# Discrete Mathematics

COMS 3203 – Fall 2017

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## Homework # 5

Due Sunday, December 10th

**Solve any ten problems for full marks**. You are encouraged to form study groups and discuss with your classmates, come to office hours and post on Piazza but the write up must be your own. Problems denoted (+) are more difficult, and ones designated by (!) may be more interesting or instructive as opposed to emphasizing computation.

### 1. Warm Up

(-) Give the sample space (the set of possible outcomes) for the following:

1. Choosing a prime number less than 30 uniformly at random.
2. Choosing one letter uniformly at random from the word PROBABILITY.
3. Repeatedly rolling a die until getting a 6.
4. Give the size of the sample space of a five card poker hand.

### 2. Definitions

(-) Give the probability mass functions and cumulative distribution functions for the following. Plot the graph of the cumulative distribution functions.

1. The sum of two fair die.
2. The product of two die.
3. The maximum of two die.

### 3. Conditional Probability

Suppose that 5 percent of men and 0.25 percent of women are colorblind. If a colorblind person is chosen at random, what is the probability of that person being male (assuming that there are an equal number of males and females)? What if the population had twice as many males as females?

## 4. (Problem 9.11)

In a famous game show on television, a prize is placed behind one of three doors, with probability 1/3 for each door. The contestant chooses a door. The host then opens one of the other doors and says "As you can see, the prize is not behind this door. Do you want to stay with your original guess or switch to the remaining door?" When the contestant has chosen a wrong door, the host opens the other wrong door. When the contestant has chosen the right door, the host opens one of the two wrong doors, each with probability 1/2. Show that the contestant should switch.

## 5. (Problem 9.25)

(+) Consider $n$ envelopes with amounts $a_1, \cdots, a_n$ in dollars, where $a_1 \leq \cdots \leq a_n$. A gambler is presented two successive envelopes, with the probability being $p_i$ that the envelopes contain $a_i$ and $a_{i+1}$ dollars, for $1 \leq i \leq n-1$. She opens one of these two envelopes at random and sees what it contains. She then can either keep that amount or switch to the other envelope. Suppose that she sees $a_k$ dollars. In terms of the data of the problem, determine whether she should switch.

## 6. Bernoulli Recurrence

Recall that independent trials that result in a success with probability $p$ and failure $(1-p)$ are called Bernoulli trials. Let $P_n$ be the probability that $n$ Bernoulli trials result in an even number of successes ($0$ being an even number). Show that:

$$P_n = p(1 - P_{n-1}) + (1-p)P_{n-1} \qquad n \geq 1$$

and use this formula to prove (by induction) that

$$P_n = \frac{1 + (1-2p)^n}{2} \tag{1}$$

## 7. Approximating a Binomial

(+) Show that the Poisson random variable can be used to approximate a Binomial random variable when $n$ is large and $p$ is small. Hint: If $X$ is a Binomial random variable with parameters $(n, p)$ and the Poisson parameter $\lambda = np$, you want to show:

$$P\{X = x\} = \frac{n!}{(n-x)!x!}p^x(1-p)^{n-x} = e^{-\lambda}\frac{\lambda^x}{x!} \tag{2}$$

## 8. (Problem 9.37)

Let $a_n$ denote the number of lattice paths of length $2n$ that never step above the diagonal (these end at some point $(k, 2n-k)$ with $k \geq n$). Prove that $a_n = \binom{2n}{n}$.

## 9. Algorithm for Unbiased Bernoulli Trials

Consider a black box that when activated returns two Bernoulli trials whose outcomes are each H or T. That is, each time you run the procedure you return HH, HT, TH or TT. Can you design an algorithm that takes as input two possibly biased Bernoulli trials and returns a fair Bernoulli trial? You are allowed to discard certain events or simply rerun the black box instantly. Prove that your algorithm returns a single unbiased random variable.

## 10. Limiting Distributions and the CLT

Prove the following:

$$\lim_{n\to\infty} e^{-n}\sum_{k=0}^{n}\frac{n^k}{k!} = \frac{1}{2} \tag{3}$$

Hint: Use the Central Limit Theorem.

## 11. Bernoulli Trials Revisited

Suppose you and a friend play a series of chess matches. Each game is independently won by you with a probability $p$ and by your friend with a probability $1-p$. You stop when the total number of wins of one person is two greater than that of the other. The individual with the greater number of total wins is declared the winner of the series.

1. Find the probability that a total of 4 games are played.
2. Find the probability that you are the winner of the series.

## 12. (Problem 10.17)

(!) Prove that the Erdos-Szekeres result is best possible by constructing for each $n$ (with proof) a list of $n^2$ distinct numbers having no monotone sublist of length $n+1$.

## 13. (Problem 10.22)

Given $m \geq 2n$, let $S$ be a set of $m$ points on a circle with no two diametrically opposite. Say that $x \in S$ is "free" if fewer than $n$ points of $S - x$ lie in the semicircle clockwise from $x$. Prove that $S$ has at most $n$ free points. (Hint: Reduce the problem to the case $m = 2n$.)

## 14. (Problem 10.31)

Let $A_1, \cdots, A_n$ be subsets of a universe $U$. Let $T \subseteq [n]$ be a collection of indices, and let $N(T)$ be the number of elements of $U$ that belong to the sets indexed by $T$ but to no others among $A_1, \cdots, A_n$. By defining a new universe, prove the following generalization of the inclusion-exclusion formula:

$$N(T) = \sum_{T\subseteq S\subseteq[n]} (-1)^{|S|-|T|}\Big|\bigcap_{i\in S} A_i\Big| \tag{4}$$

## 15. Gamma and Beta Functions

($\star$) Gamma and Beta functions play an important role in defining probability densities of the same names and are natural analogues to functions and discrete distributions we have seen. (This uses a bit of calculus).

1. Recall that the Gamma function can be used to interpolate the factorial.

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du \tag{5}$$

   Prove by induction that $\Gamma(x) = (x-1)!$ What discrete distribution is the product of an exponential and a polynomial (divided by a factorial)?

2. Perform a change of variables to show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Comment on the connection with the Gaussian function.

3. Comment on similarities between the Beta function and the Binomial coefficient.

$$B(x,y) = \int_0^1 p^{x-1}(1-p)^{y-1} dp \qquad x, y > 0 \tag{6}$$

   Apply Poisson's 'trick' (double integration in Polar coordinates) that we used in class to show the following:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{7}$$