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Lecture 13: Max Flow, Spectral Graph Theory

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1 Introduction

In the last lecture, we explored 3 algorithms that improve on the run time of the original Ford-Fulkerson algorithm. Today we will continue on Ford-Fulkerson, and also get started on the foundational concepts of Spectral Graph Theory as well.

2 Max Flow Continuation

In this section, we will continue from the last lecture and finish the last bit of the Ford-Fulkerson shortest path algorithm and obtain a few extensions from it.

Denote $d_f(s,v)$ as the distance from s to v in G_f (the residual graph).

Claim 1. *Fix f. Let P be the shortest augmenting path in G_f, and f' be the flow after augmenting P, we would have*

$$d_{f'}(s,v) \geq d_f(s,v)$$

Proof. Let $A = \{v : d_{f'}(s,v) < d_f(s,v)\}$ and $v = \min_{v \in A} d_{f'}(s,v)$. From the premise, $A \neq \varnothing$.

Let w be the preceding node of v in P, then

$$d_{f'}(s,v) = d_{f'}(s,w) + 1$$

Since $d_{f'}(s,v) < d_f(s,v)$ and $d_{f'}(s,w) \geq d_f(s,w)$, we have

$$d_f(s,v) \geq d_f(s,w) + 2$$

However, if there exists $w \to v$ in G_f, it must be

$$d_f(s,v) \leq d_f(s,w) + 1$$

This implies in G_f, path P goes from v to w, and in turn that

$$d_f(s,w) = d_f(s,v) + 1$$

which results in a contradiction with $d_f(s,v) \geq d_f(s,w) + 2$ that we obtained previously.

□

Claim 2. *For $\forall v \in V$, $d_f(s,v) \leq n$.*

Claim 3. *For every edge $v \to w$, it can be saturated at most $\frac{n}{2}$ times.*

Proof. During a saturating event, if $v \to w$ is on the (shortest) augmenting path P, then we have

$$d_f(s, w) = d_f(s, v) + 1$$

At the same time, as $v \to w$ is saturated, in the next residual graph $G_{f'}$, $v \to w$ does not exist.

Before edge $v \to w$ is saturated again, it must be created first, which means that a shortest path P' must go from $w \to v$ in some $G_{f''}$ for f''.

This implies that

$$\begin{aligned} d_{f''}(s, v) &= d_{f''}(s, w) + 1 \\ &\geq d_f(s, w) + 1 && \text{(Claim 1)} \\ &= d_f(s, v) + 1 + 1 \\ &= d_f(s, v) + 2 \end{aligned}$$

i.e. there are at most $\frac{n}{2}$ saturations of $v \to w$.

□

Claim 4. *In every step of the FF algorithm, at least* 1 *edge is saturated.*

In conclusion, for run time analysis, we have:

- Number of iterations in FF: $m \cdot \frac{n}{2}$.
- Total time: $O(\frac{mn}{2} \cdot m) = O(m^2 n)$.

There are also improved run time results from a few other references:

- $O(m^{1.5} \log n \log U)$ (up to the 90s).
- $O(m\sqrt{n} \log U)$ (Lee and Sidford [LS15]).
- $O(m^{4/3} \cdot (\log n)^{O(1)})$ for $U = O(1)$ (Kathuria, Liu and Sidford [KLS20]).

3 Spectral Graph Theory

In this section, we mainly focus on undirected graph $G = (V, E)$.

Definition 5. *Adjacency matrix A_G of graph G is an $n \times n$ matrix with elements*

$$(A_G)_{ij} = \begin{cases} 1 & \text{if}(i, j) \in E \\ 0 & \text{otherwise} \end{cases}$$

Observation 6. *For undirected graph G, A_G is symmetric.*

Definition 7. *Degree matrix D_G of graph G is an $n \times n$ diagonal matrix with elements*

$$(D_G)_{ii} = \mathit{degree}(i)$$

4 Random Walks / Diffusion

4.1 Diffusion equation

Think of a walker walking randomly in G, who at every time step, takes a step to a (uniformly) random neighbor.

Definition 8. *$x^t \in \mathbb{R}^+$ is the probability distribution of the walker's position after t steps.*

Starting from the initial position $x^0 = (1, 0, \cdots, 0)$, the evolution of x^t is dictated by diffusion operator W, in the form $x^{t+1} = Wx^t$.

From the statement of random walk process, we have

$$x_j^{t+1} = \sum_{(i,j)\in E} \frac{x_i^t}{degree(i)}$$

in which x_i^t is the probability that the walker is in position i at time step t.

Therefore, we can see W is in fact an $n \times n$ matrix with elements

$$W_{ji} = \frac{1}{degree(i)}(A_G)_{ji}$$

Further it can be written as

$$W = A_G D_G^{-1}$$

Here we have some remarks:

1) $x^t = Wx^{t-1} = \cdots = W^t x^0$

2) In the evolution process $x^{t+1} = A_G D_G^{-1} x^t$, $D_G^{-1} x^t$ can be seen as a special vector indicating how much fraction node i gives to each of its neighbor.

3) W is not necessarily symmetric even if A_G is symmetric.

4.2 Stationary distribution

Definition 9. *The stationary distribution for a random walk process is the vector $x^* \in \mathbb{R}_+^n$ s.t. $Wx^* = x^*$.*

Two questions centering on stationary distribution are:

1) What is x^*? Does it exist? Is it unique if existence is guaranteed?

2) When $t \to \infty$, does x^t converge? If so, is x^* the limit point?

5 Spectral Decomposition/Theory of symmetric matrices

5.1 Eigenvector and eigenvalue

Definition 10. *Let M be an $n \times n$ matrix. A non-zero vector $v \in \mathbb{R}^n$ is called an eigenvector of M if $Mv = \lambda v$. λ is the corresponding eigenvalue of v.*

From the definition we have the observation that for any eigenvector/eigenvalue pair (v, λ), $Mv = \lambda v \Rightarrow (M - \lambda I)v = 0$.

It further implies that the determinant $\det(M - \lambda I) = 0$. Since the determinant is a polynomial of λ of degree n (in fact a summation of products of n items from $M - \lambda I$), this equation has precisely n roots (counting multiplicities).

In addition we have the following fact:

Fact 11. *If M is a real symmetric matrix, then all n roots of $\det(M - \lambda I) = 0$.*

5.2 Spectral Theorem

Theorem 12. *For any $n \times n$ matrix M, there exist n eigenvalue/eigenvector pairs (λ_i, v_i) such that*

1) $\|v_i\| = 1$

2) $v_i \cdot v_j = 0$ *for* $i \neq j$

3) $Mv_i = \lambda_i v_i$

Here are some remarks on spectral theorem.

1) The eigenvalues $\lambda_1, \cdots, \lambda_n$ are unique. (By convention, eigenvalues are named in the order $\lambda_1 \geq \cdots \geq \lambda_n$)

2) The eigenvectors are not necessarily unique.

   If λ_i are distinct, then the corresponding eigenvectors $v_1, \cdots, v_n$ must be unique (up to negativity, i.e. v_i and $-v_i$ are the only two eigenvectors corresponding to λ_i).

   Otherwise, it is not the case. For example, given $\lambda_i = \lambda_{i+1}$, apart from v_i, v_{i+1}, $\frac{1}{\sqrt{2}}(v_i + v_{i+1})$, $\frac{1}{\sqrt{2}}(v_i - v_{i+1})$ can also be two valid eigenvectors.

3) M can be expressed as the sum of outer products of eigenvectors.

$$M = \sum_{i=1}^{n} \lambda_i v_i^T v_i$$

Some examples of spectral decomposition:

1) $M = 0$, then $\lambda_i = 0, i = 1, \cdots, n$. Any basis in $\mathbb{R}^n$ is a valid set of eigenvectors.

2) $M = I$, then $\lambda_i = 1, i = 1, \cdots, n$. Any basis in $\mathbb{R}^n$ is a valid set of eigenvectors.

Further, since we proved that n eigenvectors form a basis of $\mathbb{R}^n$, any vector $x \in \mathbb{R}^n$ can be expressed as a linear combination of them.

$$x = \sum_{i=1}^{n} \alpha_i v_i$$

Then the result of M being applied to x is just the summation of scaled eigenvectors.

$$Mx = M \sum_{i=1}^{n} \alpha_i v_i = \sum_{i=1}^{n} \alpha_i (Mv_i) = \sum_{i=1}^{n} \lambda_i \alpha_i v_i$$

5.3 Rayleigh Quotient

Definition 13. *$\forall x \neq 0$, Rayleigh Quotient $R(x)$ is a function mapping from $\mathbb{R}^n$ to $\mathbb{R}$:*

$$R(x) = \frac{x^T M x}{||x||^2} = \frac{x^T M x}{x^T x}$$

Observation 14. *For $\forall$ eigenvector v_i,*

$$R(v_i) = \frac{v_i^T M v_i}{||v_i||^2} = \frac{v_i^T \lambda_i v_i}{||v_i||^2} = \lambda_i$$

From the observation, we can easily derive the following theorems.

Theorem 15.

$$\max_{x \neq 0} R(x) = \lambda_1 \quad \textit{(maximum eigenvalue)}$$

$$\min_{x \neq 0} R(x) = \lambda_n \quad \textit{(minimum eigenvalue)}$$

Theorem 16.

$$\lambda_i = \max_{\substack{x \neq 0 \\ x \perp v_1, v_2, \ldots, v_{i-1}}} R(x)$$

e.g.

$$\lambda_2 = \max_{\substack{x \neq 0 \\ x \perp v_1}} R(x)$$

This observation in fact shows a different way to compute eigenvalue without finding the roots of determinant.

But it is worth noting that only eigenvalues are obtained. The corresponding eigenvectors still need to be computed by solving $(M - \lambda I)v = 0$.

References

[KLS20] Tarun Kathuria, Yang P. Liu, and Aaron Sidford. Unit capacity maxflow in almost \$o(mˆ{4/3})\$ time. In *61st IEEE Annual Symposium on Foundations of Computer Science, FOCS 2020, Durham, NC, USA, November 16-19, 2020*, pages 119–130. IEEE, 2020.

[LS15] Yin Tat Lee and Aaron Sidford. Path finding ii : An õ(m sqrt(n)) algorithm for the minimum cost flow problem, 2015.