Discrete Mathematics

COMS 3203 – Fall 2017

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Homework # 3

Due Sunday, November 5th

Solve any ten problems for full marks. You are encouraged to form study groups and discuss with your classmates, come to office hours and post on Piazza but the write up must be your own. Problems denoted (+) are more difficult, and ones designated by (!) may be more interesting or instructive as opposed to emphasizing computation.

5.8

1. A fair coin is flipped exactly $2n$ times. Compute the probability of obtaining exactly n heads. Evaluate the formula for $n = 10$.
2. Determine the coefficient of x^4y^5 in the expansion of $(x + y)^9$.
3. Compute the probability that a random five-card hand has a) at least three cards with the same rank, b) at least two cards with the same rank.

5.23

In terms of binomial coefficients, count the (five-card) poker hands having:

1. One pair (two cards of equal rank and no others of equal rank)
2. Full house (two cards of equal rank and three cards of another rank)
3. Straight flush (five consecutive cards from the same suit).

5.24

(!) A *bridge hand* consists of 13 cards from a standard 52-card deck. Its *distribution* is the list in non increasing order of the number of cards in each suit. Thus 5440 denotes a hand with five cards in one suit and four cards in each of two others. List the distributions, find their probabilities and rank them. Explain intuitively why 4333 ranks so low.

5.26

Use Pascal's Formula to prove the Binomial Theorem by induction on n.

5.31

(!) Count the ways to group $2n$ distinct people into pairs. (The answer is 1 when $n = 1$ and is 3 when $n = 2$.)

5.53

Let $s(f)$ be the minimum number of transpositions needed to transform the permutation f to the identity permutation. Without considering cycle structure, give a direct procedure to sort a permutation using at most $n-1$ permutations. Prove that the permutation $n\ n-1 \cdots 1$ requires at least $n/2$ transpositions to sort.

5.65

(+) The goal of this problem is to determine which polynomials p with rational coefficients have the property that $p(n) \in \mathbb{Z}$ if $n \in \mathbb{Z}$. Let I be the set of polynomials with this property. Recall that the sum $p+q$ of two functions p, q on a set $\mathcal{S}$ is the function h such that $h(x) = p(x) + q(x)$. Similarly, the scalar multiple $n \cdot p$ is the function h such that $h(x) = n \cdot p(x)$.

1. Show that if $p, q \in I$ and $n \in \mathbb{Z}$, then $p + q \in I$ and $n \cdot p \in I$.
2. Show that $p_j \in I$, where $p_j(x) = \binom{x}{j}$, and that $\sum_{j=0}^{k} n_j \binom{x}{j} \in I$ for $\{n_j\} \subseteq \mathbb{Z}$.
3. Let f be a polynomial of degree k with rational coefficients. Prove that f can be expressed as $f(x) = \sum_{j=0}^{k} b_j \binom{x}{j}$, where the b_j's are rational. (Hint: One way to prove this uses induction on the degree of the polynomial.)
4. Prove that $f \in I$ if and only if $f(x) = \sum_{j=0}^{k} b_j \binom{x}{j}$ where the b_j's are integers. (Hint: Evaluate f at the integers in the set $\{0, \cdots, k\}$. Note that $\binom{0}{0} = 1$ by our convention that $0! = 1$).

5.97

(+) In class we defined Leibniz formula for the determinant using permutations (where S_n is the symmetric group):

$$det(A) = \sum_{\sigma \in S_n} (-1)^{\#inv} \prod_{i=1}^{n} a_{\sigma(i),i} \tag{1}$$

Why is this function called a determinant? What exactly does it determine? An undirected graph $G = (V, E)$ is an ordered pair comprised of a set of nodes and edges. We wish to find a subset of edges $M \subseteq E$ of size $n/2$ such that every pair $e, e' \in M$ do not share endpoints $e \cap e' = \emptyset$. This is called a *perfect matching* and means that every node is covered by the matching M (Think of this as finding a prom date for everyone in two disjoint sets). We can construct the following matrix:

$$A_{ij} = \begin{cases} x_{ij} & (u_i, v_j) \in E \\ 0 & (u_i, v_j) \notin E \end{cases} \tag{2}$$

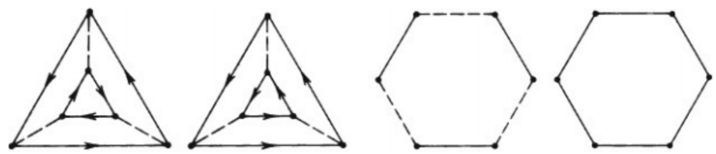

Figure 1: Cayley Diagrams for Four Groups

Describe how to construct the matrix from the graph. You may let components of A (the variables x_{ij}) take values of 1 if two people both 'swipe right.' Can you show that $det(A) \neq 0$ if G contains a perfect matching?

5.58

Consider the Cayley diagrams for the groups G_1, G_2, G_3, and G_4 in Figure 1. Find and prove all isomorphisms between the groups.

5.99

Prove that isomorphism is an equivalence relation on the set of all groups.

6.9

(-) For each diophantine equation below, find all solutions, if any exist.

- $17x + 13y = 200$.
- $21x + 15y = 93$.
- $60x + 42y = 104$.
- $588x + 231y = 63$.

6.15

What is the smallest number of American coins (values may repeat) sufficient to make change equal to any value from 1 cent through 99 cents? Is there only one optimal solution? What is the answer when the coins can be made with any desired value?

6.33

Let abc be a 3-digit natural number (written in base 10). Prove that the 6-digit number $abcabc$ has at least three distinct prime factors.

6.41

Polya's proof for infinitude of primes. Let $a_n = 2^{2^n} + 1$. Prove by induction that a_n divides $a_m - 2$ if $n < m$. Conclude that a_n and a_m have no common factors if $n \neq m$. Use this to prove that there are infinitely many primes. (This method also proves that there are at least $log_2\, log_2\, N$ primes less than N.)

6.43

Let a and b be nonnegative integers. Prove that the following algorithm computes $gcd(a, b)$. Each step of the algorithm replaces the current pair of numbers with a new pair or reports an output, according to the following rules.

1. When one number is 0 or they are equal, stop and report the maximum of the pair as output.
2. When both numbers are nonzero and at least one is even, divide the first even of the pair by 2.
3. When both numbers are odd, replace the larger one with their difference. (Comment: This algorithm runs faster than the Euclidean Algorithm).

6.55

(+) A set $S \subseteq \mathbb{Z}$ is an *ideal* in $\mathbb{Z}$ if S is nonempty and satisfies 1) if $a, b \in S$, then $a + b \in S$, and 2) if $a \in S$ and $n \in \mathbb{Z}$, then $na \in S$. Prove that every ideal in $\mathbb{Z}$ is the set of multiples of a single integer. (Comment: This strengthens Theorem 6.12, showing that every ideal in $\mathbb{Z}$ is a principal ideal - see Definition 6.25. The analogous result for $\mathbb{R}[x]$ is Theorem 6.26.)