Discrete Mathematics

COMS 3203 – Fall 2017

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Practice Exam # 3

Solve any six problems for full marks. **Good luck and don't panic!** If something is taking too long, move on to the next question. Note that this is a sample exam and while it bears some similarity with the real exam, the two are not isomorphic.

Problem 1

Solve three of the following recurrences:

1. $a_n = 6a_{n-1} - 9a_{n-2}$ when $a_0 = 2, a_1 = 21$
2. $a_n = 4a_{n-1} - 5a_{n-2}$ when $a_0 = 2, a_1 = 6$
3. $a_n = 2a_{n-1} + 1$ when $a_1 = 1$
4. $na_n = (n-2)a_{n-1} + 2$ when $a_1 = 1$

Problem 2

Suppose we select two points randomly on the unit circle $x^2 + y^2 = 1$. What is the probability that the chord joining the two points has length at least 1? How many points are necessary to guarantee that between two of them, there is a chord of length less than 1?

Problem 3

How many members of the set $S = \{1, 2, 3, \cdots, 105\}$ have nontrivial factors in common with 105? Hint: use the inclusion-exclusion principle.

Problem 4

The Poisson distribution is defined below.

$$P(X = x) = e^{-\lambda} \frac{\lambda^x}{x!}$$

Show that the variance of the Poisson distribution is equal to its mean (λ). Hint: use the computational formula for variance.

Problem 5

Show that $\mathbb{E}_Y\Big(\mathbb{E}_X(X|Y)\Big) = \mathbb{E}_X(X)$.

Problem 6

Consider two Gaussian functions:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma_f^2}} e^{-(x-\mu_f)^2/2\sigma_f^2} \qquad g(x) = \frac{1}{\sqrt{2\pi\sigma_g^2}} e^{-(x-\mu_g)^2/2\sigma_g^2}$$

Take their product $z(x) = f(x) \cdot g(x)$. Expand terms in the exponent to show that the product is also a Gaussian function. Hint: you will need to complete the square to factor the polynomial. What is the mean of $z(x)$ denoted μ_z and standard deviation σ_z as a function of $\mu_f, \mu_g, \sigma_f, \sigma_g$? Can you derive the normalization?

Problem 7

A particle moves along 12 points of a circle. At each step, it is equally likely to move one step in the clockwise or counterclockwise direction. Can you (i) derive a recurrence relation and (ii) compute the expected number of steps for the particle to return to its starting position?

Problem 8

1. Draw a connected planar graph with 4 faces, degrees: 3, 3, 4, 4. Is there only one isomorphic graph?

2. Consider the complete graph on five vertices, K_5. Is this planar? Prove or disprove.

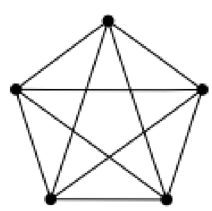

Figure 1: The Graph of K_5