Discrete Mathematics

COMS 3203 – Fall 2017

http://www.cs.columbia.edu/~amoretti/3203

Homework # 6 (Optional)

Due Wednesday, December 20th

Solve any ten problems for full marks. You are encouraged to form study groups and discuss with your classmates, come to office hours and post on Piazza but the write up must be your own. Problems denoted (+) are more difficult, and ones designated by (!) may be more interesting or instructive as opposed to emphasizing computation.

1. Recurrence Relations (12.1-12.5)

Find a formula for a_n given the stated recurrence relation and initial values.

1. $a_n = 3a_{n-1} - 2$ for $n \geq 1$ with $a_0 = 1$.
2. $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 2$ with $a_0 = 1, a_1 = 8$.
3. $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 2$ with $a_0 = a_1 = 1$.
4. $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$ with $a_0 = 1, a_1 = 3$.
5. $a_n = 3a_{n-1} - 1$ for $n \geq 1$ with $a_0 = 1$.

2. (12.6)

Suppose that $\langle a \rangle$ satisfies the recurrence $a_n = -a_{n-1} + \lambda^n$. Determine the values of λ such that $\langle a \rangle$ can be unbounded.

3. (12.7)

Let $a_n = n^3$. Find a constant-coefficient first-order linear recurrence relation satisfied by $\langle a \rangle$. Does there exist a homogeneous constant-coefficient first-order linear recurrence relation satisfied by $\langle a \rangle$? Why or why not?

4. (12.14)

Complete the proof of Corollary 12.18, solving the recurrence $a_n = ca_{n-1} + f(n)\beta^n$, where f is a polynomial and β a constant.

5. (12.41)

Let f be a polynomial of degree n. The *first difference* of f is the function $g = \Delta f$ defined by $g(x) = f(x+1) - f(x)$. The *kth difference* of f is the function $g^{(k)}$ defined inductively by $g^{(0)} = f$ and $g^{(k)} = \Delta g^{(k+1)}$ for $k \geq 1$. Obtain a formula for the nth difference of f

6. (12.43)

Let G_n be the graph consisting of a path with n vertices plus one vertex adjacent to each vertex of the path. Let a_n be the number of spanning trees in G_n.

- Prove that $a_n = a_{n-1} + \sum_{i=0}^{n-1} a_i$ for $n \geq 2$, where $a_0 = a_1 = 1$.
- Prove that $a_n = 3a_{n-1} - a_{n-2}$ for $n \geq 3$.

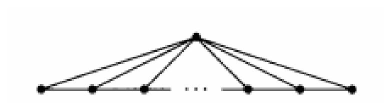

Figure 1: 12.43

7. (12.44)

Let G_n be the graph on $2n$ vertices and $3n - 2$ edges pictured below, for $n \geq 1$. Prove that the chromatic polynomial of G_n is $(k^2 - 3k + 3)^{n-1}k(k-1)$.

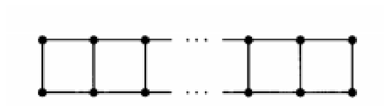

Figure 2: 12.44

8. Gambler's Ruin (12.45)

Two people gamble by flipping a fair coin until one goes broke. If the flip is heads, then A pays B \$1; otherwise B pays A \$1. Suppose A starts with r dollars and B starts with s dollars. Let $a_n(r, s)$ be the probability that A goes broke on the nth flip. Obtain (and prove) a recurrence relation for $a_n(r, s)$. (There are three parameters; be careful about the initial values.)

9. (11.12)

($-$) Among the graphs below, which pairs are isomorphic?

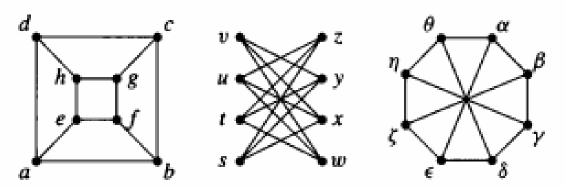

Figure 3: 11.12

10. Peterson Graph (11.17)

(!) The Peterson graph is the graph on the left below. Prove that the graphs below are pairwise isomorphic and thus all represent the Peterson graph.

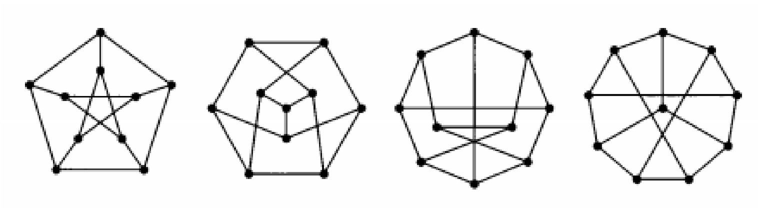

Figure 4: Peterson Graph