AA Lecture 16 3/11/21

Spectral partitioning

Thm: μ_2 of $\hat{L}$ is $0 \iff G$ discon.

$$\hat{L} = D^{-1/2}(D-A)D^{-1/2} = I - \hat{A}.$$

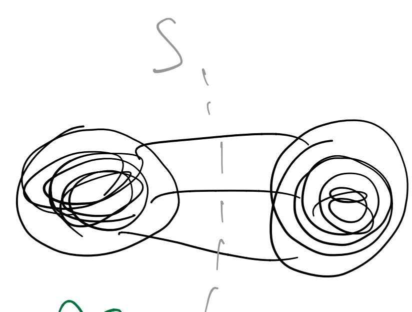
Cheeger inequality: $\mu_2 \longleftrightarrow$ combinatorial notion of conn. **conductance**

Defs for cut $S \subset V$:

∂S = # edges crossing $S \leftrightarrow \bar{S}$.

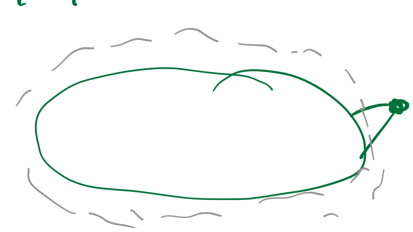
$$\mathrm{vol}(S) = \sum_{i \in S} d_i.$$

$$\phi(S) = \frac{\partial S}{\mathrm{vol}(S)}.$$



Alt: $\dfrac{\partial S}{|S|}$.

$$\phi(G) = \min_{\substack{S \neq \emptyset \\ \mathrm{vol}(S) \leq \mathrm{vol}(V)/2}} \phi(S)$$



$$= \min_{S \neq \emptyset, V} \frac{\partial S}{\min(vol(S), vol(\bar{S}))}.$$

<u>Thm [Cheeger]:</u> $\boxed{\mu_2/2 \leq \phi(G)} \leq \sqrt{2\mu_2}.$

<u>~~proof~~ #1:</u>

$$\mu_2 = \min_{x \perp v_1} R(x) = \boxed{\min_{e^T D^{1/2} x = 0} \frac{x^T \hat{L} x}{x^T x}}.$$

$$\hookrightarrow = (\sqrt{d_1}, \sqrt{d_2}, \ldots \sqrt{d_n})$$

$$= D^{1/2} \cdot \underline{e}.$$

Enough to exhibit/construct some x

s.t. 1) $e^T D^{1/2} x = 0$

2) $R(x) \leq 2\phi(G) = 2 \cdot \min_{\substack{S \neq \emptyset \\ vol(S) \leq \frac{1}{2} vol(V)}} \frac{\partial S}{vol(S)}.$

Let S = arg min of $\phi(G)$;

$$\frac{\partial S}{vol(S)} = \phi(G).$$

$$R(x) = \frac{x^T \hat{L} x}{x^T x} = \frac{x^T D^{-1/2} L D^{-1/2} x}{x^T x}$$

change of vars: $y = D^{-1/2} x \Rightarrow x = D^{1/2} y$.

$$R(x) = \frac{y^T L y}{y^T D y} = \frac{\sum_{(i,j) \in E} (y_i - y_j)^2}{y^T D y}.$$

$$L_e = \begin{array}{c} \\ i \\ j \end{array}\begin{array}{c} \begin{array}{cc} i & j \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

$$y^T L_e y = y_i^2 + y_j^2 - 2 y_i y_j = (y_i - y_j)^2.$$

$$y = \mathbb{1}_S = \begin{cases} 1, & \text{in coords } i \in S \\ 0 & \text{oth.} \end{cases}$$

$$y^T L y = \sum_{ij \in E} (y_i - y_j)^2 = \partial(S).$$

$$y' = y - \sigma \cdot e = \mathbb{1}_S - \sigma \cdot e \Rightarrow y'^T L y = \partial(S)$$

$\uparrow$ var, set it so that $e^T D^{1/2} \underbrace{D^{1/2} y'}_{x} = 0$.

$$0 = e^T D \cdot (\mathbb{1}_S - \sigma e)$$

$$= \sum_{i \in S} d_i \cdot (1 - \sigma) + \sum_{i \notin S} d_i \cdot (-\sigma)$$

$$= (1-\sigma) \cdot vol(S) - \sigma \cdot (vol(V) - vol(S))$$

$$= vol(S) - \sigma \cdot vol(V)$$

$$\Rightarrow \boxed{\sigma = \frac{vol(S)}{vol(V)}}.$$

Compute $\quad y'^T D y' = \sum_{i=1}^{n} y_i' \cdot d_i \cdot y_i'$

$$= \sum_{i \in S} d_i \cdot (1-\sigma)^2 + \sum_{i \notin S} d_i \cdot \sigma^2$$

$$= (1-\sigma)^2 \cdot \mathrm{vol}(S) + \sigma^2 \cdot (\mathrm{vol}(V) - \mathrm{vol}(S))$$

$$= \mathrm{vol}(S) - 2\sigma \cdot \mathrm{vol}(S) + \sigma^2 \cdot \mathrm{vol}(V)$$

$$= \mathrm{vol}(S) - 2 \cdot \frac{\mathrm{vol}(S)^2}{\mathrm{vol}(V)} + \frac{\mathrm{vol}(S)^2}{\mathrm{vol}(V)}$$

$$= \mathrm{vol}(S) \cdot \left(1 - \frac{\mathrm{vol}(S)}{\mathrm{vol}(V)}\right)$$

$$\mu_2 \le R(x) = \frac{y'^T L y'}{y'^T D y'} = \frac{\partial S}{\mathrm{vol}(S) \cdot \left(1 - \frac{\mathrm{vol}(S)}{\mathrm{vol}(V)}\right)} \qquad = \phi(S)$$

$$\le \frac{\partial S}{\mathrm{vol}(S) \cdot 1/2} = 2 \cdot \phi(S)$$

$$= 2 \cdot \phi(G).$$

$\boxtimes$

Intuition behind #2 $\left(\phi(G) \le \sqrt{2\mu_2}\right)$

$$\mu_2 = \min_{\substack{x \in \mathbb{R}^n \\ x \neq 0 \\ x \perp v_1}} R(x)$$

x need not look like $x = D^{1/2}(\mathbb{1}_S - \ldots)$

Algorithm (spectral partitioning):

finds a set S with $\phi(S) \leq \sqrt{2\mu_2}$.

build S from 2^{nd} eigenvector

$$V_2 = \arg\min_{\substack{x \neq 0 \\ x \perp v_1}} R(x) \in \mathbb{R}^n.$$

- sort all nodes $i \in [n]$ using $V_{2,i}$

  $\pi_1, \pi_2, \ldots \pi_n \in [n]$: $V_{2,\pi_i} \leq V_{2,\pi_{i+1}}$,

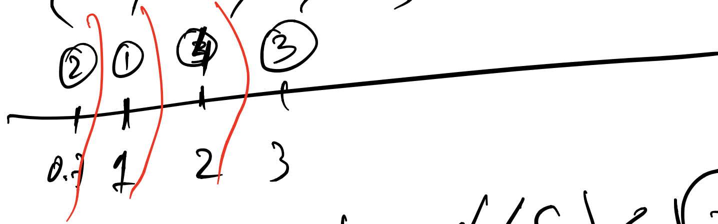
- iterate throug $i = 1 \ldots n-1$,

  consider $S_i = \{\pi_1, \ldots \pi_i\}$

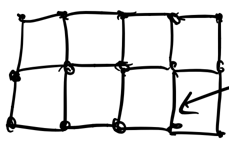
  compute $\phi(S_i) = \dfrac{\partial S}{\min\{vol(S), vol(\bar{S})\}}$,

  take the min.

$v_2 = (1, 0.7, 3, 2)$



Thm: $\exists\, i \in [n-1]$ s.t. $\phi(S_i) \leq \sqrt{2\mu_2}$.

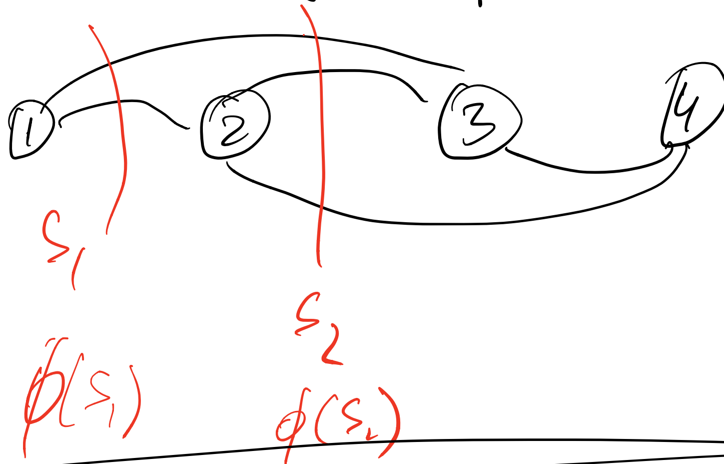


weight of edge = diff in gray scale

find a cut that minimizes weight of cut edges.

$$\phi(S) = \frac{\partial S}{\mathrm{vol}(S)}.$$

Remark: spec. part. algo. runs in $O(m + n \cdot \lg n)$ time. (after having computed v_2).



Optimization

- LP
- GD, Newton m (2^{nd}),
- IPM.

General opt problem:

$$\min/\max \quad f(x)$$

s.t. $x \in F \subset \mathbb{R}^n$ feasible set.

Example: computing $\phi(G) = \min_{S \dots} \phi(S)$

vars: $x \in \mathbb{R}^n$,

$x_i \in \{0,1\}$ = whether $i \in S$.

$$\min \quad f(x) = \min \frac{\sum_{ij \in E} (x_i - x_j)^2}{\sum_{i=1}^{n} d_i x_i}$$

s.t. $\sum_{i=1}^{n} x_i \geq 1$

$$\sum_{i=1}^{n} d_i x_i \le \frac{1}{2} \sum_{i=1}^{n} d_i$$

F.

$$x_i (x_i - 1) = 0$$

$\phi(G)$.

NP-hard.

Linear Programming: opt. problem

where f is linear: $f(x) = c^T \cdot x$

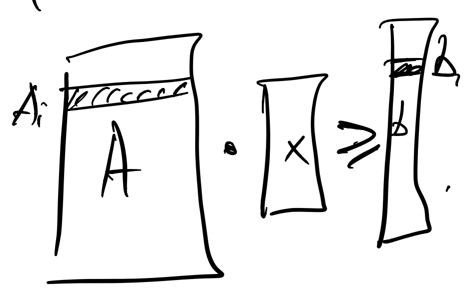
F is linear.

F is defined as follows:

$$Ax \ge b$$

$$\Updownarrow$$

$$A_i \cdot x \ge b_i$$



m = # constraints = # rows in A.

Eg: min $x_1 - 2x_2$

s.t. $x_1 \ge 3$

$x_1 + x_2 \geq 7$

$x_2 \leq 10.$

General form LP:

$$\boxed{\begin{aligned} &\min_{x \in \mathbb{R}^n} \quad c^T x \\ &\text{s.t.} \quad Ax \geq b \end{aligned}}$$