AA Lecture 17 3/16/21

Linear Programming

General Form

unknowns $x_1 \dots x_n = x \in \mathbb{R}^n$

$$\boxed{\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & \underline{A x \geq b}. \end{aligned}}$$

$c \in \mathbb{R}^n$

$b \in \mathbb{R}^m$

$A \in \mathbb{R}^{m \times n}$

(given).

n = #vars

m = #constraints.

$$\begin{aligned} A_1 x &\geq b_1 \\ A_2 x &\geq b_2 \\ &\vdots \\ A_m x &\geq b_m. \end{aligned}$$

Remark:

1) $\max\limits_{x} f(x) = -\min\{-f(x)\}$

2) $A_2 x \leq b_2 \iff (-A_2) \cdot x \geq -b_2$.

3) $A_3 x = b_3 \iff \begin{aligned} A_3 x &\geq b_3 \\ A_3 x &\leq b_3. \end{aligned}$

Max-Flow via LP:

unknowns: edge flow vars $f(u,v) \in \mathbb{R}$, $(u,v) \in E$.

$$\max \sum_{u:(s,u)\in E} f(s,u) - \sum_{u:(u,s)\in E} f(u,s) \qquad c\text{'s}$$

$$\text{s.t.} \quad f(u,v) \geq 0, \quad \forall (u,v)\in E$$

$$f(u,v) \leq c_{u,v}, \quad \forall (u,v)\in E$$

$$\sum_{v:(u,v)\in E} f(u,v) - \sum_{v:(v,u)\in E} f(v,u) = 0 \quad \forall u \neq s,t.$$

b's.

\# vars: = \# edges = m.

\# constraints: $m + m + n - 2 = 2m + n - 2$.

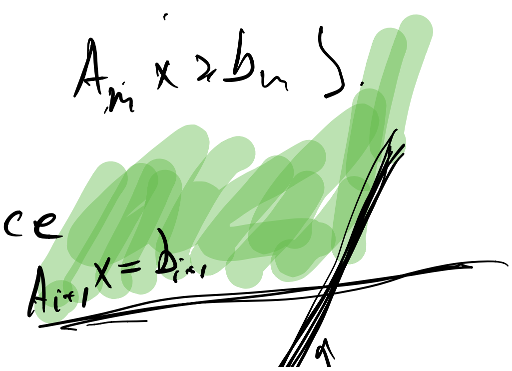
$c \in \mathbb{R}^m \qquad c_{u,v} \in \{\pm 1, 0\}$.

Structure of solutions to LP

Feasible set: $F = \{x : Ax \geq b\}$

$$= \{x : A_1 x \geq b_1,\ A_2 x \geq b_2,\ \vdots\ A_m x \geq b_m\}.$$

$\{x : A_i x \geq b_i\}$ → half-space



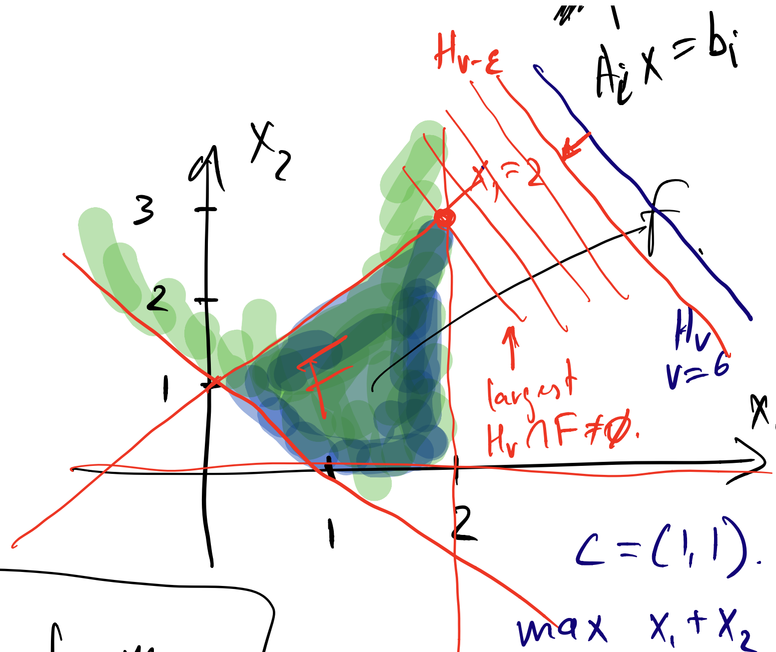
LP: x_1, x_2.

$x_1 \leq 2$

$x_1 + x_2 \geq 1$

$x_1 - x_2 \geq -1$

$x_2 \geq 0$



F = $\boxed{\text{intersection of } m \text{ half-spaces.}}$ — polytope.

3 possibilities for sol x:

1) $F = \emptyset$. (eg. adding $x_1 \leq -1$).

2) F is unbounded. (eg, remove $x_1 \leq 2$).

↳ sol is infinite: max x_1

$x_1 = +\infty$

$x_2 = 0$.

↳ sol is finite: min x_1.

$x_1 = 0$

$x_2 = 1$.

3) F is finite $\Rightarrow$ sol. is finite.

Given objective funct $f(x) = c^T x$, find opt. x?

"Algorithm $\emptyset$".

Let's guess $v = \min c^T x$ s.t. $Ax \geq b$.

$$\Rightarrow v = c^T x.$$

Consider $H_v = \{ x : c^T x = v \}$.

hyperplane.

Algo - take $v =$ very large

- decrease v continuously until $H_v \cap F \neq \emptyset$.

- x^* is opt. sol.

Conclusion: x^* is in a "corner" of polytope F.

$\hookrightarrow$ vertex : n equations of the form

$$\left.\begin{aligned} A_{i_1} x &= b_{i_1} \\ A_{i_2} x &= b_{i_2} \\ &\vdots \\ A_{i_n} x &= b_{i_n} \end{aligned}\right\}$$ linear system of n equations & n vars

<u>Def:</u> for some $x \in F$, if $A_i x = b_i$
ith constr. is <u>tight</u>.

<u>Algorithm 0':</u>
- try all possible corners/vertices
- choose best.

→ try every $\binom{m}{n}$ sets $S = \{i_1, \ldots, i_n\}$

- solve $$\boxed{\begin{aligned} A_{i_1} x &= b_{i_1} \\ A_{i_2} x &= b_{i_2} \\ &\vdots \\ A_{i_n} x &= b_{i_n} \end{aligned}} \longrightarrow x^S$$

A_S

- check that sol x^s is feasible:

$$Ax^s \ge b$$

- if feasible compute $c^T x^s$.

Runtime: $\cancel{\binom{m}{n}} \cdot \left([\text{time to solve } A_S x = b] + m \cdot n + n \right)$.

$\approx m^n$. exponential

Detour: Solving System of Lin. Equations

$$Ax = b, \quad A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m$$

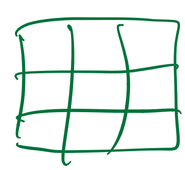
unkn. $x \in \mathbb{R}^n$.

Special Case: $m = n \Rightarrow A$ is square.

$$\boxed{A} \cdot \boxed{x} = \boxed{b}.$$

Thm: following are equiv:

1) A is invertible (A^{-1} exists).
2) $\det(A) \neq 0$



$$\det(A) = \sum_{\pi:[n]\to[n]} \operatorname{sign}(\pi) \prod_{i=1} A_{i,\pi(i)}.$$

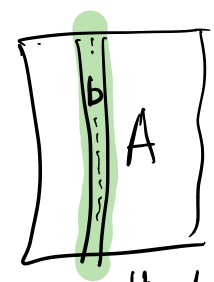
3) columns of A are linearly indep.

4) rows of A —— // ——

5) $Ax = b$ has unique sol:

$$x = A^{-1} \cdot b.$$

$$x_i = \frac{\det(A_i)}{\det(A)}$$

$A_i =$ 

$= A$ with i^{th} col. replaced by b.

Corollary: suppose A_{ij}, b_i are integer, described by B bits.

$\Rightarrow x_i$ is rational, needs only $O(n \cdot \lg n + Bn)$ bits to describe.

pf: to describe x_i, need to describe

$\lg(\text{max possible value for } \det(A_i))$

$+ \lg($ —— // —— $\det(A))$

$$|\det(A)| \le n! \cdot (2^B)^n \le n^n \cdot 2^{Bn}.$$

\# bits to describe $x_i \le$

$$O(\lg(n^n \cdot 2^{Bn})) = O(n \lg n + Bn).$$

$\boxtimes$

<u>Note</u>: - same is true for <u>LP solution</u>!

- could take $\geq n^2$ time to write down sol x^* to LP.

Assume $n \neq m$ or $\det(A) = 0$.

<u>$\mathrm{col}(A)$</u> = set of columns of $A \subset \mathbb{R}^m$.

$= \{A_1, \ldots A_n\}$.

$$\mathrm{span}(\mathrm{col}(A)) = \{y \in \mathbb{R}^m : \exists x \in \mathbb{R}^n \text{ s.t. } y = \sum_{i=1}^{n} x_i \cdot A_i\}$$

$\Leftrightarrow y = Ax$

span(col(A)) = all possible vectors

we can get on the left.

$\underline{Ax}$.

- if $b \notin span(col(A))$ $\Rightarrow$ no solution
- $b \in span(col(A))$. $\Rightarrow$ exist a solution or more.

How to find a solution then?

$S \triangleq$ set of linearly independent vectors from $col(A)$.

$= \{s_1, \dots s_k\}$.

$\Rightarrow span(S) = span(col(A)) \subseteq \mathbb{R}^m$

$\bar{S} \triangleq$ completion of S to a basis.

$= \{s_1, s_2, \dots s_k, \bar{s}_{k+1}, \dots \bar{s}_m\}$

$|\bar{S}| = m$.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

S''

$\bar{S} = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$.

$$\text{columns } \bar{S}$$

Solve
$$b = \left[s_1 \mid s_2 \mid \dots s_k \mid \bar{s}_{k+1} \dots \right] \cdot \begin{bmatrix} x' \\ y \end{bmatrix}$$

x''s corresp. to col's of A

corresp. to completion $\bar{S} \setminus S$.

by Thm from before (#3)

$\Rightarrow \exists$ unique solution $\begin{bmatrix} x' \\ y \end{bmatrix}$.

Take $x^*_i = x'_i$ if col i of A is in S.

$x^*_i = 0$ otherwise.

Proof x^* is sol to $Ax = b$:

since $b \in \text{span}(S)$

$\Rightarrow \exists x'$ s.t. $\sum x'_i \cdot s_i = b$.

$\Rightarrow y = 0$ (since $\begin{pmatrix} x' \\ y \end{pmatrix}$ is unique)

$$\Rightarrow \quad Ax = \sum x_i s_i = b. \quad \boxtimes$$