| COMS E6998: Algorithms for Massive Data | Sep 17, 2025 |
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| Lecture 5: Overview of Compressed Sensing | |
| Instructor: *Alex Andoni* | Scribes: *Chandhru Karthick* |

# 1 Problem Setup

Suppose we have an unknown signal expressed as $X \in \mathbb{R}^n$, that we have to deduce by making some minimal "measurements". Here, we define measurements as a linear projection on a m-dimensional space ($m << n$).

Goal: To recover x from Ax assuming some structure on x. (Without assuming a structure, this recovery may not be possible for a general signal)

Assumption: Specifically, we assume x is k-sparse.

> Note: While this is not always the case in standard basis, we can argue ∃basis B, where the signal is k-sparse. i.e, $x = B.z$ and $z$ is k-sparse.
> $Ax = A.B.z$ →This is where we ask the compressed sensing question for z ("k-sparse" view of the signal), where our linear measurement is $(A.B)$

## 1.1 Designing measurement A

Let $y = A.x$, since x is sparse we can ask the following question, what is the space of solutions minimizing the following objective:

$$L_0(y) = \underset{x \in \mathbb{R}^n, Ax=y}{\arg\min} \|x\|_0$$

However, this has $\infty$ solutions from a $|n-m|$-dim space as A is a dimension-reducing matrix.
Moreover, this optimization problem is provably NP-hard as we can notice a simple case of L0-ball of n=2 and k=1: The search space is non-convex.

> Instead we can look at the minimal convex cover of this search space, which is the surrounding L1 ball:
>
> $$L_1(y) = \underset{x \in \mathbb{R}^n, Ax=y}{\arg\min} \|x\|_1$$

Claim: We can solve $L_1(y)$ for any A, y in poly time

*Proof.* Formulating the problem as an LP:

$$\min \sum_{Ax=y} l_i$$

Such that,

$$Ax = y$$

$$\forall i : l_i \geq x_i$$

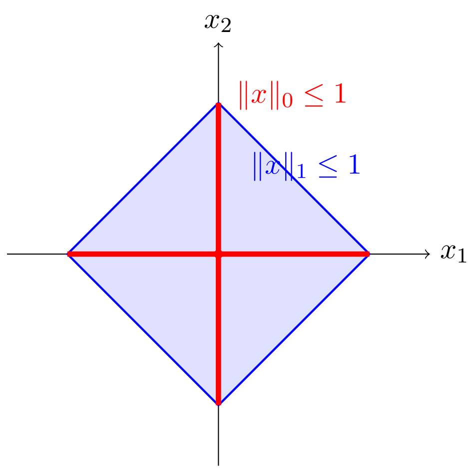
$$l_i \geq -x_i$$

The last 2 constraints are the equivalent of $l_i = |x_i|$

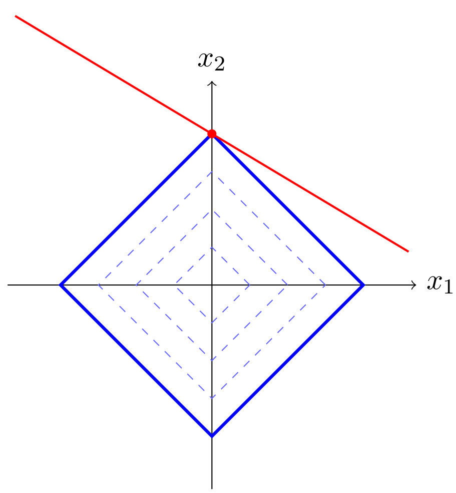
□

## 2 $L_1(y)$ as a proxy for $L_0(y)$

1) $\|x\|_0 \leq k$ $n = 2, k = 1$. Here we can do a convex relaxation of this, and the minimal convex cover is the L1 ball:




2) L1 constraints force a sparsity to solution space (stickier to corners):



Put together, fix x=original signal close to a k-sparse vector

$$x^{**} = \underset{x' \in \mathbb{R}^n, \|x'\|_0 \leq k}{\arg\min} \|x' - x\|_1$$

We hope to recover $x^{**}$ approximately close. We can define a cost or error for this sparsity as, $Err_1^k(x) = \min_{x' \in \mathbb{R}^n} \|x' - x\|_1$, This encodes distance x is away from k-sparsity
In other words, we hope to recover x' such that,

$$\|x' - x\|_1 \leq c \, \mathrm{Err}_1^k(x). \tag{1}$$

for some constant $c > 1$ perhaps as small as $c = 1 \pm \epsilon$

**Theorem 1.** *Let $x^* = L_1(y)$, $x^*$ satisfies* (1) *if $A$ is $(4k, \epsilon)$-Restricted Isometry Property(RIP), for $\epsilon < \frac{1}{2}$.*

Remarks:
1) $x^*$ is not necessarily $k$-sparse
2) We can only get $k$-sparse approximately by retaining only $k$ largest entries of x*

Definition: $A$ is $(k, \epsilon)$-Restricted Isometry Property iff $\forall x$ $k$-sparse

$$\|Ax\|_2 \in (1 \pm \epsilon)\|x\|_2$$

**Theorem 2.** *If A satisfies Oblivious Space Embedding property for a dimension k, with Pr$\geq 1 - \delta$, for $\delta = \frac{0.1}{\binom{n}{k}}$, then A is $(k, \epsilon)$-RIP with Pr$\geq$0.9:*
*Reminder of OSE property $\forall$U : $k$-dim subspace:*

$$\Pr_A \left[ \forall x \in \mathrm{U} : \frac{\|Ax\|_2}{\|x\|_2} \in 1 \pm \epsilon \right] \geq 1 - \delta$$

## 3 Approach to choosing A:

1. A=random matrix (say, gaussian) satisfying OSE. But works with 90% only.

   1.5) To get a smaller fail probability of $\delta < 0.1$, we can increase $m$ by a factor of $\log \frac{1}{\delta}$.

2. If we want no probability of failure, we simply resample A if A is not RIP (This is however NP-hard).

3. It is possible to design A as an RIP matrix in a pseudorandom fashion (though typically a bit worse bounds).

**Theorem 3.** *A:Gaussian matrix works for measurements m*

$$m = O(k \cdot \log \frac{n}{k})$$

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