AA Lecture 2. 1/20/22.

Approximate Counting: up to n.

Morris's Algo:

- init $X=0$
- @ button press: $X = \begin{cases} X+1, & w/ Pr = 2^{-X} \\ X, & oth. \end{cases}$
- estimator: $\hat{n} = 2^X - 1$.

Goal: to prove $\hat{n} \approx n$ most of time.

Claim 1: $\mathbb{E}[\hat{n}] = n$.

pf: X_n = value of X after n presses.

$\hat{n} = 2^{X_n} - 1$.

$$\mathbb{E}[\hat{n}] = \mathop{\mathbb{E}}_{X_1, X_2, \ldots X_n}[2^{X_n} - 1]$$

IH: $\mathbb{E}[2^{X_n} - 1] = n$.

Base case: $n=0$: $\mathbb{E}[2^{X_0} - 1] = 0$. ✓

Ind. step: assume that $\mathbb{E}[2^{X_{n-1}}-1]=n-1$

$$\mathop{\mathbb{E}}_{X_1 \ldots X_n}[2^{X_n}-1] = \mathop{\mathbb{E}}_{X_1 \ldots X_{n-1}}\left[\mathop{\mathbb{E}}_{X_n}[2^{X_n}-1]\right]$$

$$= \mathop{\mathbb{E}}_{X_1 \ldots X_{n-1}}\left[2^{-X_{n-1}}\cdot\left(2^{X_{n-1}+1}-1\right)+\left(1-2^{-X_{n-1}}\right)\cdot\left(2^{X_{n-1}}-1\right)\right]$$

$$= \mathbb{E}\left[\cancel{2}-\cancel{2^{-X_{n-1}}}+2^{X_{n-1}}-\cancel{1}-\cancel{1}+\cancel{2^{-X_{n-1}}}\right]$$

$$= \mathbb{E}[2^{X_{n-1}}-1]+1 = n-1+1 = n. \quad \boxtimes$$

Claim 2: space is $O(\lg\lg n)$ with prob $\geq 90\%$

pf: intuition: $\hat{n} \approx n \Rightarrow 2^{X_n} \approx n$

$$\Rightarrow X_n \approx \lg n$$
$$\Rightarrow \lg X_n \approx \lg\lg n.$$

Use Markov bound on $\hat{n}$:

$$\Pr[\hat{n} > 10\cdot n] \leq \frac{\mathbb{E}[\hat{n}]}{10n} = 1/10.$$

$\Rightarrow \hat{n} \leq 10n$ with prob $\geq 90\%$

$\Rightarrow 2^{X_n} - 1 \le 10n \Rightarrow X_n \le O(\lg n)$

$\Rightarrow \lceil \lg_2 X_n \rceil \le O(\lg \lg n)$.

\# bits takes to store X_n. $\boxtimes$

$$\boxed{2^{X_n} \le 10n+1 \Rightarrow X_n \le \lg_2(10n+1) = O(\lg n).}$$

<u>Claim 3.</u> $\mathrm{var}[\hat{n}] \le \frac{3n(n+1)}{2} + 2 = O(n^2)$.

<u>Mot:</u> can use Chebyshev bnd.

Pf: $\mathrm{var}[\hat{n}] = \mathbb{E}[(\hat{n} - \mathbb{E}[\hat{n}])^2]$

$= \mathbb{E}[\hat{n}^2] - (\mathbb{E}[\hat{n}])^2$

$= \mathbb{E}[(2^{X_n} - 1)^2] - n^2$

$= \mathbb{E}[2^{2X_n} - 2 \cdot 2^{X_n} + 1] - n^2$.

$= \mathbb{E}[2^{2X_n}] \underbrace{-2n - n^2 + 1}_{\le 0}$

$\le \mathbb{E}[2^{2X_n}] + 1$.

$\underline{\text{IH:}}$ $\mathbb{E}[2^{2X_n}] \le \frac{3n(n+1)}{2} + 1.$

$\underline{\text{Base:}}$ $n=0 \Rightarrow \mathbb{E}[2^{2\cdot 0}] = 1.$ ✓

$\underline{\text{Ind. step:}}$

$$\underbrace{\mathbb{E}_{X_1 \dots X_n}[2^{2X_n}]}_{a_n} = \mathbb{E}_{X_1 \dots X_{n-1}}\left[\mathbb{E}_{X_n}[2^{2X_n}]\right]$$

$$= \mathbb{E}_{X_1 \dots X_{n-1}}\left[2^{-X_{n-1}} \cdot 2^{2(X_{n-1}+1)} + (1-2^{-X_{n-1}}) \cdot 2^{2X_{n-1}}\right]$$

$$= \mathbb{E}\left[4 \cdot 2^{X_{n-1}} + 2^{2X_{n-1}} - 2^{X_{n-1}}\right]$$

$$= \underbrace{\mathbb{E}[2^{2X_{n-1}}]}_{a_{n-1}} + 3 \cdot \mathbb{E}[2^{X_{n-1}}]$$

$$\le \frac{3(n-1)\cdot n}{2} + 1 + 3 \cdot [n-1+1]$$

$$= \frac{3n(n+1)}{2} + 1. \quad \boxtimes$$

$$a_n = a_{n-1} + 3n = 3n + 3(n-1) + a_{n-2}$$
$$= 3 \cdot (n + n-1 + n-2 + \dots) + \underline{a_0}.$$

$\underline{\text{Est:}}$ $\hat{n}$ $\qquad E[\hat{n}] = n$

$$\mathrm{var}[\hat{n}] \le \frac{3n(n+1)}{2} + 2 \le 2n^2.$$

Use Chebyshev:

$$\Pr\left[\,|\hat{n} - n| > \lambda\,\right] \le \frac{\mathrm{var}[\hat{n}]}{\lambda^2}$$

$$\frac{\mathrm{var}[\hat{n}]}{\lambda^2} = 0.1 \Rightarrow \lambda = \sqrt{10 \cdot \mathrm{var}[\hat{n}]}$$

$$\le \sqrt{20 \cdot n^2}$$

$$\le 5 \cdot n.$$

$\Rightarrow n - 5n \le \hat{n} \le n + 5n$ with prob $\ge 90\%$.

Goal: better approx.

fix $\varepsilon > 0$. want: $(1-\varepsilon)n \le \hat{n} \le (1+\varepsilon)n$

$$\boxed{\hat{n} = (1 \pm \varepsilon)n}$$

(with Prob. $\ge 90\%$).

Morris+:

Trick: take k iid. estimators, & average them out.

keep $X^1, X^2, \dots X^k$.

est: $\hat{n}^1 \;\; \hat{n}^2 \dots \hat{n}^k$

$$\boxed{\hat{n}_k = \frac{1}{k}\sum_{i=1}^{k} \hat{n}^i.}$$ → output of Morrist.

Q: what value for k so that Goal?

Claim 1: $\mathbb{E}[\hat{n}_k] = n.$

Pf: $\mathbb{E}[\hat{n}_k] = \frac{1}{k}\sum_{i=1}^{k} \mathbb{E}[\hat{n}^i] = n.$ $\boxtimes$

Claim 2: space: $O(k \cdot \lg\lg n)$.
with prob. ≥ 90%.

Claim 3: $var[\hat{n}_k] = \frac{1}{k} \cdot var[\hat{n}^1]$

Pf: $$var[\hat{n}_k] = var\left[\frac{1}{k} \cdot \sum_{i=1}^{k} \hat{n}^i\right]$$
$$= \frac{1}{k^2} \cdot var\left[\sum_{i=1}^{k} \hat{n}^i\right]$$
$$= \frac{1}{k^2} \cdot \sum_{i=1}^{k} var[\hat{n}^i]$$
$$= \frac{1}{k^2} \cdot k \cdot var[\hat{n}^1] = \frac{var[\hat{n}^1]}{k}.$$
$\boxtimes$

Cor: $var[\hat{n}_k] \leq \frac{2n^2}{k}.$

Use Chebyshev: $\Pr[|\hat{n}^k - n| > \lambda] \le \frac{\mathrm{var}[\hat{n}^k]}{\lambda^2}$

$$\frac{\mathrm{var}[\hat{n}^k]}{\lambda^2} \le 0.1 \Rightarrow \lambda \ge \sqrt{10 \cdot \mathrm{var}[\hat{n}^k]}$$

$$\lambda \ge \frac{\sqrt{20} \cdot n}{\sqrt{k}}.$$

goal: $\lambda = \varepsilon \cdot n$

$$\Rightarrow \varepsilon \cancel{n} \ge \frac{\sqrt{20} \cdot \cancel{n}}{\sqrt{k}}$$

$$\Rightarrow \boxed{k \ge 20/\varepsilon^2}.$$

Conclusion: Morris+ algo obtains $\boxed{1 \pm \varepsilon}$

space: $\boxed{O\left(\frac{1}{\varepsilon^2} \cdot \lg \lg n\right)}$.

Chernoff bound $\rightarrow$ good for obtaining better prob. of success.

$$\Pr[\hat{n} > L \cdot n] \le 1/L$$

$\Rightarrow$ space: $\lg_2 X_n \le \lg_2 \lg L n$

$$\le \lg\lg L + \lg\lg n.$$

$$\Pr[\text{space} > \lg\lg L + \lg\lg n] \le 1/2.$$

$$\underbrace{O(1) + O(1) + \cdots}_{n} \quad O(1) \cancel{=} O(1)$$