AA Lecture 13 — 2/23/21.

Max-Flow: Ford-Fulkerson + shortest paths in G_f

$d_f(s,v)$ = distance $s \to v$ in G_f (residual graph).

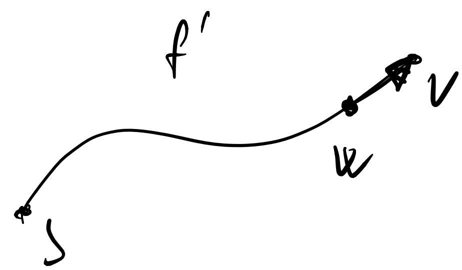
Claim: fix f. Let P = shortest augmenting path in G_f.

f' = flow after aug. P.

$$d_{f'}(s,v) \geq d_f(s,v).$$

pf: $A = \{ v : d_{f'}(s,v) < d_f(s,v) \} \neq \emptyset$

$v \in A$. min $d_{f'}(s,v)$.

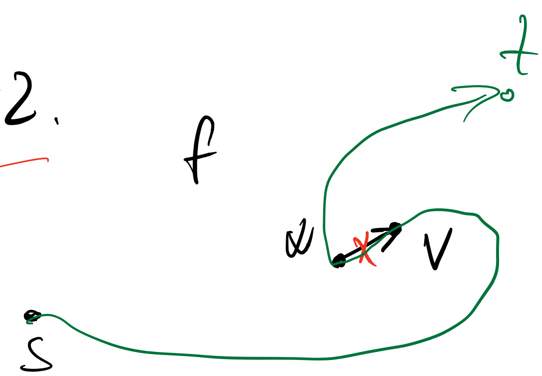


$$d_{f'}(s,v) = d_{f'}(s,u) + 1$$

$$\vee \qquad\qquad \wedge\!\!|$$

$$d_f(s,v) \qquad\qquad d_f(s,u)+1$$

$$\Rightarrow \quad d_f(s,v) \geq d_f(s,u) + 2.$$



if $\exists\ w \to v$

$\Rightarrow d_f(s,v) \le d_f(s,w)+1.$

$\Rightarrow$ in G_{f_1}, path (P) go. $v \to w$.

$\Rightarrow d_f(s,w) = d_f(s,v)+1.$

$\Rightarrow$ contradiction

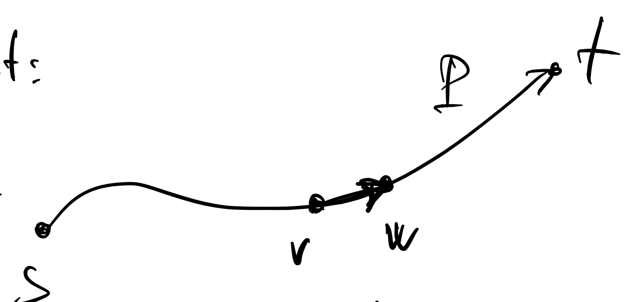
$\boxtimes$

Claim 2: $d_f(s,v) \le n. \quad \forall v.$

Claim 3: for every edge $v \to w$, can be saturated $\le n/2$ times.

Pf: saturating event:

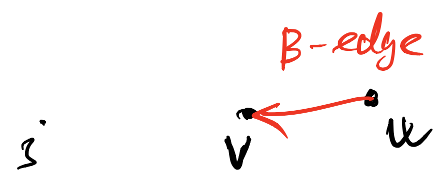
f:



$d_f(s,w) = d_f(s,v)+1.$

$\Rightarrow$ in the next $G_{f'}$, $v \to w$ does not exist.

f:



Before edge $v \to w$ is sat again,
it must be created first
$\Rightarrow$ a shortest P' must go $w \to v$
(in some $G_{f''}$, for f'').

$\Rightarrow$

$$d_{f''}(s, v) = d_{f''}(s, w) + 1$$
$$\geq d_f(s, w) + 1 \quad \text{[Claim 1]}$$
$$= d_f(s, v) + 1 + 1$$
$$= d_f(s, v) + 2.$$

$\Rightarrow$ at most $n/2$ saturations of $v \to w$.

Claim 4: in every step of FF, at least 1 edge is sat.

RT: # iterations of FF: $m \cdot \frac{n}{2}$.
total time $O\left(\frac{mn}{2} \cdot m\right) = O(m^2 n)$.

Up to '90s: $O(m^{1.5} \cdot \lg n \cdot \lg U)$.
[Lee, Sidford '12]: $O(m \cdot \sqrt{n} \cdot \lg U)$.

[Sidford '20]: $O(m^{4/3} \cdot (\lg n)^{O(1)})$,
for $U = O(1)$.

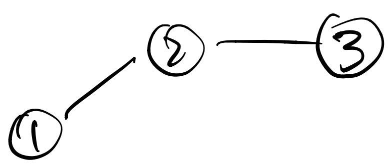
Spectral Graph Theory

Undirected graphs $G = (V, E)$.

Def: adjacency matrix A_G of graph G:

$$(A_G)_{ij} = \begin{cases} 1, & \text{if } (i,j) \in E \\ 0, & \text{oth.} \end{cases}$$

G:



$$A_G = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Obs: A_G is symmetric.

Def: degree matrix D_G:

$$(D_G)_{ii} = \text{degree } i.$$

(diagonal matrix).

$$D_G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Random walks / Diffusion in G.

Think of walker walking randomly in G:
@ every time step, take a step
to random neighbor.

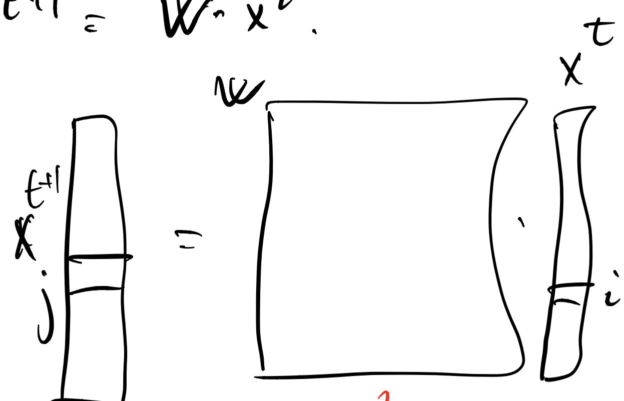
Def: $x^t \in \mathbb{R}_+^n$ probability distribution of walker after t steps

x^0 = initial posion. $x^0 = (1, 0, 0)$.

Diffusion operator: W = matrix $n \times n$.

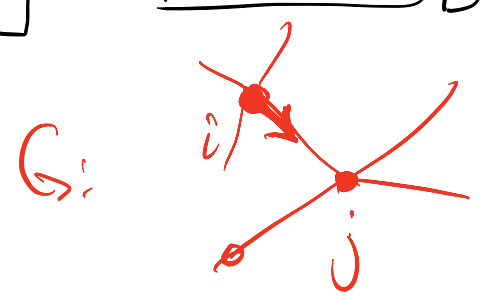
$$x^{t+1} = W \cdot x^t.$$

$$x_j^{t+1} = \sum_{i:(i,j)\in E} \frac{x_i}{\text{degree}(i)}$$



$$W_{ji} = \frac{1}{\text{degree}(i)} \cdot (A_G)_{ji}$$

$$W = A_G \cdot D_G^{-1}.$$



$$x^{t+1} = W \cdot x^t = A_G \cdot \underbrace{D_G^{-1} \cdot x^t}_{}$$

vector, i^{th} entry is how much node i gives to each neighbor

Example: G: ① — ② — ③ — ④ — ⑤

$$A_G: \begin{bmatrix} 0 & 1 & 0 & 0 & \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad D_G = \begin{bmatrix} 1 & & & & 0 \\ & 2 & & & \\ & & 2 & & \\ & & & 2 & \\ 0 & & & & 1 \end{bmatrix}$$

$$x^0 = (0, 1, 0, 0, 0)^T.$$

$$x^1 = A_G \cdot D_G^{-1} \cdot x^0 = A_G \cdot \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \\ 0 \end{bmatrix}.$$

$$x^2 = A_G \cdot D_G^{-1} \cdot x^1 = \begin{bmatrix} 0 \\ 3/4 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}.$$

<u>**Def:**</u> stationary distribution: $x^* \in \mathbb{R}^n$ s.t. $Wx^* = x^*$.

Ex: $x^* = \left(\frac{1}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}\right)$.

Questions:
- what is x^*?
- if $t \to \infty$, does $x^t \to x^*$.

$$x^t = W \cdot x^{t-1} = W^2 \cdot x^{t-2} = \dots = W^t \cdot x^0.$$

$$W = A_G \cdot D_G^{-1}.$$

Spectral Decomposition / Theorem of Sym. matrices

M = matrix. $n \times n$.

Def: $v \in \mathbb{R}^n$, $v \neq 0$ is eigenvector if $Mv = \lambda v$, with λ = its eigenvalue.

Obs: if (v, λ) are eigen v/v $\Rightarrow$

$$Mv - \lambda v = 0 \Rightarrow (M - \lambda I)v = 0.$$

$$I = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \; 1 \end{bmatrix}$$

$\Rightarrow \det(M - \lambda I) = 0$.

$\hookrightarrow$ is a polynomial in λ of deg. n.

(summation of factors of n items from $M - \lambda I$).

$\Rightarrow$ # roots of $\det(M - \lambda I) = 0$

is $= n$ (counting the multipl.).

<u>Fact:</u> if M is symmetric $\Rightarrow$ all roots are <u>real</u>.

<u>Thm:</u> [spectral theorem]: $\forall$ sym matrix M,

$\exists$ (λ_i, v_i) eigenv/v, $i = 1..n$ s.t.:

1) $\|v_i\| = 1$ $\forall i$.

2) $v_i \cdot v_j = 0$ for $i \neq j$ ortogonal.

3) $M v_i = \lambda_i \cdot v_i$.

<u>Remarks:</u> 1) $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$.

unique.

2) $v_1 \ldots v_n$ are not nec. unique:

if $\lambda_1 > \lambda_2 > \ldots > \lambda_n \Rightarrow$ unique. [up to taking $v_i \to -v_i$.]

otherwise, not.

$\lambda_i = \lambda_{i+1}$ then $\left(\lambda_i, \frac{v_i + v_{i+1}}{\sqrt{2}}\right)$

$\left(\lambda_{i+1}, \frac{v_i - v_{i+1}}{\sqrt{2}}\right)$

is a valid spectral decomp.

3) $M = \sum_{i=1}^{n} \lambda_i \cdot v_i \cdot v_i^T$.

Eg:

1) $M = 0$, $\lambda_1 = \ldots = \lambda_n = 0$

$(v_1, \ldots v_n) =$ any basis $\mathbb{R}^n$.

2) $M = I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$.

$\lambda_1 = \ldots = \lambda_n = 1$.

$(v_1, \ldots v_n) =$ any basis $\mathbb{R}^n$.

Obs: $x \in \mathbb{R}^n$.

x in basis $(v_1, \ldots, v_n)$

$$x = \sum_{i=1}^{n} \alpha_i v_i \quad \text{s.t.} \quad \sum \alpha_i^2 = \|x\|^2.$$

$$Mx = \left(\sum_{i=1}^{n} \lambda_i \cdot v_i \cdot v_i^T\right) \cdot \left(\sum_{j=1}^{n} \alpha_j v_j\right)$$

$$= \sum_{i,j} \lambda_i v_i \cdot v_i^T \cdot \alpha_j \cdot v_j$$

$$= \sum_i \lambda_i \cdot v_i \cdot \alpha_i = \sum_i \lambda_i \alpha_i \cdot v_i.$$

Rayleigh Quotients:

Def: $\forall x \neq 0$: $\quad R(x) = \dfrac{x^T M x}{\|x\|^2} = \dfrac{x^T M x}{x^T x}$.

Remark: $R(v_i) = \dfrac{v_i^T M v_i}{\|v_i\|^2} = \dfrac{v_i^T \cdot \lambda_i v_i}{\|v_i\|^2} = \lambda_i$.

(for $\forall$ eigenvector v_i).

Theorem: $\max_{x \neq 0} R(x) = \lambda_1$. (max eigenv.)

$\min_{x \neq 0} R(x) = \lambda_n$. (min).

$$\lambda_i = \max_{\substack{x \neq 0 \\ x \perp v_1, v_2, \ldots v_{i-1}}} R(x).$$

$$\lambda_2 = \max_{\substack{x \neq 0 \\ x \perp v_1}} R(x).$$

proof: consider abitrary $x \in \mathbb{R}^n$, $x \neq 0$.

$x = \sum_{i=1}^{n} \alpha_i \cdot v_i$ for some reals α_i

with $\sum \alpha_i^2 = 1$.

Hence: $R(x) = \frac{x^T M x}{\|x\|^2}$

$= \quad {}^{n}$

Obs

$$= \frac{x^T \cdot \sum_{i=1} \lambda_i \alpha_i v_i}{\|x\|^2}$$

$$= \frac{\sum_{i=1}^{n} \alpha_i v_i \cdot \sum_{i=1}^{n} \lambda_i \alpha_i v_i}{\|x\|^2}$$

$$= \frac{\sum_{i=1}^{n} \alpha_i \cdot \lambda_i \cdot \alpha_i}{\sum \alpha_i^2}$$

$$= \frac{\sum \lambda_i \alpha_i^2}{\sum \alpha_i^2} \leq \frac{\sum \lambda_1 \alpha_i^2}{\sum \alpha_i^2}$$

$$= \lambda_1 .$$

Hence $R(x) \leq \lambda_1$ always and

$R(v_1) = \lambda_1$.

hence $\max_{x \neq 0} R(x) = \lambda_1$.