Discrete Mathematics

COMS 3203 – Fall 2017

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Practice Exam # 1

Solve any four problems for full marks. Good luck and don't panic! If something is taking too long, move on to the next question. Note that this is a sample exam and while it bears some similarity with the real exam, the two are not isomorphic.

Problem 1

Prove the following by induction.

1. State $P(n)$, the *base case*, *inductive hypothesis* and *inductive step* explicitly.

$$\forall n \in \mathbb{N} \qquad \sum_{i=1}^{n} i^3 = \frac{\big(n(n+1)\big)^2}{4} \tag{1}$$

2. Recall $n!$ is defined as $n \times (n-1) \times \cdots \times 3 \times 2 \times 1$.

$$\forall n \in \mathbb{N} \text{ such that } n \geq 4 \qquad n! > 2^n \tag{2}$$

3. Prove for all $n \in \mathbb{N}$:

$$(cosx + isinx)^n = cos(nx) + isin(nx) \tag{3}$$

   Hint: you may use the angle sum/difference identity $cos(a \pm b) = cos(a)cos(b) \pm sin(a)sin(b)$ and $sin(a \pm b) = sin(a)cos(b) + cos(a)sin(b)$.

Problem 2

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{Z}$ be the set of integers. Define a relation $x \sim y$ on $\mathbb{R}$ denoted xRy $\forall x, y \in \mathbb{R}$:

$$x - y \in \mathbb{Z} \tag{4}$$

Recall that an equivalence relation is one that is *reflexive*, *symmetric* and *transitive*. Is the above an equivalence relation? Prove or provide a counter example.

Problem 3

Let $f : A \to B$ and $g : B \to C$ be bijections. Prove that their composition $h = (g \circ f) : A \to C$ is a bijection by showing i) h is injective and ii) h is surjective.

Problem 4

Consider the following game between existential and universal quantifiers. The existential player attempts to satisfy all constraints whereas the universal player aims to falsify the constraint. Is there a winning strategy, a way to assign variables such that the expression below is true, regardless of the universal quantifiers?

$$\exists x_1 \forall x_2 \exists x_3 \forall x_4 \exists x_5 \forall x_6 \exists x_7 \forall x_8 \exists x_9 \ (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_4 \vee \neg x_5 \vee x_3) \wedge (x_6 \vee \neg x_1 \vee x_7) \wedge (x_2 \vee \neg x_8 \vee \neg x_1) \wedge (x_5 \vee x_6 \vee \neg x_9) \tag{5}$$

Explain.

Problem 5

Recall the Fibonacci numbers were defined:

$$F_n = \begin{cases} 1 & n = 0 \\ 1 & n = 1 \\ F_{n-1} + F_{n-2} & n > 1 \end{cases}$$

Denote $\phi = (1 + \sqrt{5})/2$ and prove that $\forall n \in \mathbb{N}$,

$$F_n \leq \phi^n \tag{6}$$

Hint: use strong induction.

Problem 6

Recall that a function is convex if $\forall x_1, x_2 \in X$ and $\forall t \in [0, 1]$:

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2) \tag{7}$$

where the (n = 2 above) weights sum to 1 (that is, $t_1 + \cdots + t_n = 1$). Prove that if f is convex, then:

$$f(\frac{1}{n}\sum_{i=1}^{n} x_i) \leq \frac{1}{n}\sum_{i=1}^{n} f(x_i) \tag{8}$$

Hint: use the AGM inequality or induction.