COMS 4995-2: Advanced Algorithms (Spring'20) April 2, 2020

Lecture 18: Interior Point Method for Linear Programming

Instructor: *Alex Andoni* Scribe: *Jakwanul Safin* (js5142)

1 Introduction

This lecture focuses the on the Interior Point Method for solving linear programs [Karmaker '84]

A linear program is an optimization of the form,

$$\begin{aligned} &\min_{x \in \mathbb{R}^n} c^T x \\ &\text{s.t. } Ax \leq b \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. Additionally, we let m denote the optimal value if it exists and x^* be the point that attains this value.

Our approach in this lecture is to relax this problem into a continuous unconstrained optimization and apply the ideas from the previous lecture find the optima.

2 Converting To Unconstrained Optimization

Define the feasibility set as

$$k \triangleq \{x : Ax \leq b\}$$

.

We want to relax the minimization by defining a new function,

$$f' = c^T x + F(x)$$

such that

$$\begin{aligned} f'(x) &= x^T x \text{ if } x \in K \\ f'(x) &= \infty \text{ if } x \notin K \end{aligned}$$

To do so we define,

$$f_\eta = \eta c^T x + F(x)$$

where $\eta \in \mathbb{R}$ and $F(x)$ has the properties

$$\begin{aligned} F(x) &< \infty \quad \text{for } x \in K \\ F(x) &\to \infty \quad \text{as } x \to \partial K \end{aligned}$$

Let A_i denote the ith row of A. We will choose

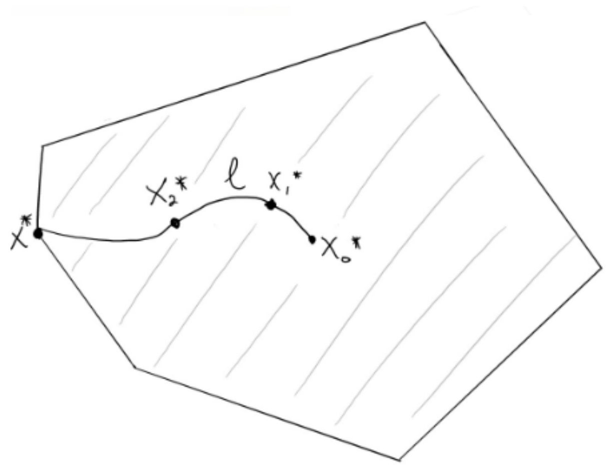
$$F(x) = \log\left[\prod_{i=1}^{m} \frac{1}{b_i - A_i x}\right] = -\sum_{i=1}^{m} \log\left[b_i - A_i x\right] \tag{2}$$

Our optimization problem is now

$$\min_{x \in \mathbb{R}^n} f_n(x) \triangleq x_n^* \tag{3}$$

Intuitively x_η^* approaches x^* as η tends to ∞.

3 Solving the optimization



$$x_0^* \triangleq \text{ the analytic center of } K$$

$$l = \{x_\eta^* | \eta \in \mathbb{R}, \eta > 0\} \triangleq \text{ the central path}$$

Claim 1. *For all $\eta > 0$, f_η is convex.*

Proof. First we compute the gradient and the hessian.

$$\begin{aligned} f_\eta(x) &= \eta c^T x + F(x) \\ \nabla f_\eta(x) &= \eta c + \nabla F(x) = \eta c + \sum_{i=1}^{m} \frac{A_i x}{b_i - A_i x} \\ \nabla^2 f_n(x) &= \nabla^2 F(x) = \sum_{i=1}^{m} \frac{A_i^\top A_i}{(b_i - A_i x)^2} \end{aligned} \tag{4}$$

For any vector $\gamma \neq 0$,

$$
\begin{aligned}
\gamma^\top \nabla^2 f_\eta(x)\gamma &= \sum_{i=1}^{m} \frac{\gamma^\top A_i^\top A_i \gamma}{(b_i - A_i x)^2} \\
&= \sum_{i=1}^{m} \frac{\|A_i\gamma\|^2}{(b_i - A_i x)^2} \geq 0
\end{aligned} \tag{5}
$$

Therefore f_η is convex.

$\square$

Note that if A is full rank ($\text{vol}(K) > 0$) then f_η is strictly convex and there is a unique minimum.

Idea 1 (Naive Approach):

Why not solve η "very large"? The problem with this approach is that if we use gradient descent then the convergence is very slow. And we can not use Newton's Method since there is no warm start.

Idea 2 (Following Central Path):

- Start at $x^*_{\eta_0}$ for small η_0
- Walk along central path. At each iteration t:
  - $\eta_{t+1} = \eta_t(1+\alpha)$
  - Run Newton's Method for $f_{\eta_{t+1}}$ with warm start $x^*_{\eta_t}$
- Terminate when η large enough

Idea 3 (Improved performance):

- Start at $x^*_{\eta_0}$ for small η_0
- Walk along central path. At each iteration t:
  - $\eta_{t+1} = \eta_t(1+\alpha)$
  - $x_{t+1} = x_t + n(x_t)$, where $n(x_t)$ is a single step of Newton's method for $f_{\eta_{t+1}}$
- Terminate when η large enough

4 Stopping Condition on η

Claim 2.

$$
c^T x^*_\eta - c^T x^* \leq \frac{m}{n}
$$

Note that with this claim, if $\eta = \frac{m}{\epsilon}$ then $c^\top x^*_n - c^\top x^* \leq \epsilon$

Proof.

$$
0 = \nabla f_\eta\left(x^*_\eta\right) = c^\top \eta + \nabla F\left(x^*_\eta\right)
$$

$$
\implies c^\top = -\frac{\nabla F\left(x^*_\eta\right)}{\eta}
$$

We want to show that

$$c^\top x_\eta^* - c^\top x^* = c^\top \left(x_\eta^* - x^*\right) = \frac{\nabla F\left(x_\eta^*\right)}{\eta}\left(x^* - x_\eta^*\right) \leqslant \frac{m}{\eta}$$

Now we show that $\forall x, y \in \mathbb{K} : \nabla F(x) \cdot (y - x) \leq m$

$$\nabla F(x)(y-x) = \sum_{i=1}^{m} \frac{A_i}{b_i - A_i x}(y-x) = \sum_{i=1}^{m} \frac{(b_i - A_i x) - (b_i - A_i y)}{b_i - A_i x} = m - \sum_{i=1}^{m} \frac{b_i - A_i y}{b_i - A_i x}$$

Since $x, y \in K$, we know that $Ax \geq b$ and $Ay \geq b$, so for all i,

$$\frac{b_i - A_i y}{b_i - A_i x} \geq 0$$

Therefore,

$$\nabla F(x)(y-x) \leq m$$

as desired. □

5 Determining $x_{\eta_0}^*$

We assume $\mathrm{vol}(K) \neq 0$. To find our starting point we start by finding some feasible point, $x' \in K$

Claim 3. *Our LP in (1) has nonempty K if and only if the LP*

$$\begin{aligned} &\min_{x \in R^n, t \in \mathbb{R}^m} t \\ &\quad s.t.\ \ Ax \leq b + t \end{aligned} \tag{6}$$

has optima equal to 0.

0 is a feasible point of (6) so we can use our interior point method to solve.

Claim 4. $\exists c'$ *s.t.*

$$x' = \min \eta c' x + F(x)$$

Proof. If x' is to be opt, then

$$\begin{aligned} &\nabla\left(\eta c' x' + F(x)\right) = 0 \\ \Leftrightarrow\quad & \eta' c' + \nabla F(x') = 0 \\ \Leftrightarrow\quad & c' = \frac{\nabla F(x')}{\eta} \end{aligned}$$

□

Idea 3.1 (Finding Initial Point) We walk backward along the central path from our feasible point.

- Find feasible point x' by solving (6) and c' from claim 4.
- Starting from c', walk backwards along central path for c'. At each iteration t:
  - $\eta_{t+1} = \eta_t(1-\alpha)$
  - $x_{t+1} = x_t + n(x_t)$, where $n(x_t)$ is a single step of Newton's method for $f_{x_{t+1}}$
- Terminate when η small enough.

Then run our main algorithm from x_η, which is approximately the analytic center.

6 Setting α and Runtime

Claim 5. $\alpha = \frac{1}{8\sqrt{m}}$ *is sufficient for correctness.*

Proof. sketch

The proof demonstrates an invariant on the distance between x_t and $x^*_{\eta_t}$. At each iteration each newton step decreases this distance, but scaling η by $(1+\alpha)$ increases it.

□

Starting from η_0, by claim 2, it is sufficient to stop at $\frac{m}{\epsilon}$.

$$\begin{aligned} T &= \log_{1+\alpha} \frac{m/\epsilon}{\eta_0} \\ &= \mathcal{O}\left(\tfrac{1}{\alpha} \log \frac{m}{\epsilon\eta_0}\right) \\ &= \mathcal{O}\left(\sqrt{m} \log \frac{m}{\epsilon\eta_0}\right) \end{aligned}$$