Discrete Mathematics: Combinatorics and Graph Theory

Practice Exam 3

Instructions. Solve any 5 questions and state which 5 you would like graded. Note that this is a sample exam, and while it bears some similarity to the real exam, the two are not isomorphic.

1. Solve the following recurrence relations:

(a) \( a_n = 6a_{n-1} - 9a_{n-2} \) when \( a_0 = 2, a_1 = 21. \)
Factor the characteristic polynomial \( x^2 - 6x + 9 = 0 \Rightarrow (x - 3)(x - 3) = 0 \Rightarrow x = -3. \) Then \( a_n = \alpha(3)^n + \beta \cdot n \cdot (3)^n. \) Applying initial conditions \( a_0 = 2 = \alpha(3)^0 + \beta \cdot 0 \cdot (3)^0 \Rightarrow \alpha = 2. \)
\( \alpha_1 = 21 = \alpha(3)^1 + \beta \cdot 1 \cdot (3)^1 \Rightarrow 21 = 6 + 3\beta \Rightarrow \beta = 5. \) The final solution is \( a_n = 2(3)^n + 5 \cdot n \cdot (3)^n. \)

(b) \( a_n = 2a_{n-1} - 2a_{n-3} \) for \( n \geq 3 \) with \( a_0 = 3, a_1 = 6, \) and \( a_2 = 0. \)
\( x^3 - 2x^2 - x + 2 = 0 \Rightarrow x^2(x - 2) - 1(x - 2) = 0 \Rightarrow (x - 1)(x + 1)(x - 2) = 0 \Rightarrow x = 1, x = -1, x = 2. \)
The solution is of the form \( a_n = A_1(1)^n + A_2(-1)^n + A_3(2)^n. \) Applying the initial conditions and solving the resulting system:
\( \alpha_1 = -2, A_2 = 6, A_3 = -3. \) Therefore the solution can be expressed
\( a_n = 2(-1)^n + 6(1)^n + (-1)(2)^n. \)

(c) \( a_n = 2a_{n-1} + 1 \) when \( a_1 = 1. \)
\( a_2 = 2 \cdot 1 + 1 = 3, a_3 = 2 \cdot 3 + 1 = 7, a_4 = 2 \cdot 7 + 1 = 15. \) Each value is twice the previous minus one: \( a_n = 2 \cdot 2^n - 1. \)

(d) \( na_n = (n - 2)a_{n-1} + 2 \) when \( a_1 = 1. \)
Recall that \( \sum_{i=1}^{n} i = n(n - 1)/2 \Rightarrow n(n - 1) = 2 \cdot \sum_{i=1}^{n} i. \) Multiply both sides by \( n - 1: \)
\[
(n - 1)a_n = (n - 1)(n - 2)a_{n-1} + 2(n - 1)
= (n - 2)(n - 3)a_{n-2} + 2(n - 2) + 2(n - 1)
= (n - 3)(n - 4)a_{n-3} + 2(n - 3) + 2(n - 2) + 2(n - 1)
= 2(1 + \cdots + n - 1)
= 2 \cdot \sum_{i=1}^{n} i.
\]
\( n(n - 1)a_n = n(n - 1) \)
\( a_n = 1 \)

(e) \( a_n = 3a_{n-1} + 10a_{n-2} + 7.5^n \) where \( a_0 = 4 \) and \( a_1 = 3. \)
The characteristic equation of the associated homogeneous relation is \( x^2 - 3x - 10 = 0 \Rightarrow (x - 5)(x - 2) = 0 \Rightarrow x = -5, x = -2. \) Therefore \( a_n^h = \alpha(5)^n + \beta(-2)^n. \) Since \( f(n) = 7.5^n \) is of the form \( cn^n \) consider \( Anx^n \) so that \( a_n^{(p)} = Anx^n = An5^n. \) Plugging into the recurrence relation \( An5^n = 3A(n - 1)5^{n-1} + 10A(n - 2)5^{n-2} + 7.5^n. \) Dividing by \( 5^{n-2} \) yields \( An5^2 = 3A(n - 1)5 + 10A(n - 2)5^2 + 7.5^n \Rightarrow 25A = 15An - 15A + 10An - 20A + 175 \Rightarrow 35A = 175 \Rightarrow A = 5. \) Therefore \( a_n^{(p)} = An5^n = 5n5^k = 5n^{k+1}. \) The solution of the recurrence relation can be written as \( a_n = a_n^{(h)} + a_n^{(p)} = \alpha(5)^n + \beta(-2)^n + n5^{n+1}. \) Applying initial conditions and solving gives \( \alpha = -2, \beta = 6 \) for \( a_n = -2 \cdot (5)^n + 6 \cdot (-2)^n + n5^{n+1}. \)

2. Assume that the probability a child is a boy is 0.51 and that the sexes of children born into a family are independent. What is the probability that a family of five children has

(a) exactly three boys?
\[
P(\text{exactly 3 boys}) = \binom{5}{3}(0.51)^3(0.49)^2
\]
(b) at least one boy?

\[ P(\text{at least 1 boy}) = 1 - P(\text{no boys}) = 1 - \binom{5}{0}(.51)^0(.49)^5 \]

(c) at least one girl?

\[ P(\text{at least 1 girl}) = 1 - P(\text{no girls}) = 1 - \binom{5}{5}(.51)^5(.49)^0 \]

(d) all children of the same sex?

\[ \binom{5}{5}(.51)^5(.49)^0 + \binom{5}{0}(.51)^0(.49)^5 \]

3. How many members of the set \( S = \{1, 2, 3, \cdots, 105\} \) have nontrivial factors in common with 105? Hint: use the inclusion-exclusion principle.

105 has a prime factorization \( 105 = 3 \times 5 \times 7 \), so elements in \( S \) will have common factors with 105 if they are divisible by 3, 5 or 7. Define \( A \) as elements of \( S \) divisible by 3, \( B \) as elements of \( S \) divisible by 5 and \( C \) as elements divisible by 7. There are 35 numbers from 1 to 105 divisible by 3 so the subset \( A \) contains 35 elements. Similarly there are 21 elements in the subset \( B \) and 15 elements in the subset \( C \). Consider \( A \cap B \), the subset of elements divisible by both 3 and 5. There are seven numbers between 1 and 105 divisible by 15 (we had worked this out in checking that our formula for the recurrence in question 1.2 was correct), therefore \( |A \cap B| = 7 \). Similarly \( A \cap C \) is the subset of elements divisible by both 3 and 7 (21) and \( B \cap C \) is the subset of elements divisible by both 5 and 7 (35). Therefore \( |A \cap C| = 5 \) and \( |B \cap C| = 3 \). The only number divisible by 105 is 105 so \( |A \cap B \cap C| = 1 \). Applying the inclusion-exclusion principle:

\[
|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|
= 35 + 21 + 15 - 7 - 5 - 3 + 1
= 57
\]

4. Suppose that 8% of the patients tested in a clinic are infected with HIV. Furthermore, suppose that when a blood test for HIV is given, 98% of the patients infected with HIV test positive and that 3% of the patients not infected with HIV test positive. What is the probability that

Let \( H \) denote HIV and \( P \) denote positive.

(a) a patient testing positive for HIV with this test is infected with it?

\[
P(H|P) = \frac{P(P|H)P(H)}{P(P|H)P(H) + P(P|\bar{H})P(\bar{H})} = \frac{0.98 \times 0.08}{0.98 \times 0.08 + 0.03 \times 0.92} = 0.740
\]

(b) a patient testing positive for HIV with this test is not infected with it?

\[
P(\bar{H}|P) = 1 - P(H|P) = 1 - 0.74 = 0.26
\]

(c) a patient testing negative for HIV with this test is infected with it?

\[
P(H|\bar{P}) = \frac{P(\bar{P}|H)P(H)}{P(\bar{P}|H)P(H) + P(\bar{P}|\bar{H})P(\bar{H})} = \frac{0.02 \times 0.08}{0.02 \times 0.08 + 0.97 \times 0.92} = 0.002
\]

(d) a patient testing negative for HIV with this test is not infected with it?

\[
P(\bar{H}|\bar{P}) = 1 - P(H|\bar{P}) = 1 - 0.002 = 0.998
\]
5. Find a recurrence relation for the number of ways to climb \( n \) stairs if the person climbing the stairs can take one, two or three stairs at a time. What are the initial conditions? How many ways can this person climb a flight of eight stairs?

\[
a_n = a_{n-1} + a_{n-2} + a_{n-3}
\]

The initial conditions are \( a_0 = a_1 = 1, a_2 = 2 \) or \( a_1 = 1, a_2 = 2 \) and \( a_3 = 4 \).

\[
\begin{align*}
a_3 &= a_0 + a_1 + a_2 = 1 + 1 + 2 = 4 \\
a_4 &= a_1 + a_2 + a_3 = 1 + 2 + 4 = 7 \\
a_5 &= a_2 + a_3 + a_4 = 2 + 4 + 7 = 13 \\
a_6 &= a_3 + a_4 + a_5 = 4 + 7 + 13 = 24 \\
a_7 &= a_4 + a_5 + a_6 = 7 + 13 + 24 = 44 \\
a_8 &= a_5 + a_6 + a_7 = 13 + 24 + 44 = 81
\end{align*}
\]

6. Show that \( \mathbb{E}_Y (\mathbb{E}_X (P(X|Y))) = \mathbb{E}_X (X) \)

\[
\mathbb{E}_Y (\mathbb{E}_X P(X|Y)) = \sum_y \mathbb{E}_X (P(X|Y = y)P(Y = y)
\]

\[
= \sum_y \sum_x x \cdot P(X = x|Y = y)P(Y = y)
\]

Recall that by Bayes rule \( P(A|B)P(B) = P(B|A)P(A) \)

\[
= \sum_x \sum_y x \cdot P(Y = y|X = x)P(X = x)
\]

\[
= \sum_x x \cdot P(X = x) \sum_y P(Y = y|X = x)
\]

\[
= \sum_x x \cdot P(X = x)
\]

\[
= \mathbb{E}_X (X)
\]

7. Consider a random walk (a drunk stumbling in one dimension) with step sizes of \( S_i \) where \( S_i \) is +1 with probability \( p \) and -2 with probability \( q = 1 - p \). Let \( T_n = \sum_{i=1}^m S_i \) be the displacement after a fixed (not random) number of steps \( n \). Find the probability \( P(T_n = t) \) and the mean and variance of \( T_n \) in terms of \( n \) and \( p \).

Denote the number of +1 moves with \( n \) steps as \( X_n \). We can then define the number of +2 moves is defined as \( n - k \) when \( k \) denotes the number of +1 moves. The total displacement is then \( t = k + 2(n - k) \Rightarrow k = 2n - t \).

(a) We want \( P(T_n = t) \) which follows a Binomial distribution.

\[
P(T_n = t) = P(X_n = 2n - t) = \binom{n}{2n-t} p^{2n-t} (1-p)^{t-n}
\]

(b) \( \mathbb{E}(S_i) = p + -2(1-p) = 3p - 2 \). Each step is iid so that \( \mathbb{E}(T_n) = n \mathbb{E}(S_i) = 3np - 2n \)

(c) \( \text{Var}(S_i) = \mathbb{E}(S_i^2) - \mathbb{E}(S_i)^2 = p + (1-p)(-2)^2 - (3p - 2)^2 = 9p(1-p) \). Each step is iid so that \( \text{Var}(T_n) = n \text{Var}(S_i) = 9np(1-p) \)