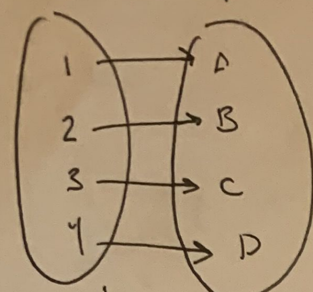
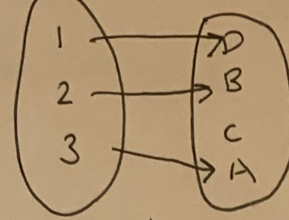


bijective



injective only

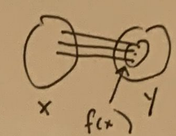
Injective

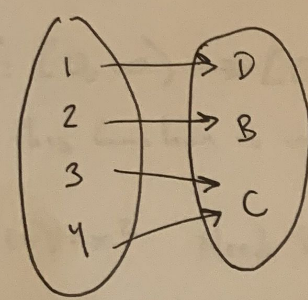
$f: X \to Y$

X: domain

Y: codomain

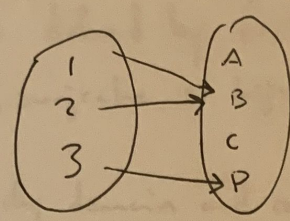
$f(x)$: image





Surjective only

Surjective



general

non-injective

non-surjective

For any function

$\forall x \in X, \exists! y \in Y$ s.t. $y = f(x)$

Def A function is injective (one-to-one) if each element in codomain mapped to at most one element of domain

or equivalently, if distinct elements of the domain map to distinct elements of the codomain

$$\forall x, x' \in X,\ f(x) = f(x') \Rightarrow x = x'$$

or contrapositive $\forall x, x' \in X,\ x \neq x' \Rightarrow f(x) \neq f(x')$

A function is surjective (onto) if each element of the codomain is mapped to at least one element of the domain.

That is, the image and the codomain are equal.

$$\forall y \in Y, \exists x \in X \text{ s.t. } y = f(x)$$

A function is bijective (one to one and onto or invertible) if each element in the codomain is mapped to exactly one element of the domain. That is, both injective and surjective

$\cancel{\forall x \in X, \exists! x}$ $\forall y \in Y, \exists! x \in X$ s.t. $y = f$

$a \to x$

$b \to y$

$c \to z$

$d \to z$

(surjection not injection)

Ex2) $a \to v$

$b \to w$

x

$c \to y$

$d \to z$

Injection not surjection

Ex 3

$f: [0,\infty) \to [0,\infty)$ defined by $f(x) = \sqrt{x}$

this function is injective, surjective so bijective

Ex 4 $f(x) = x^2$ Need to specify domain and codomain

If domain and codomain are $\mathbb{R}$, then $f(x)$ neither injection nor surjection.

Not a surjection b/c range is not equal to codomain. For example there is no number in the domain with image -1 which is an element of codomain.

It is not an injection b/c more than one distinct element in domain is mapped to the same element in the codomain. For example, $f(-1) = f(1)$ but $(-1) \neq (1)$

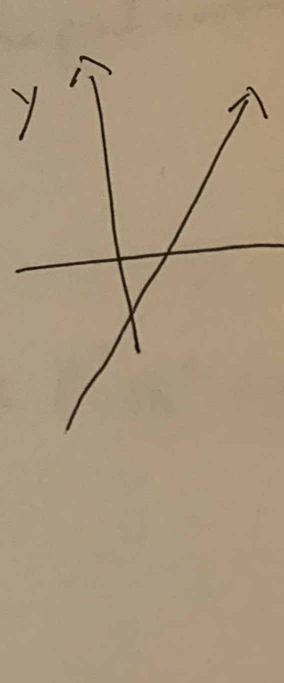
$\Rightarrow$ what if domain is $\mathbb{R}$ and codomain is $[0,\infty)$
what properties would the function have? (injective / surjective)

$\Rightarrow$ what if domain and codomain are both $[0,\infty)$

Ex show $f(x) = 3x - 7$ injective

Need to show $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ is easier to show



$3x_1 - 7 = 3x_2 - 7$

$3x_1 = 3x_2$

$x_1 = x_2$ (this is true for any value of x) $\square$

$f(x) = x^2$ injective

$f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2$

$\pm x_1 = \pm x_2$ Not true $(-x_1 \neq x_2)$

Ex Show $f(x) = 4x+3$ surjective for $\forall x \in \mathbb{R}$, $\forall x \in \mathbb{Z}$

$y = f(x)$ If we pick $y=0$ $x = \frac{-3}{4}$

$y = 4x+3$ $y=1$ $x = -\frac{4}{4} = -1$

$y-3 = 4x$ if $f: \mathbb{R} \to \mathbb{R}$, then surjective

$\frac{y-3}{4} = x$ $f: \mathbb{Z} \to \mathbb{Z}$

$y = 0$, $x = \frac{-3}{4} \notin \mathbb{Z}$.

So we cannot get $y=0$ from set of integers

$y=0$ is not in the range.

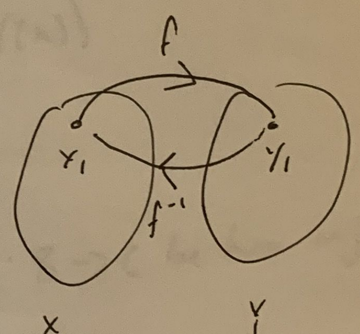
Ex $f: \mathbb{R} \to \mathbb{Z}$

Y only integers, then the above is surjection.

Remark, A bijective function implies that $|X| = |Y|$ size of domain equals size of codomain, cardinalities are the same.

Therefore, constructing a bijection is a way of

Def Given $f: X \to Y$, we define inverse as $f^{-1}: Y \to X$



$f(x) = y$

$f^{-1}(y) = x$

Ex find inverse of $f(x) = 5x+3$

To find inverse, need to prove injective and surjective (note surjective proof involving finding an inverse).

Ex ~~$f(x)$~~ $f(x) = \ln(x)$ if $a \in \mathbb{R}_+$ and $a \neq 1$ and $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$

$f: \mathbb{R} \to \mathbb{R}_+$ defined by $f(x) = a^x$ is a bijection. Its inverse $f^{-1}: \mathbb{R}^+ \to \mathbb{R}$ is the log with base a: $f^{-1}(x) = \log_a(x)$

$y = \log_a(x)$ iff $a^y = x$

LMM

If a function f is a bijection, then its inverse is also a bijection

Proof

Let $f: A \to B$ be a bijection and let $f^{-1}: B \to A$ be its inverse

Need to show: 1) Injection and 2) Surjection $\quad f(x) = f(x') \Rightarrow x = x'$
(f^{-1})

1)
$x_1, x_2 \in B$ s.t $f^{-1}(x_1) = f^{-1}(x_2)$

Then by def of inverse $x_1 = f(f^{-1}(x_2)) = x_2$ so f^{-1} is injective

2) $\forall x \in A \ \exists y \in B$ s.t $y = f(x)$

$f^{-1}(y) = f^{-1}(f(x)) = x$ since f^{-1} is inverse of f

$\Rightarrow \forall x \in A, \ \exists y \in B$ s.t. $f^{-1}(y) = x \Rightarrow f^{-1}$ surject

Def (Composition)

Let $f: A \to B$ and $g: B \to C$. The composition of g with f, denoted $g \circ f$ is a function from A to C defined by $(g \circ f)(x) = g(f(x))$

Thm The composition of two injective functions is injective

Proof Let A, B, C be sets, let $f: A \to B$ and $g: B \to C$ be two injections

Suppose $x, y \in A$ s.t. $(g \circ f)(x) = (g \circ f)(y)$

This means that $g(f(x)) = g(f(y))$

Since codomain of f is B, $f(x) \in B$ and $f(y) \in B$

Thus we have two elements of B, $f(x)$ and $f(y)$ s.t. $g(f(x)) = g(f(y))$

Since g injective, we have $f(x) = f(y)$

We know f injective so $x = y$ $\square$

Bijections for Infinite Sets

- Can use bijections to verify that two sets have same cardinality. what about sets that are infinite?

$\mathbb{N} = \{1, 2, 3, \ldots\}$

$E = \{2, 4, 6, \ldots\}$

Map $\mathbb{N} \to E: n \to 2n$ is a bijection

$\tilde{\mathbb{E}} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

What about

$\mathbb{N} \to \tilde{\mathbb{E}}$

0 1 2 3 4 5 6 ...

0 2 -2 4 -4 6 -6 ...

$$n \mapsto \begin{cases} -n & \text{if } n \text{ even} \\ n+1 & \text{if } n \text{ odd} \end{cases}$$

$\mathbb{N} \to \mathbb{Z}$

$$n \to \begin{cases} n/2 & \text{if } n \text{ even} \\ -\frac{(n+1)}{2} & \text{if } n \text{ odd} \end{cases}$$

Proof (onto)

if y positive, then

$f(2y) = y$

Therefore y has a pre-image.

If y negative, then $f(-2y+1)) = y$

Therefore y has a preimage

Therefore f is onto.

<u>Proof</u> (one-to-one)

Contradiction

Suppose $f(x) = f(y)$

Then they both must have same sign.

Then either

$f(x) = \frac{x}{2}$ and $f(y) = \frac{y}{2}$

So $f(x) = f(y) \Rightarrow$

$\frac{x}{2} = \frac{y}{2} \Rightarrow x = y$

or,

$f(x) = \frac{-(x+1)}{2}$ $f(y) = -\frac{(y+1)}{2}$

$f(x) = f(y)$

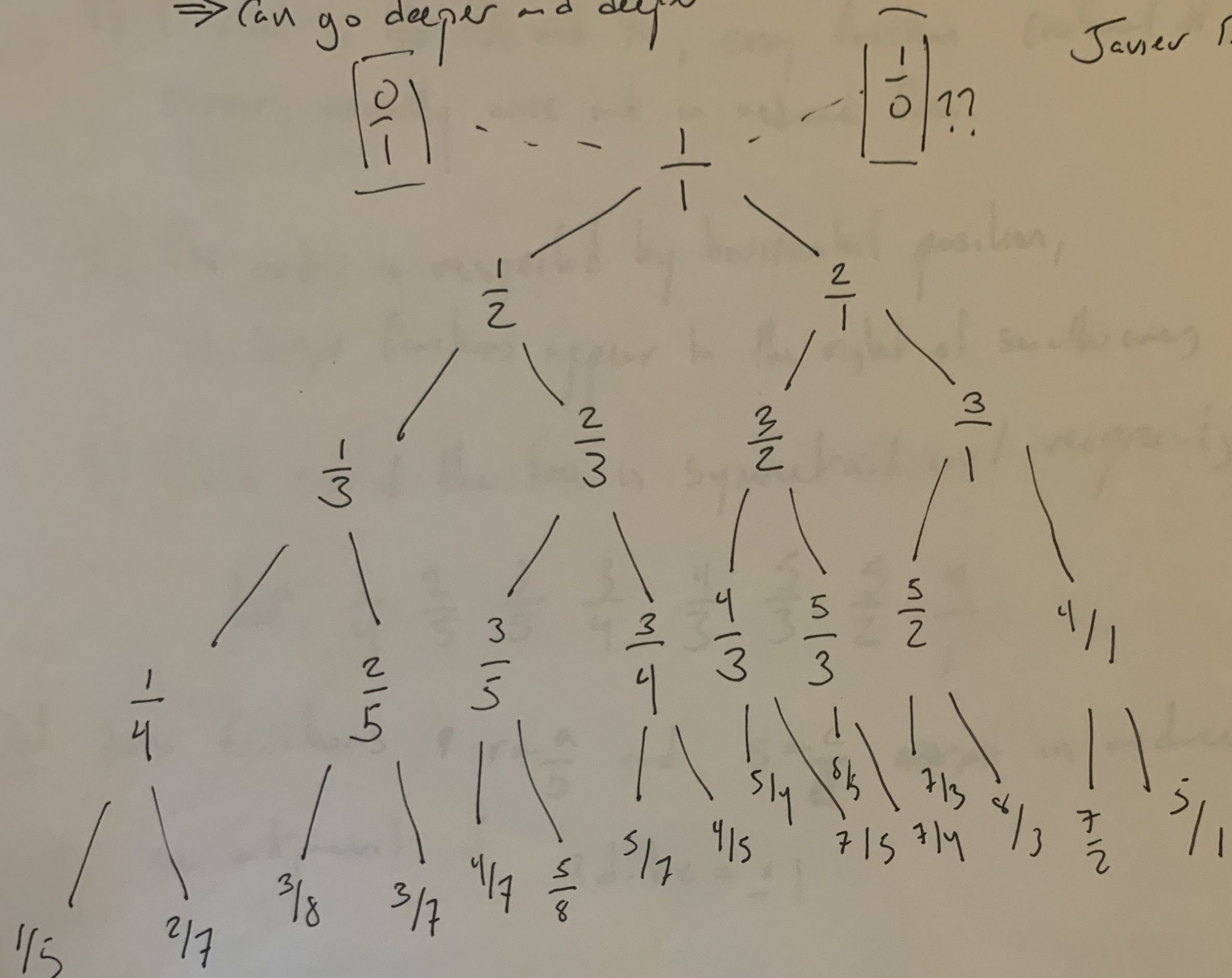
$\Rightarrow \frac{-(x+1)}{2} = -\frac{(y+1)}{2} \Rightarrow x = y$

→ Stern Brocot Tree

- Rational numbers are not uniform but have a layered structure

⇒ Can go deeper and deeper

Javier Burroni



Gives us insight into structure of Rational numbers

- Branching out at each point
- Binary Tree
- If we go down LHS 1/2, 1/3, 1/4, 1/5, RHS 1, 2, 3, 4, 5

To go from one layer to the next

Mediants $\frac{a}{b} = \frac{c+d}{e+f}$ for $\frac{c}{d}$ and $\frac{e}{f}$

~~Add two closest numbers to~~ To obtain $\frac{5}{3}$, add mediant of two closest on right and left

$\frac{5}{} = \frac{3+2}{}$

Properties of Stern Brocot Tree

1) Excluding ~~[illegible]~~ 0/1 and 1/0, every fraction (rational # > 0) appears exactly once and in reduced form.

2) The order is respected by horizontal position, i.e. larger fractions appear to the right of smaller ones

3) Each row of the tree is symmetrical wrt reciprocals

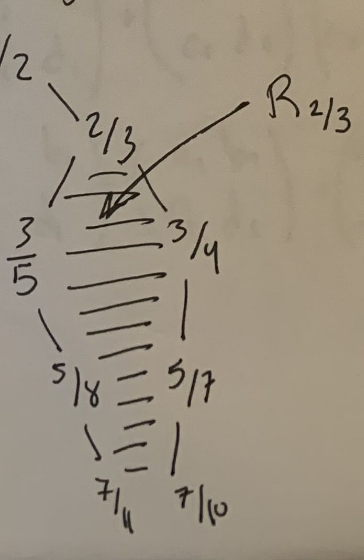
<u>Ex</u> $\frac{1}{4}\ \frac{2}{5}\ \frac{3}{5}\ \frac{3}{4}\ \frac{4}{3}\ \frac{5}{3}\ \frac{5}{2}\ \frac{4}{1}$

<u>Def</u> Two fractions $r=\frac{a}{b}$ and $s=\frac{c}{d}$ ~~[illegible]~~ in reduced form are adjacent if $ad-bc=\pm 1$

4) Any two neighbors in the Stern-Brocot tree are adjacent (connected by edge)

1/2 \ 2/3 $\quad 1\cdot 3-2\cdot 2=-1$ ✓

5) Every fraction determines a unique region in the plane below it. Thus $s \to R_s$



6) Every fraction ~~f~~ r on the boundary of R_s is adjacent to s.

(2/3)

2/3 and 5/7 adjacent

$2\cdot 7-3\cdot 5=-1$

7) If $r = a/b$ is in reduced form,

Define the simplicity of r to be $\ell(r) = \frac{1}{ab}$.

(Take product & reciprocal)
(complexity)

Then the sum of the simplicities along any row of the Stern-Brocot tree is 1

Ex $\frac{1}{3}\ \frac{2}{3}\ \frac{3}{2}\ \frac{3}{1} = \sum_r \ell(r) = \frac{1}{3} + \frac{1}{6} + \frac{1}{6} + \frac{1}{3} = 1$

8) Moving LRLRL... (Left / Right) down the tree from $\frac{1}{1}$

gives $\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \ldots$ a Fibonacci Path

---

Q How to determine elements directly below any given entry?
(children)

A) Look at Matrix form of SB tree   Concrete Math
Graham, Knuth, Patashnik

(informal)
Def A 2x2 matrix is an expression of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Operations

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{pmatrix}$$

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2+b_1c_2 & a_1a_2+b_1d_2 \\ - & - \; - \end{pmatrix}$$

Matrix form of SB Tree

Identity $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

$\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$ $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$

$\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$ $\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}$ $\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}$ $\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$

$\begin{pmatrix}1 & 0\\ 3 & 1\end{pmatrix}$ $\begin{pmatrix}1 & 1\\ 2 & 3\end{pmatrix}$ $\begin{pmatrix}2 & 1\\ 3 & 2\end{pmatrix}$ $\begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix}$ $\begin{pmatrix}3 & 1\\ 2 & 1\end{pmatrix}$ $\begin{pmatrix}2 & 3\\ 1 & 2\end{pmatrix}$ $\begin{pmatrix}3 & 2\\ 1 & 1\end{pmatrix}$ $\begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix}$

To get A new entry, look at parent

To move left:

$\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}$, parent = $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, Add 2nd column to first, leave 2nd where it is

$$\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix} = \begin{pmatrix}1+1 & 1\\ 0+1 & 1\end{pmatrix}$$

$$\begin{pmatrix}a_{n+1} & b_{n+1}\\ c_{n+1} & d_{n+1}\end{pmatrix} = \begin{pmatrix}a_n+b_n & b_n\\ c_n+d_n & d_n\end{pmatrix}$$

$L = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$ $R = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$

M

ML MR

To move right, do the opposite

To convert to fraction

$\begin{matrix}a & b\\ c & d\end{matrix} \Rightarrow \frac{a+b}{c+d}$

Every Matrix that appears is either I or a product of L's and R's

<u>Thm</u>

Every rational number $r>0$ can be associated to a unique (finite) binary sequence of L's and R's with $r=1$ associated to the empty sequence

<u>Ex</u>

i) $LLLRL \Leftrightarrow \begin{pmatrix} 2 & 1 \\ 7 & 4 \end{pmatrix} \Leftrightarrow \frac{3}{11}$

ii) $RRLLRL \Leftrightarrow \begin{pmatrix} 12 & 7 \\ 5 & 3 \end{pmatrix} \Leftrightarrow \frac{19}{8}$

Q) How to go backwards from a rational number $r>0$ to a sequence of L's and R's?

A) Theory of continued fractions

<u>Ex</u> i) $$\frac{3}{11} = \frac{1}{11/3} = \frac{1}{3+2/3} = \frac{1}{3+\frac{1}{3/2}} = \frac{1}{3+\frac{1}{1+\frac{1}{2}}}$$

$$= \frac{1}{3+\frac{1}{1+\frac{1}{1+1}}} \Longleftrightarrow LLLRL$$

ii) $$\frac{19}{8} = 2+\frac{3}{8} = 2+\frac{1}{8/3} = 2+\frac{1}{2+\frac{2}{3}} = 2+\frac{1}{2+\frac{1}{3/2}}$$

$$= 2+\frac{1}{2+\frac{1}{1+\frac{1}{2}}} = 2+\frac{1}{2+\frac{1}{1+\frac{1}{1+1}}} \Longleftrightarrow RRLLRL$$