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Lecture 14 – Spectral Graph Theory

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1 Introduction

In the last lecture, we introduced Spectral Graph Theory and the idea to examine the eigenvalues and their corresponding eigenvectors to deduce combinatorial properties of a graph. We looked at the diffusion operation and we defined the Rayleigh Quotient. In this lecture, we show that properties of the first and second eigenvalues of an adjacency matrix tell us about the connectivity properties of the corresponding undirected graph.

2 Symmetric Matrix Transformation

Let X_0 be the initial distribution of weights in the graph. We saw last lecture that

$$X_1 = AD^{-1}X_0$$

and

$$X_t = (AD^{-1})^t X_0$$

Note that (AD^{-1}) here is no longer a symmetric matrix. We would like to make this expression cleaner by writing it as a power of a symmetric matrix.

Definition 1. $\hat{A} := D^{-1/2}AD^{1/2}$
We have that $X_{t+1} = AD^{-1}X_t$, *so* $D^{-1/2}X_{t+1} = D^{-1/2}AD^{-1}X_t = D^{-1/2}X_{t+1} = (D^{-1/2}AD^{-1/2})(D^{-1/2}X_t)$.

Definition 2. $Y_t = D^{-1/2}X_t$

Definition 3. $\hat{A}_{i,j} = \frac{A_{i,j}}{\sqrt{d_i}\sqrt{d_j}}$

From the definitions above, we have:

$$Y_{t+1} = \hat{A}Y_t = \hat{A}^t Y_0$$

Claim 4. *If* $\lambda_1 \geq \lambda_2 \geq \dots \lambda_n$ *are the eigenvalues of* $\hat{A}$, *then* $\lambda_1^t, \lambda_2^t, \dots, \lambda_n^t$ *are the eigenvalues of* $\hat{A}^t$

Proof. Suppose the claim is true for $t = 1, ..., k-1$ (inductive hypothesis). The base cases are easy to show. Then,

$$\hat{A}^k v_i = \hat{A}^{k-1}\hat{A}v_i = \hat{A}^{k-1}\lambda_i v_i = \lambda_i^k v_i$$

Hence, our induction is complete and the claim holds $\forall t \in \mathbb{N}$. □

3 Properties of the graph based on Eigenvalues

Suppose we fix $Y_0 = \Sigma \alpha_i v_i$, where v_i is the i'th orthonormal eigenvector of $\hat{A}$. Note that we can do this by the Spectral Decomposition Theorem.

$$Y_t = \hat{A}^t Y_0 = (\Sigma \lambda_i^t v_i v_i^T)(\Sigma \alpha_i v_i) = \Sigma_{i=1}^n \lambda_i^t \alpha_i v_i$$

As $t \to \infty$, if $\lambda_i < 0, \lambda_i^t \to 0$ and if $\lambda_i > 0, \lambda_i^t \to \infty$.
$|Y_t|^2 = \Sigma \alpha_i^2 \lambda_i^{2t}$, so we intuitively expect that $|\lambda_i| \leq 1$.

Lemma 5. $\lambda_1 = 1$

Proof. First let's prove that $\lambda_1 \geq 1$. We assume the ordering $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. We know that

$$\lambda_1 = max_{x \neq 0} R(x)$$

from the previous lecture. If we can show that there exists an x such that $R(x) = 1$, then we know that λ_1 has to be at least 1. Indeed, this is true for $x = v = (\sqrt{d_1}, \sqrt{d_2}, \dots \sqrt{d_n})$ where d_i is the degree of the ith node.

$$R(v) = \frac{v^T \hat{A} v}{v^T v} = \frac{v^T D^{-1/2} A D^{-1/2} v}{d_1 + d_2 + \ldots + d_n} = \frac{\mathbb{1}_n^T A \mathbb{1}_n}{d_1 + \ldots + d_n} = \frac{\mathbb{1}_n^T [d_1, \ldots, d_n]^T}{d_1 + d_2 + \ldots + d_n} = 1$$

And so $\lambda_1 = max_{v \neq 0} R(v) \geq 1$.
Now we show $\lambda_1 \leq 1$.

$$R(v) = \frac{v^T \hat{A} v}{v^T v} = \frac{\sum_{(i,j) \in E} v_i \hat{A}_{ij} v_j}{||v||_2^2} = \frac{\sum_{(i,j) \in E} \frac{1}{\sqrt{d_i d_j}} v_i v_j}{||v||_2^2}$$

Using Cauchy-Schwartz, we get:

$$\frac{\sum_{(i,j) \in E} \frac{1}{\sqrt{d_i d_j}} v_i v_j}{||v||_2^2} \leq \frac{(\sum_{(i,j) \in E} (\frac{v_i}{\sqrt{d_i}})^2)(\sum_{(i,j) \in E} (\frac{v_j}{\sqrt{d_j}})^2)}{||v||_2^2} = \frac{(\sum_i \frac{v_i^2 d_i}{d_i})(\sum_j \frac{v_j^2 d_j}{d_j})}{||v||_2^2} = \frac{(\sum_i v_i^2)(\sum_j v_j^2)}{||v||_2^2} = 1$$

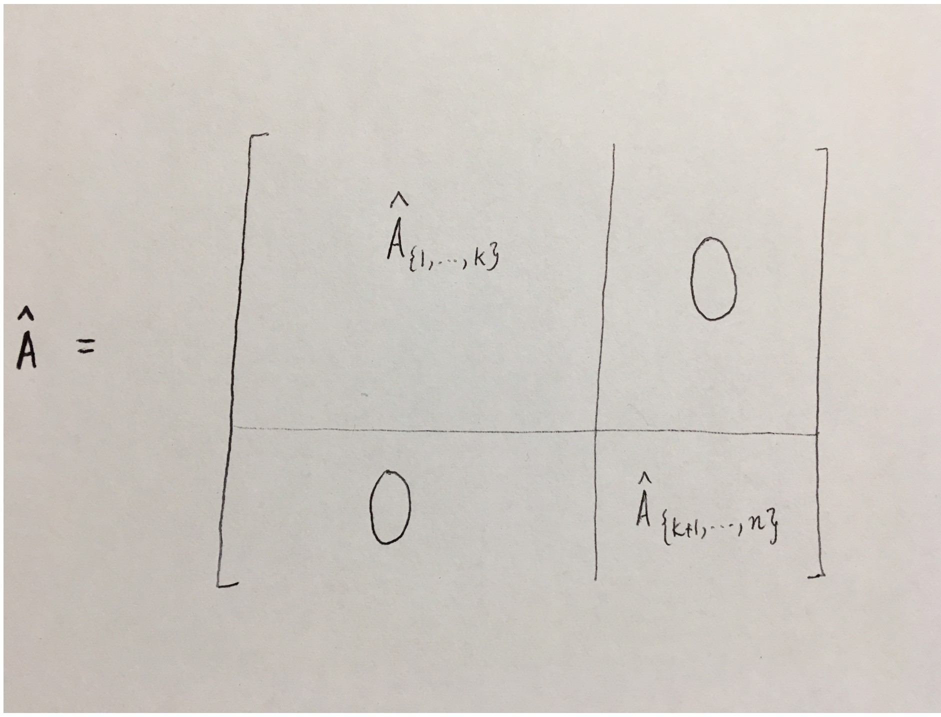
as required. Hence, $R(v) \leq 1 \forall v$. So, $\lambda_1 = 1$ since we have shown $\lambda_1 \geq 1$ before.
Note that if $R(v) = 1$, then by the condition for equality in the inequality, we must have $\frac{v_i}{\sqrt{d_i}} = \alpha \frac{v_j}{\sqrt{d_j}}$ for some constant α.
Note that we can rearrange the sums in the product of the inequality such that the v_i term is matched with the v_i term instead of an arbitrary v_j. This forces $\frac{v_i}{\sqrt{d_i}} = \alpha \frac{v_i}{\sqrt{d_i}} \implies \alpha = 1$. Hence, if $R(v) = 1$, then $\frac{v_i}{\sqrt{d_i}} = \frac{v_j}{\sqrt{d_j}} \forall i, j$ such that i, j are vertices of some edge. □

Lemma 6. $\lambda_2 < 1 \iff$ *G is connected.*

Proof. Suppose G is disconnected and has two components. Then, its vertex set can be separated into parts $\{1, 2, ..., k\}$ and $\{k+1, ..., n\}$ such that vertices with the respective indices in these two sets are disjoint and not connected by an edge. Now, consider the vectors:

$$v_1 = (\sqrt{d_1}, ..., \sqrt{d_k}, 0, ..., 0), v_2 = (0, ..., 0, \sqrt{d_{k+1}}, ..., \sqrt{d_n})$$

The corresponding matrix $\hat{A}$ for the disconnected graph looks like:



Note that:

$$R(v_1) = R(v_2) = 1 \implies \lambda_1 = \lambda_2 = 1$$

Note that v_1 and v_2 are orthogonal to each other, which implies that $\lambda_2 = 1$, by the proof of Lemma 1. Now, we show that G is connected $\implies \lambda_2 < 1$.

- Let $v_2 = argmax\ R(x)$, where $x \neq 0$, $x \perp v_1$, where v_1 is the vector associated with λ_1.
- Let v be such that $R(v) = 1$.
- From the proof of Lemma 1, we know that for vector v, all its elements satisfy:

$$\frac{v_i}{\sqrt{d_i}} = \frac{v_j}{\sqrt{d_j}}$$

.

- Hence, all vectors satisfying $R(v) = 1$ satisfy the property $\frac{v_i}{\sqrt{d_i}} = \beta \forall i \in [n]$, where β is some constant.
- All such vectors belong to the same one-dimensional space and cannot be orthogonal to each other (unless they are all the zero vector).
- So, there do not exist two distinct orthogonal vectors such that $R(v) = 1 \implies R(v_2) \neq 1$.
- We have already shown that $\lambda_1 = 1$, so $\lambda_2 \leq 1 \implies \lambda_2 < 1$, as required.

- Hence, G is connected $\iff$ $\lambda_2 < 1$! Note that the proof generalizes in an obvious way for the case when G has an arbitrary number of components (instead of just two).

$\square$

4 Next Time

- So far: $|\lambda_i| \leq 1$.
- $\lambda_n = -1 \iff G$ is bipartite. Proof sketch: Consider $v = (\sqrt{d_1}, ..., \sqrt{d_k}, -\sqrt{d_{k+1}}, ..., -\sqrt{d_n})$.
- We define the Laplacian of a Graph $L_G = D_G - A_G$.
- Taking $\hat{L}_G = D^{-1/2} L_G D^{-1/2} = I - \hat{A}$, with eigenvalues $\lambda_1 \leq ... \leq \lambda_n \rightarrow \hat{A}$, we will identify properties of the eigenvalues of $\hat{L}$ and identify their connections to the original graph G.