# Lecture 8:

# Dimension Reduction

COLUMBIA | ENGINEERING
The Fu Foundation School of Engineering and Applied Science

# Plan

- Pick up PS1 at the end of the class
- PS2 out

- Dimension Reduction
- Fast Dimension Reduction

- Scriber?

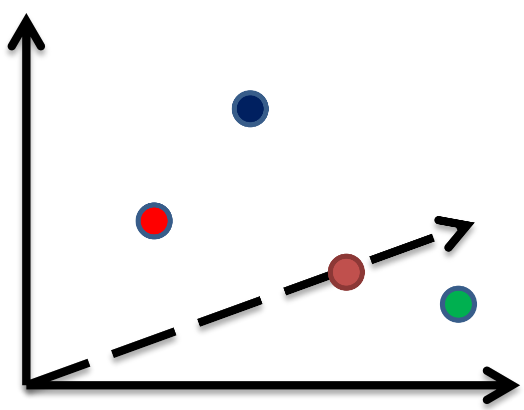
# High-dimensional case

- Exact algorithms degrade rapidly with the dimension $d$

| *Algorithm* | *Query time* | *Space* |
| --- | --- | --- |
| Full indexing | $O(\log n \cdot d)$ | $n^{O(d)}$ (Voronoi diagram size) |
| No indexing – linear scan | $O(n \cdot d)$ | $O(n \cdot d)$ |

# Dimension Reduction

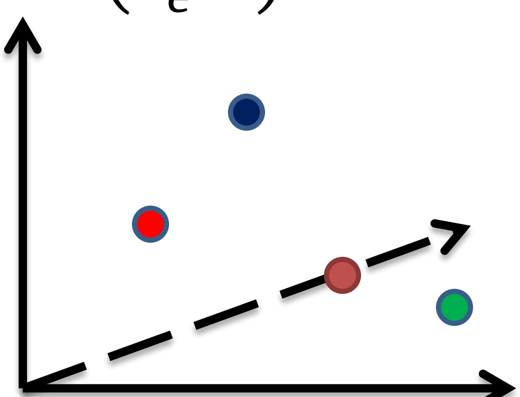
- Reduce high dimension?!
  - "flatten" dimension $d$ into dimension $k \ll d$
- Not possible in general: packing bound
- But can if: for a fixed subset of $\Re^d$



# Johnson-Lindenstrauss Lemma

- [JL84]: There is a randomized linear map $F: \ell_2^d \to \ell_2^k$, $k \ll d$, that preserves distance between two vectors $x, y$
  - up to $1+\epsilon$ factor:
    $$||x-y|| \le ||F(x)-F(y)|| \le (1+\epsilon)\cdot ||x-y||$$
  - with $1-e^{-C\epsilon^2 k}$ probability ($C$ some constant)

- Preserves distances between $n$ points for $k = O\left(\frac{\log n}{\epsilon^2}\right)$ with probability at least $1-1/n$

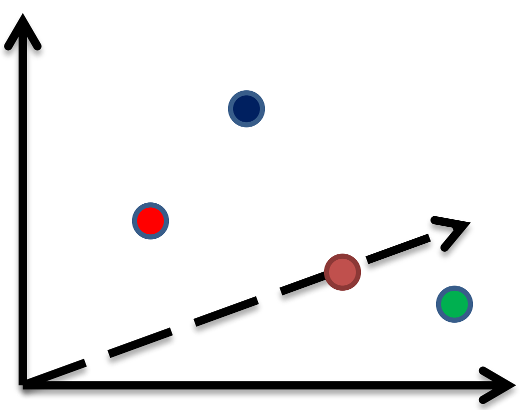


# Dim-Reduction for NNS

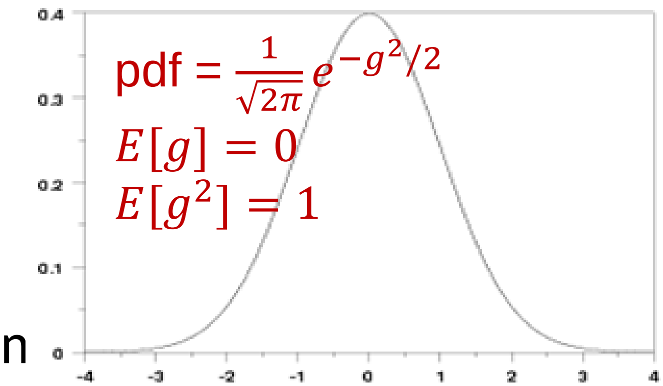
- [JL84]: There is a randomized linear map $F: \ell_2^d \to \ell_2^k$, $k \ll d$, that preserves distance between two vectors $x, y$
  - up to $1 + \epsilon$ factor:
    $||x - y|| \leq ||F(x) - F(y)|| \leq (1 + \epsilon) \cdot ||x - y||$
  - with $1 - e^{-C\epsilon^2 k}$ probability ($C$ some constant)
- Application: NNS in $\ell_2^d$
  - Trivial scan: $O(n \cdot d)$ query time
  - Reduce to $O(n \cdot k) + T_{dim-red}$ time after using dimension reduction
  - where $T_{dim-red}$ time to reduce dimension of the query point
  - Important that $F$ is *oblivious* !
- Have we seen something similar to JL84 in class?

# Idea:

- Project onto a *random* subspace of dimension $k$!
- In general, $F$ linear:
  - $F(x) - F(y) = F(x - y)$
  - Ok to prove that for $z = x - y$
  - $F(z) \approx ||z||$

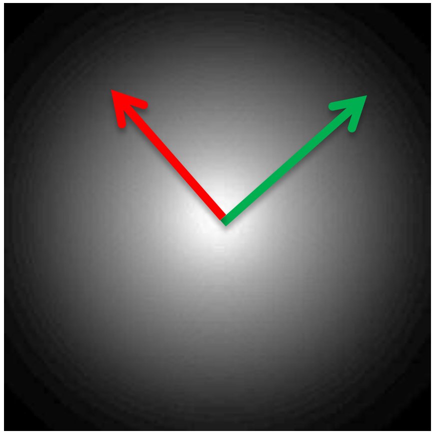


# 1D embedding

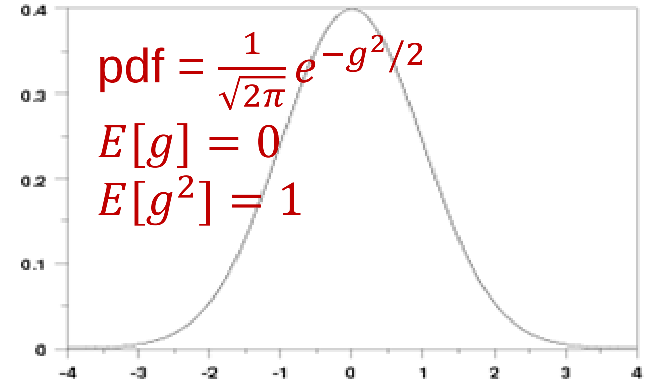



- Map $f: \ell_2^d \rightarrow \Re$
  - $f(x) = \sum_i g_i \cdot x_i$ ,
    - where $g_i$ are iid normal (Gaussian) ran
- Why Gaussian?
  - Stability property: $\sum_i g_i \cdot x_i$ is distributed as $||x|| \cdot g$, where $g$ is also Gaussian
  - Proof: $\langle g_1, \ldots, g_d \rangle$ is centrally distributed, i.e., has random direction, and projection on random direction depends only on length of $x$
  - Hence, enough to consider $x = e_1$

$$P(a) \cdot P(b) =$$
$$= \frac{1}{\sqrt{2\pi}} e^{-a^2/2} \frac{1}{\sqrt{2\pi}} e^{-b^2/2}$$
$$= \frac{1}{2\pi} e^{-(a^2+b^2)/2}$$



# 1D embedding




- Map $f(x) = \sum_i g_i \cdot x_i$ ,
  - for any $x$, $f(x) \sim ||x|| \cdot g$
  - Linear
- Want: $|f(x)| \approx ||x||$
- Claim: for any $x \in \Re^d$, we have
  - Expectation: $\mathrm{E}[|f(x)|^2] = ||x||^2$
  - Standard deviation:
    - $\sigma[|(f(x)|^2] = O(||x||^2)$
- Proof:
  - Expectation $= \mathrm{E}\left[\left(f(x)\right)^2\right] = \mathrm{E}[||x||^2 \cdot g^2]$
    $= ||x||^2$

# Full dimension reduction

- Just repeat the 1D embedding $k$ times
  - $F(x) = (g_1 \cdot x, g_2 \cdot x, \dots g_k \cdot x)/\sqrt{k} = \frac{1}{\sqrt{k}} Gx$
  - where $G$ is a $k \times d$ random Gaussian matrix

- Again, want to prove that
  - $F(z) = (1 \pm \epsilon) \cdot ||z||$
  - For fixed $z$
  - With probability $1 - e^{-\Omega(\epsilon^2 k)}$

# Concentration

- $F(z)$ is distributed as
  - $\frac{1}{\sqrt{k}}(||z|| \cdot a_1, ||z|| \cdot a_2, \ldots ||z|| \cdot a_k)$
  - where each $a_i$ is distributed as Gaussian
- Norm $||F(z)||^2 = ||z||^2 \cdot \frac{1}{k} \sum_i a_i^2$
  - $\sum_i a_i^2$ is called chi-squared distribution with $k$ degrees
- Fact: chi-squared very well concentrated:
  - Equal to $1 + \epsilon$ with probability $1 - e^{-\Omega(\epsilon^2 k)}$
  - Akin to central limit theorem

# Johnson Lindenstrauss: wrap-up

- $F(x) = (g_1 \cdot x, g_2 \cdot x, \ \dots g_k \cdot x) \,/\sqrt{k} = \frac{1}{\sqrt{k}} Gx$
- $||F(x)|| = (1 \pm \epsilon)||x||$ with high probability

- Contrast to Tug-Of-War:
  - $F(x) = \frac{1}{\sqrt{k}} Rx$ for $R$ contained of $\pm 1$
    - Only proved 90% probability
    - Would apply median to get high probability
  - Can also prove high probability [Achlioptas'01]
  - Gaussians have geometric interpretation

# Dimension Reduction for $\ell_1$

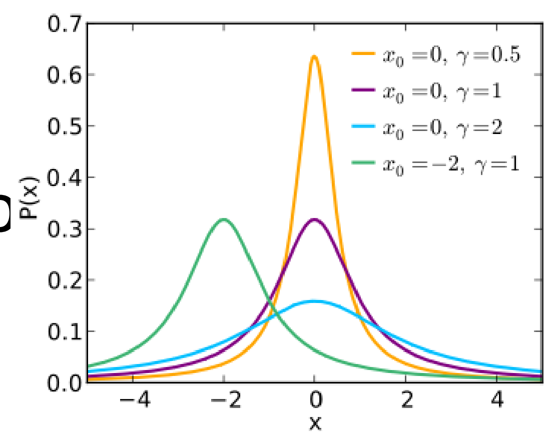
- Dimension reduction?
  - Essentially no [CS’02, BC’03, LN’04, JN’10…]
  - For $n$ points, $D$ approximation: between $n^{\Omega(1/D^2)}$ and $O(n/D)$ [BC03, NR10, ANN10…]
    - even if map depends on the dataset!
  - In contrast: [JL] gives $O(\epsilon^{-2}\log n)$, and doesn’t depend on the dataset
  - No distributional dimension reduction either
  - But can sketch!

# Sketch

- Can we do the "analog" of Euclidean projections?
- For $\ell_2$, we used: Gaussian distribution
  - has stability property:
  - $g_1 z_1 + g_2 z_2 + \cdots g_d z_d$ is distributed as $g \cdot ||z||$
- Is there something similar for 1-norm?
  - Yes: Cauchy distribution!
  - 1-stable:
  - $c_1 z_1 + c_2 z_2 + \cdots c_d z_d$ is distributed as $c \cdot ||z||_1$

$$pdf(s) = \frac{1}{\pi(s^2+1)}$$

- What's wrong then?
  - Cauchy are **heavy-tailed**…
  - doesn't even have finite expectatio



# Sketching for $\ell_1$ [Indyk'00]

- Still, can consider map as before
  - $S(x) = (C_1 x, C_2 x, \dots, C_k x) = \boldsymbol{C}x$
- Consider $S(x) - S(y) = \boldsymbol{C}x - \boldsymbol{C}y = \boldsymbol{C}(x-y) = \boldsymbol{C}z$
  - where $z = x - y$
  - each coordinate distributed as $||z||_1$ ×Cauchy
  - Take 1-norm $||\boldsymbol{C}z||_1$ ?
    - does not have finite expectation, but…
- Can estimate $||z||_1$ by:
  - *Median* of absolute values of coordinates of $\boldsymbol{C}z$ !
- Correctness claim: for each $i$
  - $\Pr[|C_i z| > ||z||_1 \cdot (1-\epsilon)] > 1/2 + \Omega(\epsilon)$
  - $\Pr[|C_i z| < ||z||_1 \cdot (1+\epsilon)] > 1/2 + \Omega(\epsilon)$

# Estimator for $\ell_1$

- Estimator: median$(|C_1 z|, |C_2 z|, \dots |C_k z|)$
- Correctness claim: for each $i$
  - $\Pr[|C_i z| > ||z||_1 \cdot (1-\epsilon)] > 1/2 + \Omega(\epsilon)$
  - $\Pr[|C_i z| < ||z||_1 \cdot (1+\epsilon)] > 1/2 + \Omega(\epsilon)$
- Proof:
  - $|C_i z| = abs(C_i z)$ is distributed as $\mathrm{abs}(||z||_1 c) = ||z||_1 \cdot |c|$
  - Need to verify that
    - $\Pr[|c| > (1-\epsilon)] > 1/2 + \Omega(\epsilon)$
    - $\Pr[|c| < (1+\epsilon)] > 1/2 + \Omega(\epsilon)$

# Estimator for $\ell_1$

- Estimator: median$(|C_1 z|, |C_2 z|, \dots |C_k z|)$
- Correctness claim: for each $i$

$L_i = 1$ if holds

$U_i = 1$ if holds

  - $\Pr[|C_i z| > ||z||_1 \cdot (1-\epsilon)] > 1/2 + \Omega(\epsilon)$
  - $\Pr[|C_i z| < ||z||_1 \cdot (1+\epsilon)] > 1/2 + \Omega(\epsilon)$
- Take $k = O(1/\epsilon^2)$
  - $E[L_i] \geq 1/2 + \Omega(\epsilon)$
  - Hence $\Pr\left[\sum_i L_i \leq \frac{k}{2}\right] < 0.05$ (by Chebyshev)
  - Similarly with $U_i$
- The above means that
  - median$(|C_1 z|, |C_2 z|, \dots |C_k z|) \in (1 \pm \epsilon)||z||_1$ with probability at least 0.90

# PS1

- Avg: 65.4
- Standard deviation: 20.5
- Max: 96

- By problems (average % points):

  1: 0.83

  2: 0.62

  3: 0.44