Def. A function is injective (one-to-one) if each element in the domain is mapped to at most one element of the codomain or, equivalently, if distinct elements of the domain map to distinct elements of the codomain.

\[ \forall x, x' \in X, \ f(x) = f(x') \Rightarrow x = x' \]

or contrapositive \[ \forall x, x' \in X, \ x \neq x' \Rightarrow f(x) \neq f(x') \]

A function is surjective (onto) if each element of the codomain is mapped to at least one element of the domain. That is, the image and the codomain are equal.

\[ \forall y \in Y, \ \exists x \in X \text{ s.t. } y = f(x) \]

A function is bijective (one-to-one and onto) if each element in the codomain is mapped to exactly one element of the domain. That is, both injective and surjective.
Ex 3

\[ f: [0, \infty) \rightarrow \mathbb{R}, \text{ defined by } f(x) = \sqrt{x} \]

This function is injective, surjective, so bijective.

Ex 4

\[ f(x) = x^2 \]

Need to specify domain and codomain.

If domain and codomain are \( \mathbb{R} \), then \( f(x) \) will be injective and surjective.

Not a surjection because range is not equal to codomain. For example, there is no number in the domain with image \(-1\) which is an element of codomain.

It is not an injection because more than one distinct element in domain is mapped to the same element in the codomain. For example, \( f(-1) = f(1) \) but \(-1 \neq 1\).

\[ \Rightarrow \text{ What if domain is } \mathbb{R} \text{ and codomain is } [0, \infty) \]

What properties would the function have? (Injective, surjective)

\[ \Rightarrow \text{ What if domain and codomain are both } \mathbb{R}, \mathbb{R} \]

Show \( f(x) = 3x - 7 \) injective

Need to show \( x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2) \)

\[ f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \] is easier to show.

\[ 3x_1 - 7 = 3x_2 - 7 \]

\[ 3x_1 = 3x_2 \]

\[ x_1 = x_2 \] (this is true for any value of \( x \)) \( \square \)

\( f(x) = x^2 \) injective

\[ f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2 \]

\[ \pm x_1 = \pm x_2 \] Not true. (\( -x_1 \neq x_2 \))
Show \( f(x) = 4x + 3 \) is surjective for \( x \in \mathbb{R}, \ y \in \mathbb{Z} \)

\[ \begin{align*}
  y &= f(x) \\
  y &= 4x + 3 \\
  y - 3 &= 4x \\
  \frac{y - 3}{4} &= x
\end{align*} \]

If we pick \( y = 0 \), \( x = -\frac{3}{4} \)

\[ y = 1 \quad x = -\frac{4}{4} = -1 \]

If \( f: \mathbb{R} \to \mathbb{R} \), then surjective

\( f: \mathbb{Z} \to \mathbb{Z} \)

\[ \begin{align*}
  y &= 0, \quad x = -\frac{3}{4} \notin \mathbb{Z}.
\end{align*} \]

So we cannot get \( y = 0 \) from set of inputs, \( y = 0 \) is not in the range.

Ex. \( f: \mathbb{R} \to \mathbb{Z} \)

If only integers, then the above is surjective.

Remark. A bijective function implies that \( |X| = |Y| \) size of domain equals size of codomain.

Therefore, considering a bijection is a way of cardinality of the set.

Def. Given \( f: X \to Y \), we define inverse as \( f^{-1}: Y \to X \)

\[ \begin{align*}
  f(x) &= y \\
  f^{-1}(y) &= x
\end{align*} \]

Ex. Find inverse of \( f(x) = 5x + 3 \)

To find inverse, need to prove injective and surjective (note surjective proof involves finding an inverse).

Ex. \( \log(a) \) \( f(x) = \log(x) \) if \( a \in \mathbb{R}^+ \) and \( a \neq 1 \) and \( \mathbb{R}^+ = \{ x \in \mathbb{R} : x > 0 \} \)

\( f: \mathbb{R} \to \mathbb{R}^+ \), defined by \( f(x) = a^x \) is a bijection. Its inverse \( f^{-1}: \mathbb{R}^+ \to \mathbb{R} \)

is the log with base \( a \): \( f^{-1}(x) = \log_a(x) \)

\[ y = \log_a(x) \iff a^y = x \]
If a function \( f \) is a bijection, then its inverse is also a bijection.

**Proof**

Let \( f: A \to B \) be a bijection and let \( f^{-1}: B \to A \) be its inverse.

1. \( f \) is injection and \( f^{-1} \) is surjection: \( f(f^{-1}(x)) = x \) for all \( x \in A \).
2. \( f \) is injection: \( f(f^{-1}(x)) = f(f^{-1}(y)) \) implies \( x = y \).

**Def.** *Composition*

Let \( f: A \to B \) and \( g: B \to C \). The composition of \( g \) with \( f \), denoted \( g \circ f \), is a function from \( A \) to \( C \) defined by \((g \circ f)(x) = g(f(x))\).

**Thm.** The composition of two injections is injective.

**Proof.** Let \( A, B, C \) be sets, let \( f: A \to B \) and \( g: B \to C \) be two injections.

Suppose \( x, y \in A \) s.t. \((g \circ f)(x) = (g \circ f)(y)\).

This means that \( g(f(x)) = g(f(y))\).

Since codomain of \( f \) is \( B \), \( f(x) \in B \) and \( f(y) \in B \)

Thus we have two elements of \( B \), \( f(x) \) and \( f(y) \) s.t. \( g(f(x)) = g(f(y)) \)

Since \( g \) is injective, we have \( f(x) = f(y) \).

We know \( f \) is injective so \( x = y \).
Bijections for Infinite Sets

* Can use bijection to verify that two sets have same cardinality
  
  What about sets that are infinite?

  \[ N = \{1, 2, 3, \ldots\} \]

  \[ E = \{-1, 1, 3, \ldots\} \]

  Map \( N \rightarrow E: n \mapsto 2n \) is a bijection

  What about \( N \rightarrow E \)

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-2</td>
<td>1</td>
<td>-4</td>
<td>2</td>
<td>-6</td>
<td>4</td>
<td>...</td>
</tr>
</tbody>
</table>

  \[ n \mapsto \begin{cases} 2n & \text{if } n \text{ even} \\ n+1 & \text{if } n \text{ odd} \end{cases} \]

  \[ N \rightarrow \mathbb{Z} \]

  \[ n \mapsto n/2 \text{ if } n \text{ even} \\ \frac{(n+1)}{2} \text{ if } n \text{ odd} \]

  Proof (one-to-one)

  \[ \text{Contradiction} \]

  Suppose \( f(x) = f(y) \)

  The maps both must have the same sign.

  Thus either \( f(x) = -\frac{x}{2} \) and \( f(y) = \frac{y}{2} \)

  So \( f(x) = f(y) \) implies \( \frac{x}{2} = \frac{y}{2} \) \( \Rightarrow x = y \)

  \[ \text{Dr.} \]

  \[ f(x) = -\frac{(x+1)}{2} \text{ and } f(y) = -\frac{(y+1)}{2} \]

  \[ f(x) = f(y) \]

  \[ \Rightarrow -\frac{(x+1)}{2} = -\frac{(y+1)}{2} \iff x = y \]

  \[ f(3) = -\frac{(3+1)}{2} \]

  \[ = -2 \]

  \[ f(1) = \frac{1}{2} \]

  \[ \frac{1}{2} = 0.5 \]

  \[ \text{Thus } x = y \]
Stern-Brocot Tree

- Rational numbers are not uniform but have a layered structure.
  - Can go deeper and deeper.
  - James Barning

```
        0/1
       /   
      1/1   1
     /     /   
    2/3   1/2  3/2
   /   /     /   
  3/5  2/4  5/3  3/1
 / / / / / / / / / /
1/4 5/3 4/3 3/1 9/1
```

Gives us insight into structure of Rational numbers.

- Branching out at each point.
- Binary Tree.

If we go down HFS 1/2, 1/3, 1/4, 1/5, RHS 1, 2, 3, 4, 5.

To go from one layer to the next:

- Mediants: \( \frac{a}{b} = \frac{c+d}{e+f} \) as \( \frac{1}{2} = \frac{1}{1} + \frac{1}{1} \)
- Add closely related. To obtain \( \frac{5}{3} \), add what is closest on right and left.
  \( \frac{5}{3} = \frac{3+2}{1+2} \)
Properties of Stern Brocot Tree

1) Excluding $\frac{0}{1}$ and $\frac{1}{0}$, every fraction (natural $\# 70$) appears exactly once and in reduced form.

2) The order is respected by horizontal position; i.e., larger fractions appear to the right of smaller ones.

3) Each row of the tree is symmetrical w.r.t. reciprocals.

\[
\frac{F}{X} = \frac{1}{4} \frac{2}{5} \frac{3}{5} \frac{3}{4} \frac{4}{3} \frac{5}{3} \frac{5}{2} \frac{4}{1}
\]

Let two fractions \(\frac{a}{b}, \frac{c}{d}\) in reduced form be adjacent if \(ad - bc = \pm 1\).

4) Any two neighbors in the Stern-Brocot tree are adjacent (consecutively by edge).

\[
\frac{1}{2}, \frac{2}{3}, 1, 3 - 2, 2 = -1, v
\]

5) Every fraction determines a unique region in the plane below it. Thus $S \rightarrow R_s$

6) Every fraction $\Phi$ on the boundary of $R_s$ is adjacent to $S$.

$\frac{2}{13}$ - $\frac{5}{14}$ adjacent
$\frac{2}{7} - \frac{3}{5} = -1$
7) If $r = \frac{a}{b}$ is in reduced form, define the simplicity of $r$ to be $l(r) = \frac{1}{ab}$.

(Take product & reciprocal)

Then the sum of the simplicities along any row of the Stern-Brocot tree is 1.

\[
\frac{2}{3} + \frac{3}{4} + \frac{3}{4} + \frac{3}{3} = \sum l(r) = \frac{1}{3} + \frac{1}{6} + \frac{1}{6} + \frac{1}{3} = 1
\]

8) Moving $LBRBLBLBL$ (Left/Right) down the tree for $\frac{1}{1}$ gives $\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \ldots$ a Fibonacci Path

---

A How to determine elements directly below a given one?

B) Look at Matrix form of SB tree Concrete Math Grahm, Knuth, Patashnik

Det A $2 \times 2$ matrix is an expression of $\text{Det} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \text{Det} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad - bc)

\[
\begin{pmatrix}
(a, b_1) + (a_1, b) & (a_1, b_1) \\
(c, d_1) + (c_1, d) & (c_1, d_1)
\end{pmatrix} = (ad - bc)
\]

\[
\begin{pmatrix}
(a, b_1) & (a, b_1) \\
(c, d_1) & (c, d_1)
\end{pmatrix} = (ad - bc)
\]
To get a new entry, look at parent

(2 1), parent: (1 1), Add 2nd column to first, leave 2nd where it is

\[
\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\]

To move right, do the opps

\[
(\text{Antiburn}) = \begin{pmatrix} a+b & b \\ c+d & d \end{pmatrix}
\]

\[
(\text{Cuts and burn}) = \begin{pmatrix} a+b & b \\ c+d & d \end{pmatrix}
\]

L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}  
R = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}  
M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}  
ML = MR
Every Matrix that appears is either $I$ or a product of $L$'s and $R$'s.

Then, every real number $r > 0$ can be associated to a unique (Gödel) binary sequence of $L$'s and $R$'s with $r = 1$ associated to the empty sequence.

Example:

1. $L L L R L L \leftrightarrow (2 \ 1) \leftrightarrow \frac{3}{11}$

2. $R R L L R L \leftrightarrow (12 \ 7) \leftrightarrow \frac{19}{8}$

Q1: How to go backwards from a real number $r > 0$ to a sequence of $L$'s and $R$'s?

A1: Theory of continued fractions.

Example:

1. $\frac{3}{11} = 3 + \frac{1}{3 + \frac{1}{3 + \frac{1}{3 + \frac{1}{1 + \frac{1}{2}}}}} = \frac{1}{1 + \frac{1}{2}} \leftrightarrow L L L R L L$

2. $\frac{19}{8} = 2 + \frac{3}{8} = 2 + \frac{1}{8} = 2 + \frac{1}{2 + \frac{2}{3}} = 2 + \frac{1}{2 + \frac{1}{3 + \frac{1}{2}}} \leftrightarrow R R L L R L$
