AA Lecture 14 2/25/2021

Projects: Team: 2-4.

PR1: 1 page.

1) Reading
2) Implementation
3) Research

PR1: 3/11
PR2: 4/6
Final: tbd.

scholar.google.com.

Spectral Graph Theory

Spectral Decomposition: for symmetric M,

$\exists (\lambda_i, v_i)$ eigenval/vect.

$$M = \sum \lambda_i v_i v_i^T.$$

Raleigh quotient: $R(x) = \frac{x^T M x}{\|x\|^2}$, $R(v_i) = \lambda_i$.

Random walks in Graph G.

$A = A_G$ = adj. matrix

$D = D_G$ = diag matrix of degrees $D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$

assumptions: $d_i > 0$.

x^0 = starting dist over $[n]$.

x^t = distrib @ time t (after t random steps).

$$x^t = \left(A \cdot D^{-1}\right)^t \cdot x^0.$$

$$x^t = A \cdot D^{-1} \cdot A \cdot D^{-1} \cdot A \cdot D^{-1} \cdot \ldots \cdot AD^{-1} x^0.$$

$$x^t = D^{1/2} D^{-1/2} \underbrace{A D^{-1/2}}_{w} \cdot \boxed{D^{-1/2} \cdot \underbrace{A \cdot D^{-1/2}}_{w}} \cdot \boxed{D^{-1/2} \cdot A \cdot D^{-1/2}} \cdot \ldots D^{-1/2} A D^{-1/2} \times D^{-1/2} \cdot x^0$$

$\hat{A} = D^{-1/2} \cdot A \cdot D^{-1/2}$. normalized adj. m.

$$x^t = D^{1/2} \cdot \left(\hat{A}\right)^t \cdot D^{-1/2} \cdot x^0$$

$$\Rightarrow D^{-1/2} \cdot x^t = \hat{A}^t \cdot D^{-1/2} \cdot x^0.$$

symmetric.

$y^t = D^{-1/2} x^t$.

$$\boxed{y^t = \hat{A}^t \cdot y^0.}$$

Apply spectral theorem to $M = \hat{A}$:

$$\hat{A} = \sum \lambda_i v_i v_i^T . \qquad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n .$$

<u>Fact:</u> $\hat{A}^t = \sum \lambda_i^t \cdot v_i v_i^T .$

$$y^0 = \sum_{i=1}^{n} \alpha_i v_i \quad , \quad \alpha_i \in \mathbb{R} \text{ unique.}$$

$$y^1 = \hat{A} \cdot y^0 = \sum_{i=1}^{n} \lambda_i v_i v_i^T \cdot \sum_{j=1}^{n} \alpha_j \cdot v_j$$

$$= \sum_{i,j} \lambda_i \alpha_j v_i \cdot \underbrace{v_i^T \cdot v_j}_{=1 \text{ iff } i=j.}$$

$$= \sum \lambda_i \alpha_i v_i .$$

$$\Rightarrow \quad y^t = \sum_{i=1}^{n} \lambda_i^t \cdot \alpha_i \cdot v_i .$$

<u>Obs:</u> * if $\lambda_1 > 1 \longrightarrow$ can't happen.

$y^t \xrightarrow{t \to \infty}$ diverges.

same with $\lambda_n < -1$

$\Rightarrow \lambda_1 \dots, \lambda_n \in [-1, 1].$

* $\lambda_i \in (-1, 1), \quad \lambda_i^t \alpha_i \cdot v_i \to 0.$

Thm: $\lambda_1 = 1$.

pf: 1) $\lambda_1 \geq 1$.

$$\lambda_1 = \max_{x \neq 0} R(x).$$

all 1's.

$$x = (\sqrt{d_1}, \sqrt{d_2}, \dots \sqrt{d_n})^T = D^{1/2} \cdot \mathbb{1} = \begin{bmatrix} d_1^{1/2} & & \\ & \ddots & \\ & & d_n^{1/2} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$R(x) = \frac{x^T \hat{A} x}{\|x\|^2} = \frac{\mathbb{1}^T \cdot \cancel{D^{1/2} \cdot D^{-1/2}} \cdot A \cdot \cancel{D^{-1/2} \cdot D^{1/2}} \cdot \mathbb{1}}{\sum_{i=1}^{n} x_i^2}$$

$$= \frac{\mathbb{1}^T \cdot A \cdot \mathbb{1}}{\sum d_i} = \frac{\mathbb{1}^T \cdot [d_1, d_2, \dots d_n]^T}{\sum d_i}$$

$$\begin{bmatrix} & & 1 \\ & & \\ 1 & & \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \qquad = \frac{\sum d_i}{\sum d_i} = 1.$$

2) $\lambda_1 \leq 1 \iff \forall x \neq 0 \quad R(x) \leq 1$.

$$R(x) = \frac{x^T \hat{A} x}{\|x\|^2} = \frac{x^T \cdot D^{-1/2} \cdot A \cdot D^{-1/2} \cdot x}{\|x\|^2}$$

$$= \frac{\left(\frac{x_1}{\sqrt{d_1}}, \dots \frac{x_n}{\sqrt{d_n}}\right) \cdot A \cdot \left(\frac{x_1}{\sqrt{d_1}}, \dots \frac{x_n}{\sqrt{d_n}}\right)^T}{\|x\|^2}$$

$$= \frac{\sum_{ij} A_{ij} \cdot \frac{x_i}{\sqrt{d_i}} \cdot \frac{x_j}{\sqrt{d_j}}}{\|x\|^2}$$

$$= \frac{\sum_{i,j \in E} \frac{x_i}{\sqrt{d_i}} \cdot \frac{x_j}{\sqrt{d_j}}}{\|x\|^2}.$$

Use Cauchy-Schwartz: $\forall$ vectors p, q:

$$p \cdot q \leq \|p\| \cdot \|q\|.$$

$$p \cdot q = \|p\| \cdot \|q\| \cdot \cos(p, q).$$

$p, q \in \mathbb{R}^E$

$p_{ij} = \frac{x_i}{\sqrt{d_i}}$.

$$\overset{(CS)}{\leq} \frac{\left(\sum_{i,j \in E} \left(\frac{x_i}{\sqrt{d_i}}\right)^2\right)^{1/2} \cdot \left(\sum_{i,j \in E} \left(\frac{x_j}{\sqrt{d_j}}\right)^2\right)^{1/2}}{\|x\|^2}$$

$$= \frac{\left(\sum_{ij \in E} \frac{x_i^2}{d_i}\right)}{\|x\|^2} = \frac{\sum_i x_i^2}{\sum_i x_i^2} = 1.$$

Proved: $R(x) \leq 1. \quad \forall x \neq 0.$ $\boxtimes$

Remarks: 1) if $\lambda_1 = 1$ is unique,

$\Rightarrow \quad v_1 = (\sqrt{d_1}, \sqrt{d_2}, \dots \sqrt{d_n})$.

2) consider x s.t. $R(x) = 1$

$\Rightarrow$ CS is tight

$\Rightarrow \exists \alpha > 0$ s.t. $p = \alpha q$.

$\Rightarrow \forall i, j: \quad \frac{x_i}{\sqrt{d_i}} = \alpha \cdot \frac{x_j}{\sqrt{d_j}}$.

$$\frac{x_j}{\sqrt{d_j}} = \alpha \cdot \frac{x_i}{\sqrt{d_i}}$$

$\Rightarrow \alpha = 1$.

$$\Rightarrow \boxed{\frac{x_i}{\sqrt{d_i}} = \frac{x_j}{\sqrt{d_j}} \quad \forall ij \in E.}$$

3) $R(x) \geq -1$.

$R(x) = -1$ iff $\frac{x_i}{\sqrt{d_i}} = -\frac{x_j}{\sqrt{d_j}}$

$\forall i, j \in E$.

<u>Theorem:</u> $\lambda_2 = 1 \iff G$ disconnected

Pf: 1) G disconnected $\Rightarrow \lambda_2 = 1$.

$G = G_1, G_2$ disconnected

$$\hat{A}_G = \begin{bmatrix} \hat{A}_{G_1} & O \\ O & \hat{A}_{G_2} \end{bmatrix}$$

G_1 has k vertices.

$\hat{A}_{G_1}$ has eigen v/v decomp.

(λ_i, v_i) $i=1..k$

$v_i \in \mathbb{R}^k$.

$\hat{A}_{G_2}$ has e'v/v : (λ_{i+k}, v_{i+k}), $i=1..n-k$

$v_{i+k} \in \mathbb{R}^{n-k}$

then $\hat{A}$ has e'v/v (λ_i, v_i'),

$v_i' = (v_i, 0, 0 \ldots 0)$, $i \le k$

$v_i' = (0, \ldots 0, v_i)$, $i > k$.

In particular: $\lambda_1 = 1$, $v_1' = (v_1, \ldots 0 \ldots 0)$

$\lambda_{k+1} = 1$, $v_{k+1}' = (0, 0, \ldots v_{k+1})$.

$$v_1' \cdot v_{k+1}' = 0.$$

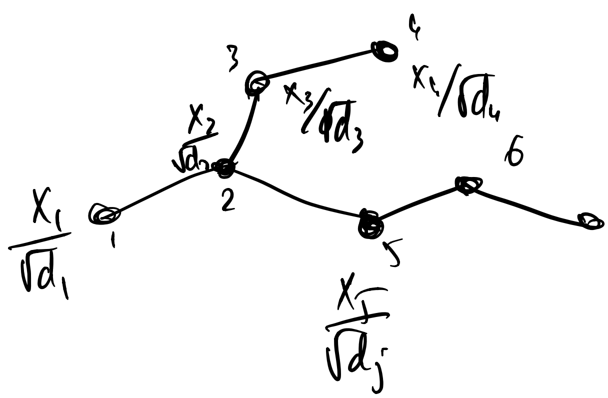
2) if G is connected $\Rightarrow \lambda_2 < 1$.

proof by contradiction: G is conn. and $\lambda_2 = 1$.

fix $v_2 \to$ e'vec corresp to λ_2.

$\Rightarrow v_2 \perp v_1$.

$$R(x = v_2) = \lambda_2 = 1 \xRightarrow{\text{Remark 2}} \boxed{\forall ij \in E \quad \frac{x_i}{\sqrt{d_i}} = \frac{x_j}{\sqrt{d_j}}.}$$



$\Downarrow$ G connected

$$\exists \beta \text{ s.t. } \forall i \in V: \frac{x_i}{\sqrt{d_i}} = \beta.$$

$$\Rightarrow x_i = \sqrt{d_i} \cdot \beta.$$

$$\Rightarrow x = v_1 \cdot \beta \qquad [v_1 = (\sqrt{d_1}, \dots \sqrt{d_n})].$$

contradiction $(v_2 \perp v_1)$. $\boxtimes$

$\lambda_1 = 1$.

$\lambda_2 = 1$ iff G disconnected.

$\lambda_n = -1$?? iff G is bipartite.

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take G conn, draw G in 2D plane

$i \mapsto ((v_2)_i, (v_3)_i)$.

Finding v_1: $\quad x^0$: $\quad x^t = A \cdot x^{t-1}$

$$x^t = \frac{x^t}{\|x^t\|}.$$

$$x^0 = \sum \alpha_i v_i \Rightarrow x^t \propto \sum \lambda_i^t \alpha_i v_i$$

Find v_2: $\quad x^0 = x^0 - \langle x^0, v_1 \rangle \cdot v_1$.

$$= \sum_{i=2}^{n} \alpha_i v_i.$$

power method

Krylov subspace method.

Runtime $\sim \dfrac{\lambda_1}{\lambda_1 - \lambda_2}$