Discrete Mathematics

COMS 3203 – Fall 2017

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Practice Exam # 1 Solution

Problem 1

1.

$$\forall n \in \mathbb{N} \qquad \sum_{i=1}^{n} i^3 = \frac{\big(n(n+1)\big)^2}{4} \tag{1}$$

Base case $n = 1$:

$$1^3 = \frac{(1 \times 2)^2}{4} = \frac{\not{4}}{\not{4}} = 1 \tag{2}$$

Assume true for n, check $n + 1$:

$$1^3 + 2^3 + \cdots + n^3 + (n+1)^3 = \frac{\big(n(n+1)\big)^2}{4} + (n+1)^3 \qquad \text{by } I.H. \tag{3}$$

$$= (n+1)^2 \times \frac{n + 4n + 4}{4} \tag{4}$$

$$= (n+1)^2 \times \frac{(n+2)^2}{4} \tag{5}$$

□

2.

$$\forall n \in \mathbb{N} \text{ such that } n \geq 4 \qquad n! > 2^n \tag{6}$$

Base case $n = 4$:

$$4! = 24 > 16 = 2^4 \tag{7}$$

Assume true for n, check $n + 1$:

$$1 \times 2 \times \cdots (n-1) \times n \times (n+1) > 2^n \times (n+1) \qquad \text{by I.H.} \tag{8}$$

$$> 2^n \times 2 = 2^{n+1} \tag{9}$$

□

3. Prove for all $n \in \mathbb{N}$:

$$(cosx + isinx)^n = cos(nx) + isin(nx) \tag{10}$$

Base case $n = 1$:

$$(cosx + isinx) = cos(x) + isin(x) \tag{11}$$

Assume true for n, check $n + 1$:

$$\underbrace{(cosx + isinx) \times (cosx + isinx) \times \cdots}_{n\ times} \times (cosx + isinx) = (cos(nx) + isin(nx)) \times (cosx + isinx) \quad \text{by I.H.} \tag{12}$$

$$\tag{13}$$

FOIL terms:

$$= cos(nx)cos(x) + icos(nx)sin(x) + isin(nx)cos(x) + i^2 sin(nx)sin(x) \tag{14}$$

$$= \underbrace{\big(cos(nx)cos(x) - sin(nx)sin(x)\big)}_{cos(nx+x)} + i \underbrace{\big(cos(nx)sin(x) + sin(nx)cos(x)\big)}_{sin(nx+x)} \tag{15}$$

Using $cos(\alpha + \beta) = cos(\alpha)cos(\beta) - sin(\alpha)sin(\beta)$ and $sin(\alpha + \beta) = sin(\alpha)cos(\beta) + cos(\alpha)sin(\beta)$:

$$= cos(nx + x) + isin(nx + x) \tag{16}$$

$$\tag{17}$$

□

Problem 2

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{Z}$ be the set of integers. Define a relation $x \sim y$ on $\mathbb{R}$ denoted xRy $\forall x, y \in \mathbb{R}$:

$$x - y \in \mathbb{Z} \tag{18}$$

Proof. Reflexive: We need to show $\forall x \in \mathbb{R}, xRx$. Note that $x - x = 0$ and $0 \in \mathbb{Z} \implies x \sim x$. Symmetric: Define $w = x - y$. Then $y - x = -w$ and $-w \in \mathbb{Z}$ so $x \sim y$ and $y \sim x$. Transitive: Let $a, b, c \in \mathbb{R}$. Suppose $a \sim b$ and $b \sim c$, then $a - b \in \mathbb{Z}$ and $b - c \in \mathbb{Z}$. Define $m = a - b$ and $n = b - c$ and note that $m, n \in \mathbb{Z}$. Then $a - c = a - b + b - c = m + n \in \mathbb{Z}$.

□

Problem 3

Let $f : A \to B$ and $g : B \to C$ be bijections. Prove that their composition $h = (g \circ f) : A \to C$ is a bijection by showing i) h is injective and ii) h is surjective.

Proof. Let's prove that $(g \circ f)(A)$ is surjective, meaning every element in the image is mapped onto.

$$\begin{aligned} h(A) &= (g \circ f)(A) && (19)\\ &= \{c \in C | (g \circ f)(a) = c, \text{ for some } a \in A\} && (20)\\ &= \{c \in C | g(f(a)) = c, \text{ for some } a \in A\} && (21)\\ &= \{c \in C | g(b) = c, \text{ for some } b \in B\} && (22)\\ &= g(f(A)) \quad \text{by definition} && (23)\\ &= g(B) && (24)\\ &= C && (25) \end{aligned}$$

□

We need to show that $(g \circ f)(A)$ is injective, meaning if $h(a) = h(a') \implies a = a'$. Suppose $h(a) = h(a')$. By definition this implies that $g(f(a)) = g(f(a'))$. We know that both g, f are injective (they are bijections) and so $g(f(a)) = g(f(a')) \implies f(a) = f(a')$ and so $a = a'$.

□

Problem 4

$$\exists x_1 \forall x_2 \exists x_3 \forall x_4 \exists x_5 \forall x_6 \exists x_7 \forall x_8 \exists x_9 \ (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_4 \vee \neg x_5 \vee x_3) \wedge (x_6 \vee \neg x_1 \vee x_7) \wedge (x_2 \vee \neg x_8 \vee \neg x_1) \wedge (x_5 \vee x_6 \vee \neg x_9) \tag{26}$$

Set all variables False.

Problem 5

Recall the Fibonacci numbers were defined:

$$F_n = \begin{cases} 1 & n = 0 \\ 1 & n = 1 \\ F_{n-1} + F_{n-2} & n > 1 \end{cases}$$

Denote $\phi = (1 + \sqrt{5})/2$ and prove that $\forall n \in \mathbb{N}$,

$$F_n \leq \phi^n \tag{27}$$

Base case $n = 0, 1$:

$$1 \leq 1 \tag{28}$$

$$1 \leq \frac{1+\sqrt{5}}{2} \tag{29}$$

Assume $P(j)$ true $\forall j$ such that $0 \leq j < k$ and show this implies $P(k)$ true:

$$F_k = F_{k-1} + F_{k-2} \qquad \text{by definition} \tag{30}$$

$$\leq \phi^{k-1} + \phi^{k-2} \qquad \text{by I.H.} \tag{31}$$

$$= \phi^{k-2}(\phi + 1) \tag{32}$$

Note $\phi^2 = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2} = 1 + \frac{1+\sqrt{5}}{2} = 1 + \phi$

$$= \phi^{k-2}\phi^2 \tag{33}$$

$$= \phi \tag{34}$$

□

Problem 6

Recall that a function is convex if $\forall x_1, x_2 \in X$ and $\forall t \in [0, 1]$:

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2) \tag{35}$$

where the (n = 2 above) weights sum to 1 (that is, $t_1 + \cdots + t_n = 1$). Prove that if f is convex, then:

$$f(\frac{1}{n}\sum_{i=1}^{n} x_i) \leq \frac{1}{n}\sum_{i=1}^{n} f(x_i) \tag{36}$$

Proof. Base case: $n = 2$ is correct by the definition of a convex function above. We now assume true for k and check $k+1$:

$$\begin{aligned}
f\Big(\sum_{i=1}^{n+1} t_i x_i\Big) &= f\Big(t_{n+1}x_{n+1} + \sum_{i=1}^{n} t_i x_i\Big) && (37)\\
&= f\Big(t_{n+1}x_{n+1} + (1-t_{n+1})\frac{1}{(1-t_{n+1})}\sum_{i=1}^{n} t_i x_i\Big) && (38)\\
&\leq t_{n+1}f(x_{n+1}) + (1-t_{n+1})f\Big(\frac{1}{(1-t_{n+1})}\sum_{i=1}^{n} t_i x_i\Big) \qquad \text{by I.H.} && (39)\\
&= t_{n+1}f(x_{n+1}) + (1-t_{n+1})f\Big(\sum_{i=1}^{n}\frac{t_i}{(1-t_{n+1})}x_i\Big) && (40)\\
&\leq t_{n+1}f(x_{n+1}) + (1-t_{n+1})\sum_{i=1}^{n}\frac{t_i}{(1-t_{n+1})}f(x_i) \qquad \text{by I.H.} && (41)\\
&= t_{n+1}f(x_{n+1}) + \sum_{i=1}^{n} t_i f(x_i) && (42)\\
&= \sum_{i=1}^{n+1} t_i f(x_i) && (43)
\end{aligned}$$

□