1 Introduction

Today’s lecture is about introduction to Linear Programming (Optimization). In general, optimization problem is considered as the following:

\[
\begin{align*}
\text{Obj} : & \quad \min \ f(x) \\
\text{s.t.} : & \quad x \in \mathbb{R}^n, \ \text{some constraints on} \ x \ (e.g. \ x \in \{0, 1\}^n)
\end{align*}
\]

Here is an example of the optimization problem:

**Example 1.** The min conductance problem in the graph \( G = (V, E) \) we discussed before is the following:

\[
\begin{align*}
& \min \ \frac{\left| \partial S \right|}{\sum_{i \in S} d_i} \\
& \text{s.t.} \quad S \neq \emptyset, \ \sum_{i \in S} d_i \leq \frac{1}{2} \sum_{i \in V} d_i,
\end{align*}
\]

where \( d_i \) is the degree of node \( i \). We can regard this problem as:

**unknown variables:** \( x_i, \ i = 1, 2, \ldots, n \)

\[
\begin{align*}
x_i & \in \{0, 1\} \quad \iff \quad \begin{cases} 
\quad x_i \in \mathbb{R} \\
\quad x_i(1 - x_i) = 0
\end{cases} \\
\text{min} \ & \ f(x) = \frac{x^T L x}{\sum_{i \in V} d_i x_i} \\
\text{s.t.} \quad \sum_{i \in V} x_i > 0, \ \sum_{i \in V} d_i x_i \leq \frac{1}{2} \sum_{i \in V} d_i,
\end{align*}
\]

In general, optimization problem is possible to formulate. But solving a problem with \( f(x) \) and all constraints = degree-2 polynomials is NP-hard.

2 Linear Programming:

**Definition 2.** LP: \( f(x) \) is linear in \( x \) and all constraints are also linear (i.e, \( ax \geq b \)):

\[
\begin{align*}
\text{Obj} : & \quad \min f(x) = c \cdot x \\
\text{s.t.} : & \quad Ax \geq b
\end{align*}
\]
Note that for maximization problems, we can convert the objective $\max f(x)$ into $\min f(x) = -c \cdot x$. For equality constraints $Ax = b$, we can convert it into $Ax \geq b, -Ax \geq -b$. For constraints $Ax \leq b$, we can convert it into $-Ax \geq -b$.

**Example 3.** Convert max-flows into a Linear Programming problem: Given $G = (V, E), (i, j) \in E, c_{ij} > 0$, we solve the following LP problem:

unknown variables: $f_{i,j}, \forall (i,j) \in E$

max $\sum_{(s,j) \in E} f_{s,j} - \sum_{(j,s) \in E} f_{j,s}$

s.t. $\forall (i,j) \in E, 0 \leq f_{i,j} \leq c_{ij}$

$\forall i \in V \setminus \{s,t\}$ $\sum_{j : (j,i) \in E} f_{j,i} = \sum_{j : (i,j) \in E} f_{i,j}$

The main goal of this module will be: How to solve a general LP?

### 2.1 General form to Standard form:

**Definition 4.** Any LP can be equivalently written in the following “standard form”:

\[
\begin{align*}
\text{min} & \quad c \cdot x \\
\text{s.t.} & \quad Ax = b \\
& \quad x_i \geq 0 \forall i.
\end{align*}
\]

For any LP problem, we can convert it into the “standard form” by doing the following two steps:

- For $\forall x_i \in \mathbb{R}$, we replace $x_i$ with $x_i^+ - x_i^-$, where $x_i^+ \geq 0, x_i^- \geq 0$ are the new unknown variables.
- Any constraint $A_i x \geq b_i$ is replaced with the constraint $\xi_i = A_i x - b_i$, where $\xi_i \geq 0$ is a new unknown. We call $\xi_i$ as slack variables.

### 2.2 Structure of Solutions to Linear Programming:

**Definition 5.** Define $x$ is a feasible solution if it satisfies all constraints. Define $x$ is optimal if it satisfies all constraints and there is no better solution for the objective.

Note that each constraint can be considered as separating the space by a hyperplane. In other words,

\[
P = \text{set of feasible solutions} = \text{intersection of half-spaces (space on a side of a half-space)} = \text{polytope/ polyhedron}
\]

We call $P$ is bounded if it is inside a box and $P$ is unbounded if otherwise. See Figure 1 for an illustration of $P$. 

2
2.3 Finding the solution for LP:

Let the optimal solution be \( x^* \), then we know the optimal value of the objective will be on the line \( c \cdot x = c x^* \) which represents a hyperplane as well. Therefore one strategy of finding the solution for LP is the following: Assume we are finding minimum of \( x_1 + 2x_2 \) over \( P \) represented in Figure 1. We do the following:

- test if the optimal value of objective can be -1000 ⇒ no feasible solution s.t. \( c \cdot x = -1000 \).
- test if the optimal value of objective can be -1000 + \( \epsilon \) · · ·
  
[Diagram of constraints]

See Figure 2 for illustration.

2.4 cases for solutions:

In general, the solution of LP falls into one of the following three options:

- There is a solution
- No solution \( P = \emptyset \) (e.g. Having constraints \( x_1 \geq 2 \) and \( x_1 \leq 1 \))
- Unbounded (e.g. \( \min x_1, x_1 \leq 1 \))

3 Simpler case: solving system of linear equations

For simple case that there is no inequalities i.e, \( Ax = b \) and \( A \) is a square matrix, we can use Gaussian Elimination process to solve the solution for \( Ax = b \). The Gaussian Elimination eliminates one variable
Figure 2: There is no feasible solution for $c \cdot x = x_1 + 2x_2 = 1 - \epsilon$. For $c \cdot x = x_1 + 2x_2 = 1$, we can find one at a time like the following example.

\[
\begin{align*}
2x_1 + x_3 &= 6 \\
-x_2 + x_3 &= 2 \\
2x_1 - x_4 &= 0 \\
&\vdots
\end{align*}
\]

Eliminate $x_1$ using $x_1 = 3 - x_3/2$, we have previous constraints become

\[
\begin{align*}
3 - x_3/2 - x_2 + x_3 &= 2 \\
6 - x_3 - x_4 &= 0 \\
&\vdots
\end{align*}
\]

Here, we review some facts about linear algebra.

**Fact 6.** The following statements are equivalent:

- $A$ is invertible
- $\det(A) \neq 0$
- $A$ has linearly independent columns
- $A$ has linearly independent rows
- $Ax = b$ has a unique solution for $\forall b$.

Now we wonder what’s the size of the solution for $Ax = b$ if there is a solution.

**Fact 7.** The solution for $Ax = b$ has polynomial description.
We’ll starting proving this now (and finish in the next lecture). First assume that $A$ is a square matrix.

- If all entries of $A$ are integers, then $x_i = \text{multiple of } \frac{1}{\det(A)}$, furthermore these multiples are determinates of minors of $A$.

- If an entry $A_{ij}$ requires at most $b$ bits to represent, then $\det(A)$ can be represented with $O(n \log n + bn)$ bits. (since $\det(A) \leq n! \cdot 2^{bn}$)

If $A$ is not square, then with some changes, we can turn it into a square matrix.

In the next lecture, we will consider the cases when matrix is non-square, $\det(A) = 0$, and when there is no solution.