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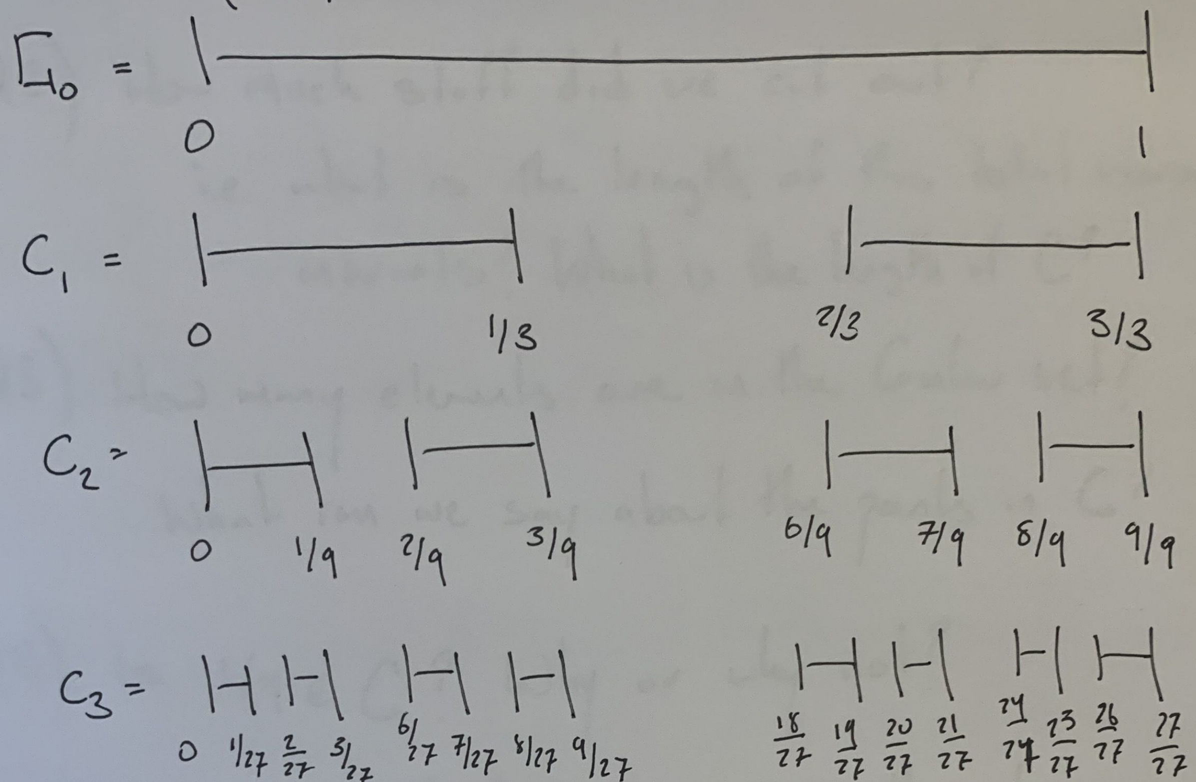
Let's construct the Cantor Set

Start with the interval $[0,1]$ denoted C_0
(closed)



For any C_n, define C_{n+1} by removing middle 1/3 of each subsequent open interval.

Ex, $C_1 = C_0 \setminus (1/3, 2/3) = [0, 1/3] \cup [2/3, 1]$

$C_2 = C_1 \setminus \left((1/9, 2/9) \cup (7/9, 8/9) \right)$

At every level, we have a power of two subintervals, where

$C_0 \supset C_1 \supset C_2 \supset C_3 \cdots \supset C_n \supset C_{n+1}$

Define the Cantor set C as $\quad C = \bigcap_{j=1}^{\infty} C_j$ where C_j has 2^j subintervals

Questions / Properties of the Cantor Set

Q1) Is the Cantor set empty (does $|C| = \phi$)?

Q2) How much stuff did we cut out?
i.e. what is the length of the total removed intervals? What is the length of C?

Q3) How many elements are in the Cantor set?
What can we say about the points in C?

Q4) Is $1/4 \in C$? Why or why not?

Lets revisit some definitions to answer the above.

__Def (Set)__
A set is a collection of distinct elements. Given set A, we say a is "an element of" A ($a \in A$) if a is one of the distinct elements (objects) in A.
Given two sets A, B, we say A is "a subset of" B ($A \subseteq B$) if every element in A is also in B
When we list elements explicitly, we put braces around the list

__Ex__ Special Chars $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ natural, integer, rational, reals
$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$ with $\mathbb{N} = \{1, 2, 3, \ldots\}$ $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2 \ldots\}$
$\mathbb{Q} = \{x \mid \exists a, b \in \mathbb{Z} \text{ s.t. } x = a/b\}$

Sets ex $\{2k : k \in \mathbb{Z}\}$ even integers

$\{2k+1 : k \in \mathbb{Z}\}$ odd integers

Def (Intervals)

When $a, b \in \mathbb{R}$, with $a \leq b$, the closed interval $[a,b]$ is the set $\{x \in \mathbb{R} : a \leq x \leq b\}$

The open interval is set $\{x \in \mathbb{R} : a < x < b\}$

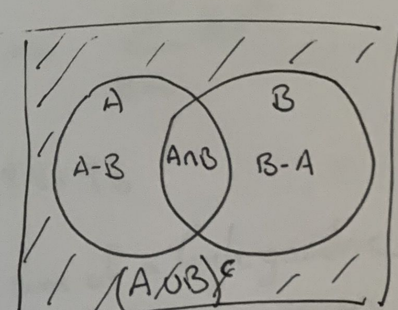
Def (Set Operations)

A, B sets. Union $A \cup B$ is all elements in A or B

Intersection $A \cap B$ " " " A and B

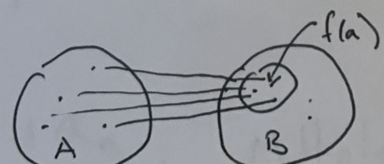
Difference $A \setminus B$ " " " A not in B

or $(A - B)$



Two sets disjoint if $A \cap B = \emptyset$

If set A contained in universe U under discussion, A^c (the compliment of A) is set of elements in U not in A: $U \setminus A$



A: domain

B: codomain or target

f(a): Image

Def (Functions)

A function f from A to B is a machine with inputs & outputs. Inputs are elements of one set, outputs are elements in a possibly different set.

$f: A \to B$ assigns to each $a \in A$ a single element $f(a) \in B$ called the image of a under f

Question: What is the length of the total removed intervals? in G

$\Longrightarrow$ We will need to define Geometric Series

Def Geometric Series

Lets say we have a sequence

$3, 6, 12, 24, 48, 96$ which we can represent as follows

$a_1, a_2, a_3, a_4, a_5, a_6, \cdots, a_n$

The a_{n+1} elements are obtained via a common ratio $a_2 = a_1 \cdot r$

A sequence is an enumerated collection of objects in which repetitions are allowed and order matters. The number of elements is called the length of the sequence.

A geometric series is one that can be written as follows

$a_1 + a_1 r + a_1 r^2 + \cdots$, or in general form as $\sum_{k=0}^{\infty} a r^k$

In our example above $S_n = \sum_{i=1}^{n} a_i = 3 + 6 + 12 + 24 + 48 + 96$

We can derive a closed form expression for a partial sum of a finite geometric series as follows:

$S_n = a_1 + a_2 + a_3 + \cdots + a_n$; Multiply by $r \Rightarrow$

$= a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1}$; $rS_n = a_1 r + a_1 r^2 + \cdots + a_1 r^n$

Difference new and original series

$$S_n - rS_n = a_1 + \cancel{a_1 r} + \cancel{a_1 r^2} + \cdots \cancel{a_1 r^{n-1}} - (\cancel{a_1 r} + \cancel{a_1 r^2} + \cdots + a_1 r^n) = a_1 + a_1 r^n$$

$$\Rightarrow (1-r) S_n = a_1 + a_1 r^n$$

$$S_n = (a_1 + a_1 r^n) / (1-r)$$

$$S_n = \frac{(a_1 + a_1 r^n)}{(1-r)} = \frac{a_1(1+r^n)}{(1-r)}$$

To find the expression when $n = \infty$ we take the limit, noting that $|r| < 1$

$$\lim_{n\to\infty} S_n = \frac{a_1}{(1-r)}$$

We cut out 1/3 of each subsequent interval, so $r = 1/3$ and $a_1 = 2/3$

$$\Rightarrow \sum_{n=1}^{\infty} ar^n = \frac{2/3}{(1-1/3)} = 1, \text{ therefore length}(C) = 0$$

$\Rightarrow$ [Mohuntes Measure theory]

Surprisingly, the length of the removed intervals to define C is 1!

Q1) Given that we remove <u>open</u> middle third, end points stay in C.

C_j consists of 2^j subintervals that we can denote with letters L and R (for Left and Right)

Ex $C_2 =$ [–] [–] [–] [–]

$0 \quad 1/9 \quad 2/9 \quad 3/9 \qquad 6/9 \quad 7/9 \quad 8/9 \quad 9/9$

LL LR RL RR

Step j involves j possible letters LL, LR, RL, RR

Given that C is the intersection of the union of these subintervals, a point in C is uniquely determined by an infinite sequence of L's and R's

Ex LL... L... R... LR... RR..

The sequence looks like an infinite binary tree, its components tell us if we are in the left or right for subinterval K

$$a_1 a_2 a_3 \cdots a_K \cdots$$

a_1 tells us if we are in left or right of first subinterval

a_2 " " " second "

and so on... $\Rightarrow |C| = \infty$

<u>Claim</u> The set of infinite binary sequences is <u>uncountable</u>.

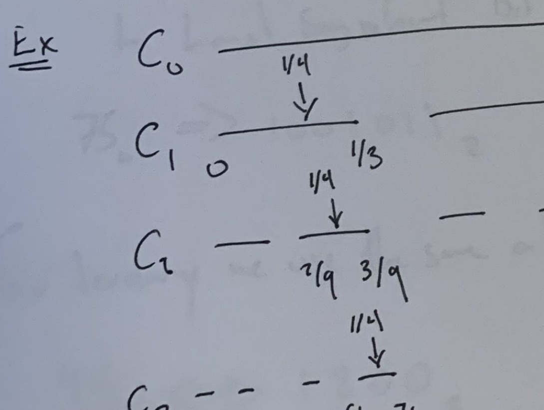
$\Rightarrow$ We will define and prove this in a lecture or two...

Not only is $|C| = \infty$, but it is a special kind of infinity which we will prove.

Q4) Is $1/4$ in C?

<u>Remark</u>: $1/4$ flips from bottom to top thirds of the interval it was last in

<u>Ex</u>



$6/27 \approx 0.222$

$7/27 \approx 0.259259$

<u>Remark</u>: $\mathbb{R}$ is continuous

$\mathbb{Q}$ is discrete

Cantor set is strangely ("in between") $\mathbb{Q}$ and $\mathbb{R}$

Q: How do we know whether $1/4 \in C$?

C is defined taking a limit so how can we answer this?

Lets Review Binary and Ternary Arithmetic

Recall to convert a number from decimal (base 10) to binary (base 2)

Apply the algorithm

64	32	16	8	4	2	1
2^6	2^5	2^4	2^3	2^2	2^1	2^0

$75 \rightarrow 64$

$75-64 = 11$ highest # under 11 is $\underline{\underline{8}}$

$11-8 = 3$ highest # under 3 is $\underline{\underline{2}}$

$3-2 = 1$ highest # under 1 is $\underline{\underline{1}}$

$\boxed{64}$ 32 16 $\boxed{8}$ 4 $\boxed{2}$ $\boxed{1}$

So in binary we have $\Rightarrow$ 1 0 0 1 0 1 1

$75 = 64+8+2+1$

$75_{10} = 1001011_2$; or alternatively

$75 \% 2 = 37 R 1 \leftarrow LSB$

$37 \% 2 = 18 R 1$

$18 \% 2 = 9 R 0$

$9 \% 2 = 4 R 1$

$4 \% 2 = 2 R 0$

$2 \% 2 = 1 R 0$

$1 \% 2 = 0 R 1 \leftarrow MSB$

We read the number from (Most Sig Bit) MSB to Least Significant Bit (LSB)

$75_{10} \Rightarrow 1001011_2$

For ternary we use the same algorithm with letters in $\{0,1,2\}$

$45_{10} \Leftrightarrow 1200_3$

$45 \% 3 = 15 R 0 \leftarrow$ ~~LSB~~

$15 \% 3 = 5 R 0$

$5 \% 3 = 1 R 2$

$1 \% 3 = 0 R 1 \leftarrow 1$

$$\left.\begin{matrix} 1 & 2 & 0 & 0 \\ 3^3 & 3^2 & 3^1 & 3^0 \end{matrix}\right\} \Rightarrow 1200_3 = 1\times3^3 + 2\times3^2 + 0\times3^1 + 0\times3^0 = \underline{\underline{45}}$$

Claim

Any point $x \in [0,1]$ can be expressed as an infinite sum of a geometric progression in ternary

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \quad \text{with } a_n = \{0,1,2\}$$

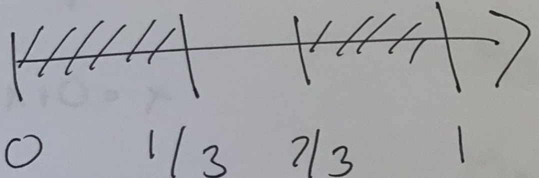
$$x = (0.a_1a_2a_3\ldots)_3 ;$$

Note that

$$1 = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = (0.2222\ldots)_3$$

Akin to $(0.9999\ldots)_{10}$

Remark Set C_1 does not have x's with $a_1 = 1$



$0 = (0.000\ldots)_3 \qquad 1/3 = (0.0222\ldots)_3$

Any number between 0 and 1/3 can be written as a fraction such that digit a_1 is zero

Remark Set C_2 does not have x's with $a_1 = 1$ or $a_2 = 1$

$2/3 = (0.2)_3 \qquad 1 = (0.222\ldots)_3$

Q) Can you show that $x \in C$ iff (if and only if) $\forall a_i \in \{0,2\}$

i.e. the sequence does not have any 1's in it?

Def^n (Field) To study $\mathbb{R}$, lets define a field as satisfying the following Field Axioms

A set with operations addition (+) and multiplication ($\cdot$) and distinguished elements 0 and 1 (with $0 \neq 1$) is a field if the following properties hold $\forall x, y, z \in S$

A_0: $x + y \in S$ — M_0: $x \cdot y \in S$ — Closure

A_1 $(x+y)+z = x+(y+z)$ — M_1: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ — Associativity

A_2 $x + y = y + x$ — M_2: $x \cdot y = y \cdot x$ — Commutativity

A_3 $x + 0 = x$ — M_3: $x \cdot 1 = x$ — Identity

A_4 Given x, $\exists w \in S$ s.t. $x + w = 0$ — M_4: For $x \neq 0$ $\exists w \in S$ s.t. $x \cdot w = 1$ — Inverse

DL: $x \cdot (y+z) = x \cdot y + x \cdot z$ — Distributive Law.