Discrete Mathematics: Combinatorics and Graph Theory

Homework 1: Due 10/7

Instructions. Solve any 10 questions. Typeset or write neatly and show your work to receive full credit.

1. Tell us a little bit about yourself. Why are you majoring in computer science (or something else)? What is the relationship between mathematics and computer science? Why are you required to take *discrete mathematics*? Why is there an emphasis on problem solving and proofs rather than on memorizing how to perform computations? What do you hope to get out of the course?

2. Prove that for every integer x and for every integer y, if x is odd and y is odd then xy is odd. Translate into symbols using quantifiers.

3. Given a real number x, let A be the statement $\frac{1}{2} < x < \frac{5}{2}$, let B be the statement $x \in \mathbb{Z}$, let C be the statement $x^2 = 1$, and let D be the statement $x = 2$. Which statements below are true for all $x \in \mathbb{R}$?
   (a) $A \to C$
   (b) $B \to C$
   (c) $(A \wedge B) \to C$
   (d) $(A \wedge B) \to (C \vee D)$
   (e) $C \to (A \wedge B)$
   (f) $D \to (A \wedge B \wedge \neg C)$
   (g) $(A \vee C) \to B$

4. Let $P(x)$ be the assertion "x is odd", and let $Q(x)$ be the assertion "$x^2 - 1$ is divisible by 8". Determine whether the following statements are true:
   (a) $(\forall x \in \mathbb{Z})[P(x) \to Q(x)]$
   (b) $(\forall x \in \mathbb{Z})[Q(x) \to P(x)]$

   Let P(x) be the assertion "x is odd", and let $Q(x)$ be the assertion "x is twice an integer". Determine whether the following statements are true:
   (a) $(\forall x \in \mathbb{Z})[P(x) \to Q(x)]$
   (b) $(\forall x \in \mathbb{Z})((P(x)) \to (\forall x \in \mathbb{Z})(Q(x))$

5. Prove the following by induction. State base case, inductive hypothesis and inductive step to receive full credit.
   (a) The sum of the first n odd natural numbers equals n^2.
   (b) $\forall n \in \mathbb{N}$, $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.
   (c) $\forall n \in \mathbb{N}$, if $n \in \mathbb{N}$, then $n^2 - n$ is even.
   (d) $\forall n \in \mathbb{N}$, $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$.
   (e) $\forall n \in \mathbb{N}$, $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}$.

6. Injections, surjections and bijections.
   (a) Give an example of a function $f : \mathbb{Z} \to \mathbb{Z}$ where (i) f is one-to-one but not onto; and (ii) f is onto but not one-to-one.
   (b) For $A = \{1, 2, 3, 4, 5, 6, 7\}$, how many bijective functions $f : A \to A$ satisfy $f(1) \neq 1$? What if $A = \{x | x \in \mathbb{Z}^+, 1 \leq x \leq n\}$ for some fixed $n \in \mathbb{Z}^+$?

(c) For $A = (-2, 7] \subseteq \mathbb{R}$ define the functions $f, g : A \to \mathbb{R}$ by $f(x) = 2x - 4$ and $g(x) = \frac{2x^2-8}{x+2}$. Verify that $f = g$. Is the result affected if we change A to $[-7, 2)$?

7. Define a function $f : \mathbb{Z} \times \mathbb{Z}^+ \to \mathbb{Q}$ by $f(p, q) = \frac{p}{q}$. Is f (i) injective, (ii) surjective, and (iii) bijective? Prove or disprove.

8. Verify whether the function $g : \mathbb{R} \to (0, 1)$ where $g(x) = \frac{1}{1+e^{-x}}$ defines a bijection. Verify whether the function $f : (-\pi/2, \pi/2) \to \mathbb{R}$ with $f(x) = tan(x)$ defines a bijection.

9. Let $f : \mathbb{Z} \to \mathbb{N}$ be defined below. Prove that f is one-to-one and onto, and determine f^{-1}.

$$f(x) = \begin{cases} 2x - 1 & \text{if } x > 0 \\ -2x & \text{if } x \leq 0 \end{cases}$$

10. Construct an explicit bijection from the open interval (0, 1) to the closed interval [0, 1].

11. Prove the following proposition (hint: use the contrapositive). Suppose a is an integer. If a is odd, then $x^2 + x - a^2 = 0$ has no integer solution.

12. Prove the following proposition (hint: use proof by contradiction). There does not exist a smallest positive rational number.

13. Prove the following:

    (a) $\sqrt{5}$ is irrational.

    (b) $\sqrt{20}$ is irrational.

    (c) $\sqrt{2} + \sqrt{5}$ is irrational.

    (d) There are infinitely many composite numbers.

14. Describe a correspondence between binary real numbers $x = (.b_1b_2b_3\cdots)_2$ in $[0, 1)$ and Stern-Brocot real numbers $\alpha = B_1B_2B_3\cdots$ in $[0, \cdots, \infty)$. If x corresponds to α and $x \neq 0$, what number corresponds to $1 - x$? Give a simple rule for comparing rational numbers based on their representations as L's and R's in the Stern-Brocot number system.

15. The Stern-Brocot representation of π is $\pi = R^3L^7R^{15}LR^{292}LRLR^2LR^3LR^{14}L^2R\cdots$. Use it to find the simplest rational approximations to π whose denominators are less than 50. Is $\frac{22}{7}$ one of them?

16. Show that the union of a countable number of countable sets is countable.

17. Use Cantor's diagonalization argument to show that the Cantor set is uncountable.

18. Is 1/108 in the Cantor set? What about $\pi/12$? Prove or disprove.

19. Show that if S is a set, there does not exist an onto function f from S to $\mathcal{P}(S)$ (the power set of S). Conclude that $|S| < |\mathcal{P}(S)|$. Suppose such a function f existed. Let $T = \{s | s \in S, s \notin f(s)\}$ and show that no element s can exist for which $f(s) = T$. This is known as Cantor's Theorem.

20. We defined Stern's diatomic series as $a_0 = 0, a_1 = 1, a_{2n} = n, a_{2n+1} = a_n + a_{n+1}$. Show that the function $f(x) = \frac{a_n}{a_{n+1}}$ defines a bijection from $\mathbb{Z}^+$ to $\mathbb{Q}^+$.