Discrete Mathematics

COMS 3203 – Fall 2017
http://www.cs.columbia.edu/~amoretti/3203

Practice Exam # 3

Solve any six problems for full marks. **Good luck and don't panic!** If something is taking too long, move on to the next question. Note that this is a sample exam and while it bears some similarity with the real exam, the two are not isomorphic.

Problem 1

1. $a_n = 6a_{n-1} - 9a_{n-2}$ when $a_0 = 2, a_1 = 21$

$$r^2 - 6r + 9 = 0 \tag{1}$$
$$(r-3)(r-3) = 0 \tag{2}$$
$$a_n = \alpha(3)^n + \beta \cdot n \cdot (3)^n \tag{3}$$
$$a_0 = 2 \ \ = \alpha(3)^0 + \beta \cdot 0 \cdot (3)^0 \tag{4}$$
$$\alpha = 2 \tag{5}$$
$$a_1 = 21 = \alpha(3)^1 + \beta \cdot 1 \cdot (3)^1 \tag{6}$$
$$21 = 6 + 3\beta \tag{7}$$
$$\beta = 5 \tag{8}$$
$$a_n = 2(3)^n + 5 \cdot n \cdot (3)^n \tag{9}$$

2. $a_n = 4a_{n-1} + 5a_{n-2}$ when $a_0 = 2, a_1 = 6$

$$r^2 - 4r - 5 = 0 \tag{10}$$
$$(r+1)(r-5) = 0 \tag{11}$$
$$a_n = \alpha(-1)^n + \beta(5)^n \tag{12}$$
$$a_0 = 2 = \alpha + \beta \tag{13}$$
$$a_1 = 6 = -\alpha + 5\beta \tag{14}$$

Add the two equations to solve the system

$$8 = 6\beta \qquad \beta = \frac{8}{6}, \alpha = \frac{4}{6} \tag{15}$$
$$a_n = \frac{4}{6}(-1)^n + \frac{8}{6}(5)^n \tag{16}$$

3. $a_n = 2a_{n-1} + 1$ when $a_1 = 1$

$$a_2 = 2 \cdot 1 + 1 = 3 \tag{17}$$
$$a_3 = 2 \cdot 3 + 1 = 7 \tag{18}$$
$$a_4 = 2 \cdot 7 + 1 = 15 \tag{19}$$

Each value is twice the previous minus one.

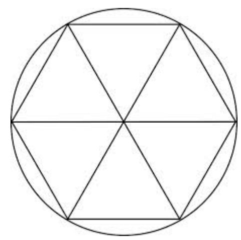
$$a_n = 2 \cdot 2^n - 1 \tag{20}$$

4. $na_n = (n-2)a_{n-1} + 2$ when $a_1 = 1$
   Recall Gauss' formula $n(n-1) = 2\sum_{i=1}^{n} i$. Multiply by $(n-1)$.

$$\begin{aligned}
n(n-1)a_n &= (n-1)(n-2)a_{n-1} + 2(n-1) && (21)\\
&= (n-2)(n-3)a_{n-2} + 2(n-2) + 2(n-1) && (22)\\
&= (n-3)(n-4)a_{n-3} + 2(n-3) + 2(n-2) + 2(n-1) && (23)\\
&= 2(1 + \cdots + n - 1) && (24)\\
n(n-1)a_n &= n(n-1) && (25)\\
a_n &= 1 && (26)
\end{aligned}$$

Problem 2

Suppose we select two points randomly on the unit circle $x^2 + y^2 = 1$. What is the probability that the chord joining the two points has length at least 1? How many points are necessary to guarantee that between two of them, there is a chord of length less than 1?



Proof. Consider inscribing equilateral triangles within the unit circle such that each shares a vertex at the origin and the remaining vertices are located on the circumference of the circle. The radius of the unit circle is 1 and so for *any* inscribed triangle, the two remaining vertices define a chord of length 1 and an inscribed angle $\theta = 60$. Given one point randomly chosen on the circumference, the inscribed angle formed with the next point chosen must be ≥ 60 degrees in order to define a chord of length ≥ 1. The set of points that define a chord of length 1, conditioned on the location of a point on the circumference, form an inscribed angle of 240. The ratio of 240/360 yields 2/3 as the probability. Seven points are needed to guarantee there is a chord of length less than 1 between them. □

Problem 3

How many members of the set $S = \{1, 2, 3, \cdots, 105\}$ have nontrivial factors in common with 105? Hint: use the inclusion-exclusion principle.

Proof. 105 has a prime factorization $105 = 3 \cdot 5 \cdot 7$, so elements in S will have common factors with 105 if they are divisible by $3, 5$ or 7. Define A as elements of S divisible by 3, B as elements of S divisible by 5 and C as elements divisible by 7. There are 35 numbers from 1 to 105 divisible by 3 so the subset A contains 35 elements. Similarly there are 21 elements in the subset B and 15 elements in the subset C. Consider A$\cap B$, the subset of elements divisible by both 3 and 5. There are seven numbers between 1 and 105 divisible by 15 (we had worked this out in checking that our formula for the recurrence in question 1.2 was correct), therefore $|A \cap B|$ = 7. Similarly $A \cap C$ is the subset of elements divisible by both 3 and 7 (21) and $B \cap C$ is the subset of elements divisible by both 5 and 7 (35). Therefore $|A \cap C| = 5$ and $|B \cap C| = 3$. The only number divisible by 105 is 105 so $|A \cap B \cap C| = 1$. Applying the inclusion-exclusion principle:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \tag{27}$$
$$= 35 + 21 + 15 - 7 - 5 - 3 + 1 \tag{28}$$
$$= 57 \tag{29}$$

□

Problem 4

The Poisson distribution is defined below.

$$P(X = x) = e^{-\lambda}\frac{\lambda^x}{x!}$$

Show that the variance of the Poisson distribution is equal to its mean (λ).

Proof.

$$\begin{aligned} Var(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 && (30)\\ &= \mathbb{E}(X(X-1) + X) - \mathbb{E}(X)^2 && (31)\end{aligned}$$

Apply linearity of expectation:

$$= \mathbb{E}(X(X-1)) + \mathbb{E}(X) - \mathbb{E}(X)^2 \tag{32}$$

Recall $\mathbb{E}(X) = \lambda$

$$= \sum_{x=0}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{x!} + \lambda - \lambda^2 \tag{33}$$

Note $\sum_{x=0}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=2}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{x!}$ since the first two values of the series are zero.

$$= \sum_{x=2}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{x\cdot(x-1)\cdot(x-2)!} + \lambda - \lambda^2 \tag{34}$$

Cancel terms:

$$= \sum_{x=2}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-2)!} + \lambda - \lambda^2 \tag{35}$$

Express as Taylor series by pulling out λ^2:

$$= \sum_{x=2}^{\infty} \lambda^2\frac{e^{-\lambda}\lambda^{x-2}}{(x-2)!} + \lambda - \lambda^2 \tag{36}$$

Pull out of the summation and simplify:

$$= \lambda^2 e^{-\lambda}\sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda - \lambda^2 = \lambda \tag{37}$$

□

Problem 5

Show that $\mathbb{E}_Y\Big(\mathbb{E}_X(X|Y)\Big) = \mathbb{E}_X(X)$.

Proof. Recall $\mathbb{E}f(x) = \sum_x f(x)p(x)$

$$\mathbb{E}_Y\Big(\mathbb{E}_X(X|Y)\Big) = \mathbb{E}_Y\Big(\sum_x x \cdot P(X = x|Y = y)\Big) \tag{38}$$

$$= \sum_y \Big(\sum_x x \cdot P(X = x|Y = y)\Big)P(Y = y) \tag{39}$$

Recall $P(X, Y) = P(X|Y)P(Y)$

$$= \sum_y \sum_x x \cdot P(X = x, Y = y) \tag{40}$$

$$= \sum_x \sum_y x \cdot P(X = x, Y = y) \tag{41}$$

$$= \sum_x x \cdot P(x = x) \tag{42}$$

$$= \mathbb{E}_X(x) \tag{43}$$

□

Problem 6

Consider two Gaussian functions:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma_f^2}} e^{-(x-\mu_f)^2/2\sigma_f^2} \qquad g(x) = \frac{1}{\sqrt{2\pi\sigma_g^2}} e^{-(x-\mu_g)^2/2\sigma_g^2}$$

Take their product $z(x) = f(x) \cdot g(x)$. Expand terms in the exponent to show that the product is also a Gaussian function. Hint: you will need to complete the square to factor the polynomial. What is the mean of $z(x)$ denoted μ_z and standard deviation σ_z as a function of $\mu_f, \mu_g, \sigma_f, \sigma_g$? Can you derive the normalization?

$$z(x) = \frac{1}{\sqrt{4\pi^2\sigma_f^2\sigma_g^2}} exp\Big\{ -\frac{(x-\mu_f)^2}{2\sigma_f^2} - \frac{(x-\mu_g)^2}{2\sigma_g^2} \Big\} \tag{44}$$

Let γ denote the term in the exponent so that $z(x) \propto e^{-\gamma}$:

$$\gamma = \frac{x^2 - 2\mu_f x + \mu_f^2}{2\sigma_f^2} + \frac{x^2 - 2\mu_g x + \mu_g^2}{2\sigma_g^2} \tag{45}$$

$$\gamma = \frac{(x^2 - 2\mu_f x + \mu_f^2)\sigma_g^2 + (x^2 - 2\mu_g x + \mu_g^2)\sigma_f^2}{2\sigma_f^2\sigma_g^2} \tag{46}$$

We want to show that this is quadratic in x. Let's organize powers of x:

$$\gamma = \frac{(\sigma_f^2 + \sigma_g^2)x^2 - 2(\mu_f\sigma_g^2 + \mu_g\sigma_f^2)x + \mu_f^2\sigma_g^2 + \mu_g^2\sigma_f^2}{2\sigma_f^2\sigma_g^2} \tag{47}$$

Divide by $(\sigma_f^2 + \sigma_g^2)$

$$\gamma = \frac{x^2 - 2\frac{\mu_f\sigma_g^2 + \mu_g\sigma_f^2}{\sigma_f^2+\sigma_g^2}x + \frac{\mu_f^2\sigma_g^2 + \mu_g^2\sigma_f^2}{\sigma_f^2+\sigma_g^2}}{\frac{2\sigma_f^2\sigma_g^2}{\sigma_f^2+\sigma_g^2}} \tag{48}$$

We want to express γ, the exponent of $z(x)$ as $(x-\mu_z)^2/2\sigma_z^2$ for functions μ_z and σ_z. This requires completing the square in γ to factorize. Let δ denote the term in the coefficient of x so that $\delta = \frac{\mu_f\sigma_g^2+\mu_g\sigma_f^2}{\sigma_f^2+\sigma_g^2}$. To complete the square we need two numbers who sum to -2δ and whose product is δ^2.

$$\gamma = \frac{x^2 - 2x\frac{\mu_f\sigma_g^2+\mu_g\sigma_f^2}{\sigma_f^2+\sigma_g^2} + \big(\frac{\mu_f\sigma_g^2+\mu_g\sigma_f^2}{\sigma_f^2+\sigma_g^2}\big)^2}{\frac{2\sigma_f^2\sigma_g^2}{\sigma_f^2+\sigma_g^2}} + \frac{\frac{\mu_f\sigma_g^2+\mu_g\sigma_f^2}{\sigma_f^2+\sigma_g^2} - \big(\frac{\mu_f\sigma_g^2+\mu_g\sigma_f^2}{\sigma_f^2+\sigma_g^2}\big)^2}{\frac{2\sigma_f^2\sigma_g^2}{\sigma_f^2+\sigma_g^2}} \tag{49}$$

Simplifying terms:

$$\gamma = \frac{\left(x - \frac{\mu_f \sigma_g^2 + \mu_g \sigma_f^2}{\sigma_f^2 + \sigma_g^2}\right)^2}{\frac{2\sigma_f^2 \sigma_g^2}{\sigma_f^2 + \sigma_g^2}} + \frac{(\mu_f - \mu_g)^2}{2(\sigma_f^2 + \sigma_g^2)} \tag{50}$$

The mean of z is defined $\mu_z = \frac{\mu_f \sigma_g^2 + \mu_g \sigma_f^2}{\sigma_f^2 + \sigma_g^2}$ and the variance of z is defined $\sigma_z^2 = \frac{2\sigma_f^2 \sigma_g^2}{\sigma_f^2 + \sigma_g^2}$. Multiply by σ_z / σ_z to simplify the expression:

$$f(z) = \frac{1}{\sqrt{2\pi\sigma_z^2}} exp\Big\{ - \frac{(x - \mu_z)^2}{2\sigma_z^2} \Big\} \frac{1}{\sqrt{2\pi(\sigma_f^2 + \sigma_g)^2}} exp\Big\{ - \frac{(\mu_f - \mu_g)^2}{2(\sigma_f^2 + \sigma_g^2)} \Big\} \tag{51}$$

This shows that the product of two Gaussians is itself a Gaussian. The normalization on the RHS is also a Gaussian function.

Problem 7

A particle moves along 12 points of a circle. At each step, it is equally likely to move one step in the clockwise or counterclockwise direction. Can you (i) derive a recurrence relation and (ii) compute the expected number of steps for the particle to return to its starting position?

This question requires some information on stochastic processes beyond our treatment of probability. You will not see a question outside of the scope of what we covered on the exam.

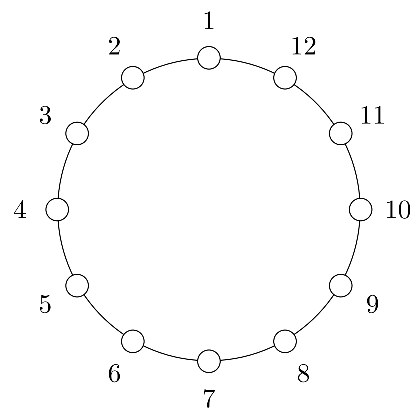


Figure 1: Graph of Markov Chain

Consider the following graph where the vertex 1 denotes the starting position of the particle. Define $f(v)$ as the expected number of steps to return to vertex 1 as a function of position v where $v \in \{1, 2, 3 \cdots, 11, 12\}$. First observe that can only enter vertex 1 by taking one additional step from the adjacent vertices 2 and 12. Now observe that the expected number of steps to the origin can be defined as one step plus the expected number of steps to return to vertices 2 and 12. By linearity of expectation we can define the recurrence:

$$f(1) = 1 + \frac{1}{2}f(2) + \frac{1}{2}f(12) \tag{52}$$

We can express this as follows:

$$a_v = 1 + \frac{1}{2}a_{v+1} + \frac{1}{2}a_{v+11} \tag{53}$$

We are interested in how this process behaves over time and must distinguish between the first visit to vertex 1 and subsequent visits. Define a set of states $s \in \{1, 2, 3, \cdots, 11, 12, \cdots\}$ to denote subsequent visits to each state so that $f(13)$ defines the expected number of steps to the second visit of 1. Observe that

$f(13) = 0$ since the process must move away first. We now have a system of recurrent functions.

$$f(1) = 1 + \frac{1}{2}f(2) + \frac{1}{2}f(12) \tag{54}$$
$$f(2) = 1 + \frac{1}{2}f(3) \tag{55}$$
$$f(3) = 1 + \frac{1}{2}f(2) + \frac{1}{2}f(4) \tag{56}$$
$$f(4) = 1 + \frac{1}{2}f(3) + \frac{1}{2}f(5) \tag{57}$$
$$f(5) = 1 + \frac{1}{2}f(4) + \frac{1}{2}f(6) \tag{58}$$
$$f(6) = 1 + \frac{1}{2}f(5) + \frac{1}{2}f(7) \tag{59}$$
$$f(7) = 1 + \frac{1}{2}f(6) + \frac{1}{2}f(5) \tag{60}$$
$$f(8) = 1 + \frac{1}{2}f(7) + \frac{1}{2}f(9) \tag{61}$$
$$f(9) = 1 + \frac{1}{2}f(8) + \frac{1}{2}f(10) \tag{62}$$
$$f(10) = 1 + \frac{1}{2}f(9) + \frac{1}{2}f(11) \tag{63}$$
$$f(11) = 1 + \frac{1}{2}f(10) + \frac{1}{2}f(12) \tag{64}$$
$$f(12) = 1 + \frac{1}{2}f(11) \tag{65}$$
$$\tag{66}$$

We can apply the result from the Gambler's ruin problem to show that the expected number of steps to first return is $\frac{1}{1/12} = 12$.

Problem 8

1. Consider the complete graph on five vertices, K_5. Is this planar? Prove or disprove.

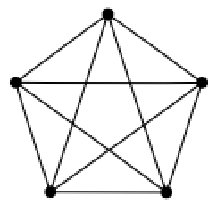


Figure 2: The Graph of K_5

We need an extension of Euler's theorem that is not in D'Angelo to solve the above. For all planar graphs, $3|F| \leq 2|E|$ where $|F|$ is the number of faces and $|E|$ is the number of edges. Equivalently $|E| \geq \frac{3}{2} \times |F|$.

Proof. We can use Euler's formula to prove that the clique K_5 is not planar. There are $v = 5$ vertices and $e = 10$ edges so Euler's formula implies that there should be $f = 7$ faces. By the above for any planar graph $e \geq \frac{3}{2} \times f$ implying that K_5 must have at least $\frac{3}{2} \times 7 = 10.5$ edges. There are $10 < 10.5$ edges therefore K_5 cannot be planar. $\square$