x) = \|x\|$, $x \in \mathbb{R}^d$. In this paper we extend the results of [9] for more general g and other classes of functions f.

We will assume that the function g is such that the probability measure defined by

$$\mu(A) = \int_{\mathbb{R}^d} 1_A(g(x))\phi_d(x)\,dx, \quad A \in \mathcal{B}(\mathbb{R}), \tag{2}$$

is equivalent to the Lebesgue measure. Then the integral (1) can be reduced by a change of variable to a one-dimensional integral, which sometimes can be solved analytically. We do not do this because we want to study the performance of quasi-Monte Carlo methods for d-dimensional integration.

Without using (2), we have the equivalent integral over the cube $[0,1]^d$,

$$I_{d,g}(f) = \int_{\mathbb{R}^d} f(g(x))\phi_d(x)\,dx = \int_{[0,1]^d} f(g(\Phi^{-1}(t_1),\ldots,\Phi^{-1}(t_d)))\,dt,$$

where Φ is the cumulative normal distribution function with mean zero and variance one,

$$\Phi(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-s^2/2}\,ds, \quad y \in [-\infty,\infty]. \tag{3}$$

For any deterministic points $t_i = (t_{i1},\ldots,t_{id}) \in (0,1)^d$, $i = 1,\ldots,n$, we define the points $x_i = (x_{i1},\ldots,x_{id}) \in \mathbb{R}^d$, $i = 1,\ldots,n$, by $x_{ij} = \Phi^{-1}(t_{ij})$, $j = 1,\ldots,d$, $i = 1,\ldots,n$. Then we approximate the integral (1) by the quasi-Monte Carlo method

$$I_{d,g,n}(f) = \frac{1}{n}\sum_{i=1}^{n} f(g(x_i)). \tag{4}$$

We study the error and the convergence rate of the method $I_{d,g,n}$ under the following conditions.

Definition 2.1. *Let x_0, γ and β be given positive numbers. Assume that the function g is such that the tails of the measure μ satisfy:*

$$1 - \mu(x) \;\le\; \gamma e^{-\beta x^2},\ x \ge x_0, \tag{5}$$

$$\mu(-x) \;\le\; \gamma e^{-\beta x^2},\ x \ge x_0, \tag{6}$$

where $\mu(x) := \mu(-\infty, x)$, $x \in \mathbb{R}$.

Let F_1 be the class of functions $f : \mathbb{R} \to \mathbb{R}$, such that $I_{d,g}(f) < \infty$, f is absolutely continuous, f' exists a.e., and

$$\left\{\int_{\mathbb{R}} [f'(x)]^2\, e^{-\alpha|x|}\,dx\right\}^{1/2} \le M,$$

where M and α are given positive numbers.

Let F_2 be the class of functions defined in the same way as F_1 with the exception that the condition above is replaced by

$$\operatorname{ess\,sup}\left\{|f'(x)|e^{-\alpha|x|} : x \in \mathbb{R}\right\} \le M.$$

Definition 2.2. *Let x_0, γ, β and δ be given positive numbers. Assume that the function g is such that the tails of the measure μ satisfy:*

$$\begin{aligned} 1 - \mu(x) &\leq \gamma e^{-\beta(\log(\delta x))^2}, \ x \geq x_0, && (7) \\ \mu(-x) &\leq \gamma e^{-\beta(\log(\delta x))^2}, \ x \geq x_0. && (8) \end{aligned}$$

In this case, the tails of μ are qualitatively different than before. They decay proportionally to $x^{-\beta \log x}$. This is much slower than the decay of the tails of the normal distribution [16].

Let F_3 be the class of functions $f : \mathbb{R} \to \mathbb{R}$, such that $I_d(f) < \infty$, f is absolutely continuous, f' exists a.e., and

$$\int_{\mathbb{R}} |f'(x)| \, dx \leq M,$$

where M is a given positive number.

Let F_4 be the class of functions defined in the same way as F_3 with the exception that the above condition is replaced by

$$\operatorname{ess\,sup} \{|f'(x)| : x \in \mathbb{R}\} \leq M.$$

3 Low Discrepancy Sequences

We briefly discuss low discrepancy sequences. Discrepancy is a measure of uniformity of a sequence of points. The discrepancy of n points $x_1, \ldots, x_n \in [0,1]^d$, $d \geq 1$, is defined by

$$D_n^{(d)} = D^{(d)}(x_1, \ldots, x_n) = \sup_E \left| \frac{A(E;n)}{n} - \lambda(E) \right|,$$

where the supremum is taken over all subsets of $[0,1]^d$ of the form $E = [0, t_1) \times \cdots \times [0, t_d)$, λ denotes the Lebesgue measure, and $A(E;n)$ is the number of the x_j that are contained in E. A detailed study of low discrepancy sequences can be found in [4, 7, 15].

A sequence $x_1, x_2, \ldots$ of points in $[0,1]^d$ is a low discrepancy sequence if

$$D_n^{(d)} \leq c(d) \frac{\log^d n}{n}, \quad \forall n > 1,$$

where $c(d)$ depends only on the dimension d.

The Koksma-Hlawka inequality establishes a relation between low discrepancy sequences and multivariate integration [7]. If f is a real function, defined on $[0,1]^d$, of bounded variation $V(f)$ in the sense of Hardy and Krause, then for any sequence $x_1, \ldots, x_n \in [0,1)^d$ we have

$$\left| \int_{[0,1]^d} f(x) \, dx - \frac{1}{n} \sum_{i=1}^{n} f(x_i) \right| \leq V(f) D_n^{(d)}.$$

In the above, the uniformity of a sequence is assessed with respect to the Lebesgue measure on the cube $[0,1]^d$. However, the discrepancy of a sequence can be considered with respect to any probability measure μ.

We introduce some definitions and notation that we will use in the rest of this paper. Let μ be a probability measure on $\mathbb{R}$. For $n > 1$, let $x_i \in \mathbb{R}$, $i = 1, \ldots, n$, be any points. Define the difference between the empirical distribution (approximating μ using the points x_i) and the measure μ by

$$R_\mu(E) = \frac{A(E;n)}{n} - \mu(E), \quad E \subset \mathbb{R},$$

where $A(E;n)$ is the number of the x_i contained in E and does not depend on μ but depends only on the points x_i. The discrepancy of the points x_i, $i = 1, \ldots, n$, with respect to μ is defined by

$$D_{\mu,n} = D_{\mu,n}(x_1, \ldots, x_n) = \sup_E |R_\mu(E)|,$$

where the supremum is taken over all sets of the form $E = (-\infty, x)$, $x \in \mathbb{R}$.

For $x \in \mathbb{R}$ and $x_i \in \mathbb{R}$, $i = 1, \ldots, n$, we use the notation

$$R_\mu(x; x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} 1_{(-\infty,x)}(x_i) - \mu(x).$$

When μ is equivalent to the Lebesgue measure, $\mu \sim \lambda$, as in this paper, or when it is absolutely continuous with respect to the Lebesgue measure, $\mu \ll \lambda$, and supported on an interval, as in [9], we can derive a low discrepancy sequence with respect to μ from a low discrepancy sequence with respect to the Lebesgue measure.

Indeed, given a low discrepancy sequence (with respect to the Lebesgue measure) $t_i \in [0,1]$, $i = 1, 2, \ldots$, the sequence $x_i = \mu^{-1}(t_i) \in \mathbb{R}$, $i = 1, 2, \ldots$, has discrepancy $D_{\mu,n}$ with respect to the measure μ, and satisfies $D_{\mu,n}(x_1, \ldots, x_n) = D_n^{(1)}(t_1, \ldots, t_n)$, $n > 1$. For brevity, when $d = 1$ we write D_n instead of $D_n^{(1)}$.

4 Quasi-Monte Carlo Error

In this section we derive the quasi-Monte Carlo error. Given a sample set consisting of function evaluations at points $x_i \in \mathbb{R}^d$, $i = 1, \ldots, n$, we will show that the error depends on the discrepancy $D_{\mu,n}$, with respect to μ, of the points $g(x_i) \in \mathbb{R}$, $i = 1, \ldots, n$.

Lemma 4.1. *Consider the integral (1) and assume that either the conditions of Definition 2.1 or those of Definition 2.2 are satisfied. Let $x_i \in \mathbb{R}^d$, $i = 1, \ldots, n$, be any points, $n \geq 1$. Then for $f \in F_i$, $i = 1, \ldots, 4$, we have*

$$I_{d,g}(f) - I_{d,g,n}(f) = \int_{\mathbb{R}} R_\mu(x) f'(x)\, dx,$$

where $R_\mu(x) = R_\mu(x; g(x_1), \ldots, g(x_n))$, $x \in \mathbb{R}$.

Proof: The error of a quasi-Monte Carlo method approximating the integral of a differentiable function is shown in [7, Ch. 2]. We proceed in a similar way. For $n, d \geq 1$ and $x_i \in \mathbb{R}^d$,

$i = 1, \dots, n$ we have

$$
\begin{aligned}
&\int_{\mathbb{R}^d} f(g(x))\phi_d(x)\,dx - \frac{1}{n}\sum_{i=1}^{n} f(g(x_i)) \\
&= \int_{\mathbb{R}} f(t)\mu'(t)\,dt - \frac{1}{n}\sum_{i=1}^{n} f(g(x_i)) \\
&= \int_0^1 f(\mu^{-1}(t))\,dt - \frac{1}{n}\sum_{i=1}^{n} f(g(x_i)) \\
&= \int_0^1 h(s)\,ds - \frac{1}{n}\sum_{i=1}^{n} h(s_i), \quad h = f\circ\mu^{-1},\ s_i = \mu(g(x_i)) \\
&= \int_0^1 R(t)h'(t)\,dt, \quad R(t) = \frac{1}{n}\sum_{i=1}^{n} 1_{[0,t)}(s_i) - t \\
&= \int_0^1 R(t)\left.\frac{df(z)}{dz}\right|_{z=\mu^{-1}(t)} \left(\mu^{-1}\right)'(t)\,dt \\
&= \int_0^1 R(t)\left.\frac{df(z)}{dz}\right|_{z=\mu^{-1}(t)} d\mu^{-1}(t) \\
&= \int_{\mathbb{R}} f'(x)R(\mu(x))\,dx \\
&= \int_{\mathbb{R}} f'(x)R_\mu(x)\,dx,
\end{aligned}
$$

and this completes the proof. ■

We remark that one can derive an equivalent lemma for the quasi-Monte Carlo error when μ is absolutely continuous with respect to the Lebesgue measure and its support is an interval, e.g., $(0,\infty)$. In this case, the domain of integration in the right hand side of the error equation is not $\mathbb{R}$ but is the support of μ.

For a class of functions F we define the worst case error of the method $I_{d,g,n}$ by

$$
\begin{aligned}
e(I_{d,g,n}, f) &= |I_{d,g}(f) - I_{d,g,n}(f)|, \quad f \in F, \\
e(I_{d,g,n}, F) &= \sup_{f\in F} e(I_{d,g,n}, f).
\end{aligned}
$$

Proposition 4.1. *For the classes of functions F_i, $i = 1, \dots, 4$, under the conditions of Definition 2.1 and Definition 2.2, the error of the method $I_{d,g,n}$ in (4) satisfies*

$$
e(I_{d,g,n}, F_i) = \sup_{f\in F_i} \int_{\mathbb{R}} |f'(x)R_\mu(x)|\,dx, \quad i = 1, \dots, 4,
$$

and

$$\begin{aligned}
e(I_{d,g,n}, F_1) &\leq M \left\{ \int_{\mathbb{R}} e^{\alpha|x|} R_\mu^2(x)\, dx \right\}^{1/2}, \\
e(I_{d,g,n}, F_2) &\leq M \int_{\mathbb{R}} e^{\alpha|x|} |R_\mu(x)|\, dx, \\
e(I_{d,g,n}, F_3) &\leq M \sup_{x\in\mathbb{R}} |R_\mu(x)| = M D_{\mu,n}, \\
e(I_{d,g,n}, F_4) &\leq M \int_{\mathbb{R}} |R_\mu(x)|\, dx.
\end{aligned}$$

Proof: From Lemma 4.1 we have $e(I_{d,g,n}, F_i) \leq \sup_{f\in F_i} \int_{\mathbb{R}} |f'(x)R_\mu(x)|\, dx$, $i = 1, \ldots, 4$. For $f \in F_i$ we consider a corresponding function w such that $w'(x) = f'(x) \cdot \mathrm{sign}(f'(x)R_\mu(x))$ a.e., where $\mathrm{sign}(x) = \pm 1$ when $x \geq 0$ and $x < 0$, respectively. Then $w'(x)R_\mu(x) \geq 0$ a.e., w is absolutely continuous and $w \in F_i$, which allows us to conclude

$$e(I_{d,g,n}, F_i) = \sup_{f\in F_i} \int_{\mathbb{R}} |f'(x)R_\mu(x)|\, dx, \quad i = 1, \ldots, 4.$$

The remaining inequalities are straightforward. The first one is obtained using the definition of F_1 and applying the Cauchy-Schwarz inequality to the above integral, while the second, third and fourth inequalities follow from the definition of F_2, F_3, and F_4, respectively. ■

Corollary 4.1. *If $\mu \ll \lambda$ and is supported on a bounded or semi-bounded interval, then Proposition 4.1 holds by replacing the domain of integration by the support of μ,* $\mathrm{supp}(\mu)$*. Moreover, when* $\mathrm{supp}(\mu)$ *is bounded Proposition 4.1 implies*

$$e(I_{d,g,n}, F_i) = O(D_{\mu,n}), \quad i = 1, \ldots, 4. \quad ■$$

We have seen that the quasi-Monte Carlo error depends on the discrepancy, $D_{\mu,n} = D_{\mu,n}(g(x_1), \ldots, g(x_n))$, of the points $g(x_i)$, $x_i \in \mathbb{R}^d$, $i = 1, \ldots, n$, with respect to the measure μ. Selecting the points x_i in a way that $D_{\mu,n}$ is minimized will lead to small quasi-Monte Carlo error. We use this in deriving the convergence of quasi-Monte Carlo in the next section. We can construct the points $x_i \in \mathbb{R}^d$, $i = 1, \ldots, n$ for which $D_{\mu,n}$ is minimized when μ is known. Otherwise, we know their existence.

There is another practical reason for which $D_{\mu,n}$ is important. When μ is known, for any given sample $x_i \in \mathbb{R}^d$, $i = 1, \ldots, n$, it is easy to compute the value of $D_{\mu,n}$. Then we can use its value along with the conditions about the decay of the tails of μ (see equations (5), (6) in Definition 2.1, and equations (7), (8) in Definition 2.2) to estimate the integrals of Proposition 4.1 from above. Therefore, we can estimate the quasi-Monte Carlo error and assess the quality of any sample.

5 Speed of Convergence

We now proceed to derive the convergence rate of the method (4) for the respective classes of functions. We have already made it clear that the error depends on the discrepancy of the sequence $g(x_i)$, $i = 1, \ldots, n$, with respect to the measure μ. It also depends on the properties of the tails of the distribution μ. The convergence rates, and the respective constant factors in the error estimates, that we obtain in this section hold for integration problems defined by any function g for which the corresponding distribution μ has tails satisfying the conditions (5), (6) of Definition 2.1, or (7), (8) of Definition 2.2.

Recall that we assumed throughout the paper that μ is equivalent to the Lebesgue measure on $\mathbb{R}$. However, in view of Corollary 4.1 the results obtained here apply also to the case where $\mathrm{supp}(\mu)$ is an interval.

Consider the sample points $x_i \in \mathbb{R}^d$, $i = 1, \ldots, n$, and let $l = \min_{1 \le i \le n}\{g(x_i)\}$ and $u = \max_{1 \le i \le n}\{g(x_i)\}$. Using $x_0 > 0$ (see Definition 2.1 and Definition 2.2) we define $l^* = \min\{-x_0, l\}$, $u^* = \max\{x_0, u\}$, and $r^* = \max\{-l^*, u^*\}$. Then

$$|R_\mu(-x)| = \mu(-x), \quad \text{and} \quad |R_\mu(x)| = 1 - \mu(x), \quad \forall\, x > r^*. \tag{9}$$

We use this fact in Lemma 5.1 and in Lemma 5.2 below.

Lemma 5.1. *Assume that the tails of μ satisfy (5) and (6). For any $h \in (0,1)$ and $r \ge \max\{(\alpha+1)/(2h\beta), r^*\}$, $\alpha, \beta > 0$, we have*

$$\int_{\mathbb{R}} R_\mu^2(x) e^{\alpha|x|}\, dx \le \int_{-r}^{r} R_\mu^2(x) e^{\alpha|x|}\, dx + (\gamma+1)e^{\alpha r}\left\{[\mu(-r)]^{2-h} + [1-\mu(r)]^{2-h}\right\}.$$

Proof: For $h \in (0,1)$, $r \ge r^*$ and $\alpha, \beta > 0$ we have

$$\begin{aligned}
\int_{-\infty}^{-r} \mu^2(x) e^{\alpha|x|}\, dx &\le [\mu(-r)]^{2-h} \int_{-\infty}^{-r} [\mu(x)]^h e^{-\alpha x}\, dx \\
&\le (\gamma+1)[\mu(-r)]^{2-h} \int_{-\infty}^{-r} e^{-(h\beta x^2 + \alpha x)}\, dx, \quad \text{since } \gamma^h \le \gamma + 1, \\
&= (\gamma+1) e^{\alpha^2/(4h\beta)} [\mu(-r)]^{2-h} \int_{-\infty}^{-r} e^{-h\beta(x+\alpha/(2h\beta))^2}\, dx \\
&= (\gamma+1) e^{\alpha^2/(4h\beta)} [\mu(-r)]^{2-h} \int_{-\infty}^{-r+\alpha/(2h\beta)} e^{-h\beta y^2}\, dy \\
&= \frac{\gamma+1}{\sqrt{2h\beta}} e^{\alpha^2/(4h\beta)} [\mu(-r)]^{2-h} \int_{-\infty}^{-\sqrt{2h\beta}(r-\alpha/(2h\beta))} e^{-t^2/2}\, dt \\
&= \frac{\gamma+1}{\sqrt{2h\beta}} e^{\alpha^2/(4h\beta)} [\mu(-r)]^{2-h} \int_{\sqrt{2h\beta}(r-\alpha/(2h\beta))}^{\infty} e^{-t^2/2}\, dt \\
&< \frac{\gamma+1}{2h\beta} e^{\alpha^2/(4h\beta)} [\mu(-r)]^{2-h} \frac{e^{-h\beta(r-\alpha/(2h\beta))^2}}{r - \alpha/(2h\beta)} \\
&= (\gamma+1)[\mu(-r)]^{2-h} \frac{e^{-h\beta r^2 + \alpha r}}{2h\beta r - \alpha},
\end{aligned}$$

where the last inequality is obtained using the estimate in [16, p. 175] for the tail of the normal distribution, i.e.,

$$\frac{1-\Phi(y)}{\Phi'(y)} < y^{-1}, \quad y > 0,$$

where Φ is given in (3).

For $r \geq \max\{(\alpha+1)/(2h\beta), r^*\}$ and since $e^{-h\beta r^2} \leq 1$ we obtain

$$\int_{-\infty}^{-r} \mu^2(x) e^{\alpha|x|}\, dx \leq (\gamma+1)[\mu(-r)]^{2-h} e^{\alpha r}. \tag{10}$$

Similarly, we derive an upper bound for $\int_r^\infty [1-\mu(x)]^2 e^{\alpha x}\, dx$. Since $\gamma^h \leq \gamma+1$ we have

$$\begin{aligned}
\int_r^\infty [1-\mu(x)]^2 e^{\alpha x}\, dx &\leq (\gamma+1)[1-\mu(r)]^{2-h} \int_r^\infty e^{-h\beta x^2+\alpha x}\, dx \\
&= (\gamma+1) e^{\alpha^2/(4h\beta)} [1-\mu(r)]^{2-h} \int_r^\infty e^{-h\beta(x-\alpha/(2h\beta))^2}\, dx \\
&= (\gamma+1) e^{\alpha^2/(4h\beta)} [1-\mu(r)]^{2-h} \int_{r-a/(2h\beta)}^\infty e^{-h\beta y^2}\, dy \\
&= \frac{\gamma+1}{\sqrt{2h\beta}} e^{\alpha^2/(4h\beta)} [1-\mu(r)]^{2-h} \int_{\sqrt{2h\beta}(r-\alpha/(2h\beta))}^\infty e^{-t^2/2} dt \\
&< \frac{\gamma+1}{2h\beta} e^{\alpha^2/(4h\beta)} [1-\mu(r)]^{2-h} \frac{e^{-h\beta(r-\alpha/(2h\beta))^2}}{r-\alpha/(2h\beta)} \\
&= (\gamma+1)[1-\mu(r)]^{2-h} \frac{e^{-h\beta r^2+\alpha r}}{2h\beta r-\alpha},
\end{aligned}$$

where the last inequality is obtained using the estimate in [16] for the tail of the normal distribution as before.

Thus for $r \geq \max\{(\alpha+1)/(2h\beta), r^*\}$ and since $e^{-h\beta r^2} \leq 1$ we obtain

$$\int_r^\infty [1-\mu(x)]^2 e^{\alpha x}\, dx \leq (\gamma+1)[1-\mu(r)]^{2-h} e^{\alpha r}. \tag{11}$$

Combining (9), (10) and (11) completes the proof. ■

Proposition 5.1. *Assume that the tails of μ satisfy the conditions (5) and (6). Then for $n \geq \max\{c[1-\mu(x_0)]^{-1}, c[\mu(-x_0)]^{-1}, (4\gamma)^{-1} e^{(\alpha+1)^2/\beta}\}$ there exist deterministic points $x_i \in \mathbb{R}$, $i = 1, \ldots, n$, for which*

$$\left\{\int_{\mathbb{R}} R_\mu^2(x) e^{\alpha|x|}\, dx\right\}^{1/2} \leq c_1 \sqrt{2(4\gamma+1)(\alpha^{-1}+\gamma+1)}\; n^{-1+\frac{3\alpha+1}{4}\{\beta\log(4\gamma n)\}^{-1/2}},$$

where $c_1 = \max\{c, 1\}$ and $c \geq 1/2$ is a constant such that the discrepancy $D_{\mu,n}(x_1, \ldots, x_n) \leq c/n$. Moreover, for the specific choice $x_i = \mu^{-1}[(2i-1)/(2n)]$, $i = 1, \ldots, n$, the inequality holds with $c = 1/2$ and $c_1 = 1$.

Proof: Let $t_i \in [0,1)$, $i = 1, \dots, n$, be n numbers with discrepancy $D_n = c/n$, where $c \geq 1/2$ is a constant. For instance, these numbers can be n terms of a low discrepancy sequence or a $(t, m, 1)$-net. It is shown in [7, Ch. 2] that the discrepancy of these points is given by

$$D_n = \frac{1}{2n} + \max_{1 \leq i \leq n} \left| t_{(i)} - \frac{2i-1}{2n} \right|,$$

where $t_{(1)} \leq \cdots \leq t_{(i)} \leq \cdots \leq t_{(n)}$ denotes the ordered sequence of points.

This implies that the discrepancy of the sequence

$$\tau_i = \begin{cases} t_i + (4n)^{-1} & \text{if } t_i < (4n)^{-1} \\ t_i - (4n)^{-1} & \text{if } t_i > 1 - (4n)^{-1} \\ t_i & \text{otherwise} \end{cases}, i = 1, \dots, n,$$

cannot exceed c/n, its minimum term satisfies $\tau_{(1)} > (4n)^{-1}$ and its maximum term satisfies $\tau_{(n)} < 1 - (4n)^{-1}$. Hence, without loss of generality, we assume that $t_{(1)} \geq (4n)^{-1}$ and that $t_{(n)} \leq 1 - (4n)^{-1}$.

Consider $x_i \in \mathbb{R}$ such that $\mu(x_i) = t_i$, $i = 1, \dots, n$. Let $u = u(n) = \max_{1 \leq i \leq n}\{x_i\}$, $l = l(n) = \min_{1 \leq i \leq n}\{x_i\}$. Since $|R_\mu(x)| \leq D_n = c/n$, $\forall\, x \in \mathbb{R}$, we have

$$\begin{array}{rcl} (4n)^{-1} \leq & \mu(l) & \leq cn^{-1} \\ (4n)^{-1} \leq & 1 - \mu(u) & \leq cn^{-1}. \end{array}$$

Since $n \geq c \max\{[1-\mu(x_0)]^{-1}, [\mu(-x_0)]^{-1}\}$ we have $\mu(l) \leq cn^{-1} \leq \mu(-x_0)$ and $1-\mu(u) \leq cn^{-1} \leq 1-\mu(x_0)$. (Note that $\mu(-x_0) \neq 0 \neq 1-\mu(x_0)$ because μ is equivalent to the Lebesgue measure.) Thus, $u, -l \geq x_0 > 0$.

Let $q = m(x) = \gamma e^{-\beta x^2}$ for some $x > 0$. Then, it is easy to verify that the inverse of m is given by

$$m^{-1}(q) = \left\{\beta^{-1} \log(\gamma/q)\right\}^{1/2}, \quad 0 < q \leq \gamma.$$

Observe that m and m^{-1} are decreasing functions, therefore, the above inequalities and equations (5) and (6) imply $m^{-1}[(4n)^{-1}] \geq m^{-1}[1-\mu(u)] \geq u$ and $m^{-1}[(4n)^{-1}] \geq m^{-1}[\mu(l)] \geq l$.

Let $r = r(n) = m^{-1}[(4n)^{-1}]$. Clearly $r \geq r^* = \max\{-l, u\}$. Assume that n is sufficiently large so that the remaining condition of Lemma 5.1 is satisfied for some $h \in (0,1)$, i.e., $r \geq (\alpha+1)/(2h\beta)$. Then

$$\begin{aligned} \int_{\mathbb{R}} R_\mu^2(x) e^{\alpha|x|}\, dx &\leq \int_{-r}^{r} R_\mu^2(x) e^{\alpha|x|}\, dx + (\gamma+1) e^{\alpha r} \left\{[\mu(-r)]^{2-h} + [1-\mu(r)]^{2-h}\right\} \\ &\leq \left(\frac{c}{n}\right)^2 \int_{-r}^{r} e^{\alpha|x|}\, dx + 2(\gamma+1)\left(\frac{c}{n}\right)^{2-h} e^{\alpha r} \\ &= \left(\frac{c}{n}\right)^2 \frac{2}{\alpha}(e^{\alpha r} - 1) + 2(\gamma+1)\left(\frac{c}{n}\right)^{2-h} e^{\alpha r} \\ &\leq 2c_1^2(\alpha^{-1} + \gamma + 1)\left(\frac{1}{n}\right)^{2-h} e^{\alpha r}, \; c_1 = \max\{c, 1\}. \end{aligned}$$

Using the condition $r \geq (\alpha+1)/(2h\beta)$, see Lemma 5.1, we define h by

$$h = \frac{\alpha+1}{2\beta}\left\{\beta^{-1}\log(4\gamma n)\right\}^{-1/2} = \frac{\alpha+1}{2}\left\{\beta\log(4\gamma n)\right\}^{-1/2}.$$

Thus $n \geq (4\gamma)^{-1}e^{(\alpha+1)^2/\beta}$ implies $h \leq 1/2$.

We will now estimate $e^{\alpha r}$. We have

$$\begin{aligned}
e^{\alpha r} &= \exp\left\{\alpha m^{-1}[(4n)^{-1}]\right\} \\
&= \exp\left\{\alpha\sqrt{\beta^{-1}\log(4\gamma n)}\right\} \\
&= \exp\left\{\alpha\beta^{-1/2}\log(4\gamma n)^{1/(\log(4\gamma n))^{1/2}}\right\} \\
&= (4\gamma n)^{\alpha/\sqrt{\beta\log(4\gamma n)}} \\
&\leq (4\gamma+1)n^{\alpha/\sqrt{\beta\log(4\gamma n)}},
\end{aligned}$$

where the last inequality holds for $n \geq (4\gamma)^{-1}e^{\alpha^2/\beta}$.

Hence,

$$\begin{aligned}
\int_{\mathbb{R}} R_\mu^2(x)e^{\alpha|x|}\,dx &\leq 2c_1^2(4\gamma+1)(\alpha^{-1}+\gamma+1)\; n^{-2}\; n^{(\alpha+1)\{\beta\log(4\gamma n)\}^{-1/2}/2}\; n^{\alpha\{\beta\log(4\gamma n)\}^{-1/2}} \\
&= 2c_1^2(4\gamma+1)(\alpha^{-1}+\gamma+1)\; n^{-2+\frac{3\alpha+1}{2}\{\beta\log(4\gamma n)\}^{-1/2}},
\end{aligned}$$

which completes the proof. ■

Remark 5.1. If $\mu \ll \lambda$ and its support is a semi-bounded interval as in [9], i.e., $\operatorname{supp}(\mu) = (-\infty, z)$ or $\operatorname{supp}(\mu) = (z, \infty)$, for some $z \in \mathbb{R}$, then it is possible that either $\mu(-x_0) = 0$ or $\mu(x_0) = 1$. In such a case, we can slightly modify the statement of Proposition 5.1 and its proof.

In particular, when $\mu(-x_0) = 0$ and $\mu(x_0) \neq 1$ it suffices to require $n \geq \max\{c[1-\mu(x_0)]^{-1}, (4\gamma)^{-1}e^{(\alpha+1)^2/\beta}\}$, and in the proof we need to consider only the maximum term u. Similarly, when $\mu(-x_0) \neq 0$ and $\mu(x_0) = 1$ we require $n \geq \max\{c[\mu(-x_0)]^{-1}, (4\gamma)^{-1}e^{(\alpha+1)^2/\beta}\}$, and in the proof we need to consider only the minimum term l. In either case, we partition $\operatorname{supp}(\mu)$ into two intervals, using u or l, and estimate $\int_{\operatorname{supp}(\mu)} R_\mu^2(x)e^{\alpha|x|}\,dx$ by estimating the resulting two integrals. The bound of Proposition 5.1 follows easily.

Similar considerations apply to the remaining parts of this paper.

Proposition 5.2. *Assume that the tails of μ satisfy the conditions (5) and (6). Then for $n \geq \max\{c[1-\mu(x_0)]^{-1}, c[\mu(-x_0)]^{-1}, (4\gamma)^{-1}e^{(\alpha+1)^2/\beta}\}$ there exist deterministic points $x_i \in \mathbb{R}$, $i = 1, \ldots, n$, for which*

$$\int_{\mathbb{R}} |R_\mu(x)|e^{\alpha|x|}\,dx \leq 2c_1(4\gamma+1)(\alpha^{-1}+\gamma+1)\; n^{-1+\frac{3\alpha+1}{2}\{\beta\log(4\gamma n)\}^{-1/2}},$$

where $c_1 = \max\{c, 1\}$ and $c \geq 1/2$ is a constant such that the discrepancy $D_{\mu,n}(x_1, \ldots, x_n) \leq c/n$. Moreover, for the specific choice $x_i = \mu^{-1}[(2i-1)/(2n)]$, $i = 1, \ldots, n$, the inequality holds with $c = 1/2$ and $c_1 = 1$.

Proof: In a way similar to that of Lemma 5.1, for $h \in (0,1)$ and $r \geq \max\{(\alpha+1)/(2h\beta), r^*\}$, $\alpha, \beta > 0$, we can show that

$$\int_{\mathbb{R}} |R_\mu(x)| e^{\alpha|x|}\, dx \leq \int_{-r}^{r} |R_\mu(x)| e^{\alpha|x|}\, dx + (\gamma+1) e^{\alpha r} \left\{ [\mu(-r)]^{1-h} + [1-\mu(r)]^{1-h} \right\}.$$

The rest of the proof is almost identical to that of Proposition 5.1 and we omit it. ■

We now turn our attention to the case of Definition 2.2.

Lemma 5.2. *Assume that the tails of μ satisfy (7) and (8). For any $h \in (0,1)$ and $r \geq \max\left\{\delta^{-1} e^{1/(h\beta)}, r^*\right\}$, $\beta > 0$, we have*

$$\int_{\mathbb{R}} |R_\mu(x)|\, dx \leq \int_{-r}^{r} |R_\mu(x)|\, dx + (\gamma+1) r \left\{ [\mu(-r)]^{1-h} + [1-\mu(r)]^{1-h} \right\}.$$

Proof: The proof is very similar to that of Lemma 5.1. For $h \in (0,1)$ and $r \geq r^*$ we have

$$\begin{aligned}
\int_{r}^{\infty} [1-\mu(x)]\, dx &\leq (\gamma+1)[1-\mu(r)]^{1-h} \int_{r}^{\infty} e^{-h\beta(\log(\delta x))^2}\, dx, \quad \text{since } \gamma^h \leq \gamma+1, \\
&= \frac{\gamma+1}{\delta} [1-\mu(r)]^{1-h} \int_{\log(\delta r)}^{\infty} e^{-h\beta y^2 + y}\, dy \\
&\leq (\gamma+1)[1-\mu(r)]^{1-h} \frac{r}{2h\beta \log(\delta r) - 1},
\end{aligned}$$

where the last inequality is obtained in a way similar to that of Lemma 5.1 using the estimates of [16] for the tails of the normal distribution.

Thus, for $r \geq \max\left\{\delta^{-1} e^{1/(h\beta)}, r^*\right\}$ we have

$$\int_{r}^{\infty} [1-\mu(x)]\, dx \leq (\gamma+1)[1-\mu(r)]^{1-h} r.$$

Also

$$\begin{aligned}
\int_{-\infty}^{-r} \mu(x)\, dx &\leq (\gamma+1)[\mu(-r)]^{1-h} \int_{-\infty}^{-r} [\mu(x)]^h\, dx, \quad \text{since } \gamma^h \leq \gamma+1, \\
&= (\gamma+1)[\mu(-r)]^{1-h} \int_{r}^{\infty} [\mu(-x)]^h\, dx \\
&\leq (\gamma+1)[\mu(-r)]^{1-h} \frac{r}{2h\beta \log(\delta r) - 1}.
\end{aligned}$$

Thus, for $r \geq \max\left\{\delta^{-1} e^{1/(h\beta)}, r^*\right\}$ we have

$$\int_{-\infty}^{-r} \mu(x)\, dx \leq (\gamma+1)[\mu(-r)]^{1-h} r.$$

Combining the above with equation (9) completes the proof. ■

Proposition 5.3. *Assume that the tails of μ satisfy the conditions (7) and (8). Then for $n \geq \max\left\{c[1-\mu(x_0)]^{-1}, c[\mu(-x_0)]^{-1}, (4\gamma)^{-1}e^{1/\beta}\right\}$ there exist deterministic points $x_i \in \mathbb{R}$, $i = 1, \ldots, n$ for which*

$$\int_{\mathbb{R}} |R_\mu(x)|\, dx \leq 2c_1(\gamma+2)(4\gamma+1)\delta^{-1} n^{-1+2\{\beta \log(4\gamma n)\}^{-1/2}},$$

where $c_1 = \max\{c, 1\}$ and $c \geq 1/2$ is a constant such that the discrepancy $D_{\mu,n}(x_1, \ldots, x_n) \leq c/n$. Moreover, for the specific choice $x_i = \mu^{-1}\left[(2i-1)/(2n)\right]$, $i = 1, \ldots, n$, the inequality holds with $c = 1/2$ and $c_1 = 1$.

Proof: The proof is almost identical to that of Proposition 5.1 and we omit some of the details. Once more, we assume that $n \geq c \max\{[1-\mu(x_0)]^{-1}, [\mu(-x_0)]^{-1}\}$. From Lemma 5.2 for $h \in (0,1)$, $r \geq \max\left\{\delta^{-1}e^{1/(h\beta)}, r^*\right\}$, and by considering a low discrepancy sequence with respect to μ with discrepancy c/n, $c \geq 1/2$, we have

$$\int_{\mathbb{R}} |R_\mu(x)|\, dx \leq 2c_1(\gamma+2)\left(\frac{1}{n}\right)^{1-h} r, \quad c_1 = c+1.$$

Let $q = m(x) = \gamma e^{-\beta(\log(\delta x))^2}$ for some $x > 0$. Then the inverse of m is given by

$$m^{-1}(q) = \delta^{-1} e^{\sqrt{\beta^{-1}\log(\gamma/q)}}, \quad 0 < q \leq \gamma.$$

Just like before, we define $r = r(n) = m^{-1}[(4n)^{-1}] = \delta^{-1}e^{\sqrt{\beta^{-1}\log(4\gamma n)}}$. Using the condition $r \geq \delta^{-1}e^{1/(h\beta)}$ we define $h = \{\beta \log(4\gamma n)\}^{-1/2}$, and $n \geq (4\gamma)^{-1}e^{1/\beta}$ implies $h < 1$.

We further estimate r by carrying out the calculations in the expression for $m^{-1}[(4n)^{-1}]$, i.e., $r \leq (4\gamma+1)n^{1/\sqrt{\beta \log(4\gamma n)}}$, which holds for $n \geq (4\gamma)^{-1}e^{1/\beta}$.

We combine everything to obtain

$$\int_{\mathbb{R}} |R_\mu(x)|\, dx \leq 2c_1(\gamma+2)(4\gamma+1)\delta^{-1} n^{-1+2\{\beta \log(4\gamma n)\}^{-1/2}},$$

which holds for $n \geq \max\left\{c[1-\mu(x_0)]^{-1}, c[\mu(-x_0)]^{-1}, (4\gamma)^{-1}e^{1/\beta}\right\}$. ■

For any choice of the sample points the quasi-Monte Carlo error is given in Proposition 4.1. For the different classes of functions, it depends either on the the discrepancy $D_{\mu,n}$ of the sample points or on $\left\{\int_{\mathbb{R}} e^{\alpha|x|} R_\mu^2(x)\, dx\right\}^{1/2}$, or on $\int_{\mathbb{R}} e^{\alpha|x|}|R_\mu(x)|\, dx$, or on $\int_{\mathbb{R}} |R_\mu(x)|\, dx$. In Proposition 5.1, Proposition 5.2 and Proposition 5.3 we have shown bounds for these integrals. We summarize our results in the following theorem.

Theorem 5.1. *There exist deterministic points $x_i \in \mathbb{R}^d$, $i = 1, \ldots, n$, such that the error of the quasi-Monte Carlo method $I_{d,g,n}$ approximating the integral $I_{d,g}$ is bounded as follows:*

$$\begin{aligned}
e(I_{d,g,n}, F_1) &\leq Mc_1\sqrt{2(4\gamma+1)(\alpha^{-1}+\gamma+1)}\, n^{-1+\frac{3\alpha+1}{4}\{\beta\log(4\gamma n)\}^{-1/2}}, \quad n \geq n_0, \\
e(I_{d,g,n}, F_2) &\leq 2Mc_1(4\gamma+1)(\alpha^{-1}+\gamma+1)\, n^{-1+\frac{3\alpha+1}{2}\{\beta\log(4\gamma n)\}^{-1/2}}, \quad n \geq n_0, \\
e(I_{d,g,n}, F_3) &\leq Mcn^{-1}, \quad n \geq 1, \\
e(I_{d,g,n}, F_4) &\leq 2Mc_1(\gamma+2)(4\gamma+1)\delta^{-1} n^{-1+2\{\beta\log(4\gamma n)\}^{-1/2}}, \quad n \geq n_1,
\end{aligned}$$

where $n_0 = k(\alpha+1)$ *and* $n_1 = k(1)$ *with* $k(s) = \max\left\{c[1-\mu(x_0)]^{-1}, c[\mu(-x_0)]^{-1}, (4\gamma)^{-1}e^{s^2/\beta}\right\}$, $c_1 = \max\{c, 1\}$ *and* $c \geq 1/2$ *is a constant such that the discrepancy* $D_{\mu,n}(g(x_1), \ldots, g(x_n)) \leq c/n$. *Moreover, when* $g(x_i) = \mu^{-1}[(2i-1)/(2n)]$, $i = 1, \ldots, n$, *the inequalities hold with* $c = 1/2$ *and* $c_1 = 1$.

Proof: Let $t_i \in (0,1)$, $i = 1, \ldots, n$, be points with discrepancy c/n, $c \geq 1/2$. For instance, the t_i may belong to a one dimensional low discrepancy sequence. The points $z_i = \mu^{-1}(t_i) \in \mathbb{R}$, $i = 1, \ldots, n$, have discrepancy $D_{\mu,n} = c/n$ with respect to the measure μ. Therefore, there exist points $x_i \in \mathbb{R}^d$ such that $g(x_i) = z_i$, $i = 1, \ldots, n$, which can be used as sample points.

Thus $e(I_{d,g,n}, F_3) \leq Mcn^{-1}$ is clearly true. The rest of the proof follows by combining Proposition 4.1 with Proposition 5.1, Proposition 5.2 and Proposition 5.3. ■

Corollary 5.1. *The results of Theorem 5.1 for the convergence rate of quasi-Monte Carlo hold when the support of the measure* μ, $\mathrm{supp}(\mu)$, *is a semi-bounded interval. Moreover, when* $\mathrm{supp}(\mu)$ *is bounded, we have*

$$e(I_{d,g,n}, F_i) = O(n^{-1}),\ i = 1, \ldots, 4.$$

Proof: The proof follows directly from Corollary 4.1. ■

6 Applications

In this section we show how our results apply to some of the problems in computational finance. Assuming a standard lognormal model for asset prices [6] the value of a financial derivative is the expected value of its payoff function with respect to the normal distribution. Often this integral can be written in the form of equation (1), i.e., $I_{d,g}(f)$, for appropriately chosen functions $g\colon \mathbb{R}^d \to \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$.

Excluding constant factors, the price of a number of call options is given by the integral of the function

$$\left(e^{\sum_{j=1}^d a_j x_j} - K\right)_+,$$

$a_j \in \mathbb{R}$, $K > 0$, with respect to the normal distribution, where $x_+ = x$ if $x \geq 0$, and is zero otherwise. Roughly speaking, the quantity $e^{\sum_{j=1}^d a_j x_j}$ corresponds to the price of an asset at some future time, and depends on d random market factors. The option payoff is the amount, if any, by which the asset price exceeds a given amount K. The expected payoff, i.e., the integral of the function above, is the value of the option. In particular, the payoff functions of vanilla European options with the stock price monitored over d intervals, forward options, compound options, and geometric mean options, and geometric mean portfolio options, to name just a few, have this form. Boyle et al. [2] considered some of these options in tests comparing Monte Carlo and quasi-Monte Carlo. Interest rate caplets [6] under different interest rate models, e.g., Vasicek, Heath Jarrow Morton and others, also have payoff functions of the above form.

We can set $g(x_1,\dots,x_d)=\sum_{j=1}^d a_jx_j$. Since g is linear its distribution is normal with mean zero and variance $\sigma^2=\sum_{j=1}^d a_j^2$. Therefore, it is equivalent to the Lebesgue measure on $\mathbb{R}$, and the decay of its tails is consistent with the conditions (5), (6) of Definition 2.1 with $x_0=1$, $\gamma=(2\pi\sigma^2)^{-1/2}$ and $\beta=(2\sigma^2)^{-1}$. We set $f(x)=(e^x-K)_+$. (We can deal with put options by considering $f(x)=(K-e^x)_+$, $K>0$.) In either case, the function f is absolutely continuous and $f\in F_1$ (or, $f\in F_2$) with appropriately chosen $\alpha>2$ and M, for example, with $\alpha=3$ and $M=\sqrt{2}$. Therefore, by Theorem 5.1 there exists a quasi-Monte Carlo method with error

$$O(n^{-1+p\{\beta\log(4\gamma n)\}^{-1/2}}),$$

with $p=(3\alpha+1)/4$ for F_1 and $p=(3\alpha+1)/2$ for F_2.

Moreover, since the distribution of g is known we can numerically calculate error bounds for any quasi-Monte Carlo method (4), i.e., for any choice of the sample points. This is accomplished by first computing how well the empirical distribution of the sample points approximates the distribution of g, i.e., by computing the discrepancy $D_{\mu,n}$, and then using the result of this computation in combination with Proposition 4.1 to obtain an upper bound for the error.

This procedure can provide insight and explain some of the differences in performance resulting from various choices of low discrepancy sequences for this type of integration problems.

It is interesting to observe that a slight modification of the payoff function yields $f\in F_3$. Indeed,

$$\begin{aligned}
&\int_{\mathbb{R}^d}\left(e^{\sum_{j=1}^d a_jx_j}-K\right)_+\phi_d(x)\,dx=\int_{\mathbb{R}^d}\left(1-Ke^{-\sum_{j=1}^d a_jx_j}\right)_+e^{\sum_{j=1}^d a_jx_j}\phi_d(x)\,dx\\
&=e^{\sum_{j=1}^d a_j^2/2}\cdot(2\pi)^{-d/2}\int_{\mathbb{R}^d}\left(1-Ke^{-\sum_{j=1}^d a_jx_j}\right)_+e^{-\sum_{j=1}^d(x_j-a_j)^2/2}\,dx_1\dots dx_d\\
&=e^{\sigma^2/2}\cdot(2\pi)^{-d/2}\int_{\mathbb{R}^d}\left(1-Ke^{-\sum_{j=1}^d a_j(x_j+a_j)}\right)_+e^{-\sum_{j=1}^d x_j^2/2}\,dx_1\dots dx_d\\
&=e^{\sigma^2/2}\int_{\mathbb{R}^d}\left(1-Ke^{-g(x)-\sigma^2}\right)_+\phi_d(x)\,dx.
\end{aligned}$$

Then the function $f(x)=(1-Ke^{-x-\sigma^2})_+$ belongs to F_3 with $M=1$.

By Theorem 5.1 there exists a quasi-Monte Carlo method with error

$$O(n^{-1}),$$

and the discrepancy of the sample points used by this quasi-Monte Carlo method also satisfies

$$D_{\mu,n}=O(n^{-1}).$$

Recall that for the class F_3, Proposition 4.1 shows that the quantity $D_{\mu,n}$ completely characterizes the integration error because $e(I_{d,g,n},F_3)=O(D_{\mu,n})$. Therefore, it is even easier than before to assess the quality of any given sample.

Observe that for our problem we considered $g(x_1,\dots,x_d)=\sum_{j=1}^d a_jx_j$ and $f(x)=(e^x-K)_+$. We could have used instead $g(x_1,\dots,x_d)=e^{\sum_{j=1}^d a_jx_j}$, and $f(x)=(x-K)_+$. The difference is that the distribution of g is now lognormal, it is supported on $(0,\infty)$, and $f \in F_4$ with $M = 1$. It is possible to show that conditions (7) and (8) are satisfied for some β and γ and then use Corollary 4.1 and Corollary 5.1 to obtain similar results.

We carry out such an analysis for a more general g by considering an arithmetic mean option with a more complicated payoff function.

Arithmetic mean options are financial instruments whose payoff is based on the arithmetic average of the stock price over a period within the option's lifetime. The average stock price, monitored over d equally spaced time intervals, is given by a function that has the form

$$g(x) = \sum_{j=1}^{d} a_j e^{q\sum_{k=1}^{j} x_k},\ x \in \mathbb{R}^d,$$

where $q > 0$ and $a_j > 0$, $j = 1, \ldots, d$. Financial instruments based on the arithmetic average of a stock price provide a payoff $f(g(x))$. We assume that f is absolutely continuous. Examples are the arithmetic mean call option, where $f(t) = (t - K)_+$, and the arithmetic mean put, where $f(t) = (K - t)_+$. In either case, $f \in F_4$ with $M = 1$. Pricing such instruments requires the computation of

$$I_{d,g}(f) = \int_{\mathbb{R}^d} f(g(x))\phi_d(x)\,dx.$$

Lemma 6.1. *Let* $S = \sum_{j=1}^{d} a_j e^{q\sum_{k=1}^{j} x_k}$, $q > 0$, $a_j > 0$, *where the* x_j *are independent random variables and follow the normal distribution with mean zero and variance one,* $N(0,1)$, $j = 1, \ldots, d$. *Consider the probability* $P(z) = \mathrm{Prob}\{S \le z\}$, $z > 0$. *Then*

$$1 - P(z) \le d(2\pi)^{-1/2} e^{-[q(d+1)]^{-2}[\log(z/a)]^2/2},$$

for $z \ge a2^d e^{d\max\{q,1\}}$, $a = \max_{1\le j\le d}\{a_j\}$.

Proof:

$$\begin{aligned}\mathrm{Prob}\{S \le z\} &\ge \mathrm{Prob}\left\{\sum_{j=1}^{d} e^{q\sum_{k=1}^{j} x_k} \le za^{-1})\right\} \\ &\ge \mathrm{Prob}\left\{x_j \le q^{-1}\log(\rho),\ \forall\, j = 1, \ldots, d\right\},\end{aligned}$$

where ρ is the solution of the equation $(\rho^{d+1} - \rho)/(\rho - 1) = za^{-1}$.

Thus,

$$1 - \mathrm{Prob}\{S \le z\} \le 1 - \Phi^d(q^{-1}\log(\rho)),$$

where Φ denotes the cumulative distribution function of $N(0,1)$. Let $\eta = q^{-1}\log(\rho)$, and assume that z is sufficiently large so that $\eta > 0$ or, equivalently, $\rho > 1$. Let $z/a \ge 2^d e^{d\max\{q,1\}} > 1$ then $\rho > 1$. Indeed, if we assume that $\rho \le 1$ then $\rho^d \le \rho$ and $d\rho \ge z/a \ge 2^d e^{d\max\{q,1\}}$. This implies that $\rho \ge 2^d/d > 1$ which is a contradiction. Since $\rho > 1$ we have $2\rho \ge d^{1/d}\rho \ge (z/a)^{1/d}$ which implies $\rho \ge e^{\max\{q,1\}} > 1$.

From [16, p. 175] we know that $1 - \Phi(y) \le (2\pi)^{-1/2} e^{-y^2/2} y^{-1}$, $y > 0$, and, therefore, we have

$$\left[1 - (2\pi)^{-1/2} e^{-y^2/2} y^{-1}\right]^d \le \Phi^d(y).$$

We can also show, by induction on d, that

$$1 - [1 - w]^d \le dw, \quad 0 < w < 1.$$

Since $\rho \ge e^{\max\{q,1\}}$ we have $\eta \ge 1$ and $(2\pi)^{-1/2} e^{-\eta^2/2} \eta^{-1} < 1$.

Combining the above we obtain

$$1 - P(z) \le d(2\pi)^{-1/2} e^{-\eta^2/2} \eta^{-1} \le d(2\pi)^{-1/2} e^{-\eta^2/2}.$$

Also $\log(\rho - 1) \ge 0$ because $\rho \ge e^{\max\{q,1\}}$. This implies

$$\begin{aligned} \log(\rho) &= (d+1)^{-1} \log\left(\frac{\rho^{d+1}}{\rho - 1}\right) + (d+1)^{-1} \log(\rho - 1) \\ &\ge (d+1)^{-1} \log\left(\frac{\rho^{d+1} - \rho}{\rho - 1}\right) \\ &= (d+1)^{-1} \log(z/a). \end{aligned}$$

Thus

$$1 - P(z) \le d(2\pi)^{-1/2} e^{-[q(d+1)]^{-2} [\log(z/a)]^2/2},$$

for $z \ge a 2^d e^{d \max\{q,1\}}$. ■

The distribution of g defines the measure μ, see equation (2). Then $\mu \ll \lambda$ and it is supported on $(0, \infty)$. By Lemma 6.1, it satisfies the conditions (7) and (8). The first part of Corollary 5.1 implies the existence of a quasi-Monte Carlo method with error

$$O(n^{-1+2\{\beta \log(4\gamma n)\}^{-1/2}}),$$

where $\beta = [q(d+1)]^{-2}/2$ and $\gamma = d(2\pi)^{-1/2}$.

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