Let's define a seq using SB Tree
Stern's Diclonic sequence

let $a_0 = D_0, a_1 = 1$, $a_{2n} = a_n$, $a_{2n+1} = a_n + a_{n+1}$

let's check $a_1 = 1$

$a_2 = a_1 = 1$

$a_3 = a_1 + a_2 = 1 + 1 = 2$

$a_4 = a_2 = 1$

$a_5 = a_3 + a_2 = 2 + 2 = 3$

$a_6 = a_3 = 2$

$a_7 = a_3 + a_4 = 2 + 1 = 3$

$a_8 = a_4 = 1$

So $(a_k^2) = 0, 1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, ...$

We can visualize this series with a 2D array (Stern's Diclonic Array)

```
1
1 2
1 2 3
1 4 3 5
1 5 4 3 8 5 7 2 7 5 8 3 7 9 5 1
```

- $a_1$
- $a_2$
- $a_3$
- $a_4$
- $a_5$
- $a_6$
- $a_7$
- $a_8$
- $a_9$
- $a_{10}$
- $a_{11}$
- $a_{12}$
- $a_{13}$
- $a_{14}$
- $a_{15}$
- $a_{16}$

First row has two terms after zero in SPS (a1, a2)

Given n'th row, generate n+1 th by copying n'th row, but inserting a new term between two terms that is their sum.

Similar to Pascal's Triangle (we will define when we introduce the Binomial Theorem). Each row is palindrome, also

Claim: $a_{2^n + k} = a_k + a_{2^n - k}$
Let's sketch a bijection from $\mathbb{N}$ to $\mathbb{Q}^+$: $f: \mathbb{N} \rightarrow \mathbb{Q}^+$

Using the series

\[ f(n) = \frac{a_n}{a_{n+1}} \]

Let $f(1) = \frac{a_1}{a_1} = \frac{1}{1}$

\[ f(2) = \frac{a_2}{a_{2+1}} = \frac{1}{2} \]

\[ f(3) = \frac{a_3}{a_{3+1}} = \frac{2}{1} \]

\[ f(4) = \frac{a_4}{a_{4+1}} = \frac{1}{3} \]

\[ f(5) = \frac{a_5}{a_{5+1}} = \frac{5}{2} \]

Wait.

There is an algorithm that will give us a bijection from $\mathbb{N} \rightarrow \mathbb{Q}^+$

Using the SB tree.

It is an infinite binary tree

(actually it's an infinite binary search tree)

So what tree traversal algorithms have you heard of?

Let $f(n) = \frac{a_n}{a_{n+1}}$

We can now say that the $5^{th}$ rational number is $\frac{5}{2}$.

Claim $f: \mathbb{N} \rightarrow \mathbb{Q}^+$ is a bijection [show this in HW]

Since $f$ a bijection, $f^{-1}$ exists. Given rational, we can find natural

**Ex.** Given $q = \frac{1}{4}$, Can we find $n$ s.t. $f(n) = \frac{1}{4}$?

Verify on HW

\[ f^{-1}(1) = 1 \]

\[ f^{-1}(q) = 2f^{-1}\left(\frac{q}{1-q}\right) \text{ if } q < 1 \]

\[ f^{-1}(q) = 2f^{-1}(q-1)+1 \text{ if } q > 1 \]

**Ex.** $f^{-1}\left(\frac{3}{2}\right) = 2f^{-1}\left(\frac{3/2}{1-3/2}\right)+1 = 2f^{-1}\left(\frac{1}{2}\right)+1 = 5$
Now that we have a bijection from $\mathbb{N} \to \mathbb{Q}^+$, we can extend to $\mathbb{N} \to \mathbb{Q}$ by using $\mathbb{N} \to \mathbb{Z}$ and writing $\mathbb{Z} \to \mathbb{Q}$.

Our bijection from $\mathbb{N} \to \mathbb{E}$:

\[
\begin{align*}
0, 1, 2, 0, 2, 4, \\
& n \mapsto \left\{ \\
& \begin{array}{ll}
& n \text{ even?} \quad \text{or} \quad \frac{n}{2} \text{ even?} \\
& \text{yes} \quad \text{or} \quad \text{yes} \\
& \text{no} \quad \text{or} \quad \text{no} \\
& \end{array} \\
& \end{align*}
\]

Take $g(z) = \begin{cases} 
\frac{a_2}{a_2 + 1} & \text{if } z > 0 \\
-\frac{a_2}{a_2 - 1} & \text{if } z < 0 \\
0 & \text{if } z = 0
\end{cases}$.

This is a bijection from $\mathbb{Z} \to \mathbb{Q}$.

We can write $\mathbb{Q} \to \mathbb{N}$ by using function composition.

Take inverse of $g$, $g^{-1}$:

\[
g^{-1}(q) = \begin{cases} 
2f^{-1}(q - 1) + 1 & \text{if } q > 1 \\
1 & q = 1 \\
2f^{-1}(\frac{q}{1-q}) & 0 < q < 1 \\
0 & q = 0 \\
-2 \left[ f^{-1}\left(\frac{-q}{1+q}\right) \right] & -1 \leq q < 0 \\
-1 & q = -1 \\
-2 \left[ f^{-1}(q-1) \right] + 1 & q > -1
\end{cases}
\]

Define inverse $h^{-1}$:

\[
h^{-1}(z) = \begin{cases} 
\mathbb{Z} & z > 0 \\
1 & z = 0 \\
-\mathbb{Z} & z < 0
\end{cases}
\]

Then $h^{-1} \circ g^{-1} : \mathbb{Q} \to \mathbb{N}$.
Given any two numbers \( a, b \in \mathbb{R} \), \( \exists r \in \mathbb{Q} \) s.t. 
\[
a < r < b
\]
rationals are dense in \( \mathbb{R} \).

Even though \( \mathbb{Q} \) is not all of \( \mathbb{R} \), it kind of fills up space b/c the implication is that every real number \( (\pi, e, \sqrt{2}) \) can be approximated by rational numbers 
\[
a, r, b
\]
\[
\mathbb{Q}
\]
\[
\mathbb{R}
\]

Proof

Suppose \( a < b \), we want to show
\[
\exists r \text{ s.t. } r = \frac{m}{n} \text{ and } a < \frac{m}{n} < b
\]

1) Look at denominator

We know \( a < b \) so \( b - a > 0 \) (small positive number that is our unit of measurement)

Def (Archimedean Property)

For, every positive rational \( \frac{m}{n} \) where \( m, n \in \mathbb{Z}^+ \), if we add

More than \( n \) copies, the resulting sum is more than 1.

\[
\Rightarrow \exists n \in \mathbb{Z}^+ \text{ s.t. } n(b - a) > 1 \Rightarrow b - a > \frac{1}{n} > a
\]

This says that even though \( b \) and \( a \) are close, they're actually not that close. Therefore we find the denominator.

2) Look at numerator (assume \( b > a > 0 \) for the other case)

We'll adapt by flipping equally and negate signs.

Idea. Start with 0 remembering \( N \) is fixed.

\[
0 \leq \frac{1}{N} \leq a \leq \frac{m}{n} \leq b \leq \frac{M}{N}
\]

Successively add \( \frac{1}{N} \) until we reach the last number \( \frac{M}{N} \) that is less than \( b \).

i.e. \( \frac{M}{N} > b \)
It turns out that the last element is what solves the problem
\[
(r = \frac{M}{N})
\]
Let \( S = \left\{ \frac{M}{N} \mid M=0,1,2,3; -\frac{M}{N} < b^3 \right\} \)
\( S \) is non-empty b/c 0 ∈ \( S \) (b is positive)
by construction, \( S \) is bounded above by \( b \).
We'll use this idea about real numbers called a HUB or supremum.
⇒ intuitively think of it like a maximum; with one caveat.
⇒ heh say you did not get best grade in class. You think who got a better grade than me. If something is not a supremum, something else is bigger than it.

Sketch: \( x \) is an upper bound of \( S \) if \( x \geq S \).

If \( x \) a partially ordered set and \( S \) a subset then \( S \) is \( \text{sup}(S) \)

1) \( S \leq S \) \( \forall S \) \( S \leq S \) \( \forall S \)
2) if \( x \in S \), \( S \leq x \) \( \forall S \leq x \)

Max
\( m = \text{max} \) of \( S \)
1. \( S \leq S \) \( \forall S \)
2. \( S \leq S \)
\( x \in \mathbb{R} \) \( S \leq S \) \( \forall S \leq S \)
\( x \in \mathbb{R} \) \( S \neq S \) \( \forall S \neq S \)

Note: If \( S \) has a max, \( M \) must be the \( \text{sup} \).
If \( x \in X \), \( S \leq x \) \( \forall S \leq x \)
But in \( \mathbb{R}^1 \), possible to have a sup but not a max.
Ex. \( \mathbb{R}^- \) does not have a max, but does have a sup 0.
Let \( S = \{ \frac{M}{N} : M = 0, 1, 2, \ldots, \frac{M}{N} \leq b \} \).

Then:
1) \( S \neq \emptyset \) (\( 0 \in S \))
2) \( S \) bounded above by \( b \)

\( \text{Sup}(S) = r \)

Claim: \( r \) solves our problem. \( r \) is related \( b \) by \( a \).

\( S \) is a finite set. Process has to stop when \( r \) goes above \( b \).

Add \( 1/N \) each time \( r \) goes over \( b \).

\[ \Rightarrow S \text{ is finite} \Rightarrow \text{Sup}(S) = \text{max}(S) = r \Rightarrow r \in S \]

Thus, \( r \) related and \( r \leq b \)

3) Next, show \( r \geq a \).

Suppose \( r = \frac{M}{N} \leq a \). Then \( b - a > \frac{1}{N} \Rightarrow b > a + \frac{1}{N} \)

\[ \geq \frac{M + 1}{N} = \frac{M + 1}{N} \]

So \( \frac{M}{N} \leq b \Rightarrow \frac{M + 1}{N} \in S \)

but \( \frac{M + 1}{N} > \frac{M}{N} = r \)

So \( r \) had a limit in \( S \) that is bigger than \( r \)

but that contradicts the fact that \( r = \text{max}(S) \). \( \square \)
The set \( \mathbb{R} \) is uncountable (there is no bijection between \( \mathbb{N} \to \mathbb{R} \)).

Let's look at a simpler problem and focus on the interval \((0,1)\).

**Claim:** \( \mathcal{P} \times \mathcal{P} (0,1) \) is uncountable.

**Proof:** Cantor’s Diagonalization Argument

Suppose we can list all the numbers in

\[
\begin{align*}
a_1 &= 0.a_{11} a_{12} a_{13} \ldots \\
a_2 &= 0.a_{21} a_{22} a_{23} \\
a_3 &= 0.a_{31} a_{32} a_{33} \ldots
\end{align*}
\]

where \( a_{ij} \) are the decimal digits

\[a_{ij} \in \{0, \ldots, 9\}\]

Consider the diagonal \( d = 0.a_{11} a_{22} a_{33} a_{44} \ldots \)

Construct \( x = 0.x_1 x_2 x_3 \) with the property that \( x_i \neq a_{ii} \) and \( x_i \neq 9 \).

**Q:** Why \( x_i \neq 9 \)? Why avoid \( 0.999\ldots = 1.000 \ldots = 1-0.000\ldots = 1 \)?

For the same reason that \( 1 = (0.222\ldots)_3 \) in ternary

\[
1 = \sum_{n=1}^{\infty} \left( \frac{2}{3} \right)^n
\]

The claim is that \( x \) is not on the list of reals.

- \( x \neq a_{ii} \) by def, so \( x \) cannot be \( a_i \).
- \( x \neq a_{jj} \) by def, so \( x \) cannot be \( a_j \).
- \( x \) is not equal to \( a_i \) \( \forall i \), therefore \( x \) is not on the list.

If \( x = a_i \) for some \( i \), then the \( i \)th digit of \( x \) would be equal to \( a_i \).

But \( x \neq a_i \), by construction.

\[\Rightarrow\] Even if you ranked all the reals, there will always be one missing.