Discrete Mathematics

COMS 3203 – Fall 2017

http://www.cs.columbia.edu/~amoretti/3203

Homework # 2

Due Monday, October 9th

To receive credit for induction please explicitly state: (1) the base case, (2) inductive hypothesis, (3) inductive step.
Solve any ten problems for full marks (no extra credit here but good practice for the midterm). You are encouraged to form study groups and discuss with your classmates, come to office hours and post on Piazza but the write up must be your own. Problems denoted (+) are more difficult, and ones designated by (!) may be more interesting or instructive as opposed to emphasizing computation.

3.5

In the five exercises below, determine whether the statement is true or false. If true, provide a proof. If false, provide a counterexample:

1. For $n \in \mathbb{N}$, $\sum_{k=1}^{n}(2k+1) = n^2 + 2n$.
2. If $P(2n)$ is true for all $n \in \mathbb{N}$, and $P(n) \to P(n+1)$ for all $n \in \mathbb{N}$, then $P(n)$ is true for all $n \in \mathbb{N}$.
3. For $n \in \mathbb{N}$, $2n - 8 < n^2 - 8n + 17$.
4. For $n \in \mathbb{N}$, $2n - 18 < n^2 - 8n + 8$.
5. For $n \in \mathbb{N}$, $\frac{2n-18}{n^2-8n+8} < 1$.

3.15

For $n \in \mathbb{N}$, prove that $\sum_{i=1}^{n}(-1)^i i^2 = (-1)^n \frac{n(n+1)}{2}$

3.16

For $n \in \mathbb{N}$, prove that $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

3.27

For $n \in \mathbb{N}$, prove that $\sum_{i=1}^{n} \frac{1}{(3i-2)(3i+1)} = \frac{n}{3n+1}$

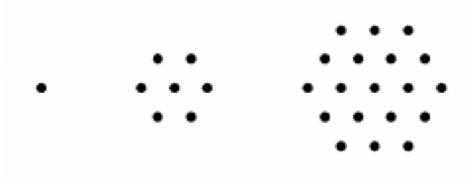

Figure 1: Hexagonal Arrangement S_n

3.43

(!) Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(xy) = xf(y) + yf(x)$ for all $x, y \in \mathbb{R}$, Prove that $f(1) = 0$ and that $f(u^n) = nu^{n-1}f(u)$ for all $n \in \mathbb{N}$ and $u \in \mathbb{R}$.

3.55

Let $\langle a \rangle$ be a sequence satisfying $a_1 = 1$, $a_2 = 8$, and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$. Prove for $n \in \mathbb{N}$ that $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$.

3.65

(!) In the village of Perfect Reasoning, each employer has an apprentice. At least one apprentice is a thief. To remedy this without embarrassment, the mayor proclaims the following true statements: ”At least one apprentice in this town is a thief. Every thief is known to be a thief by everyone except his or her employer, and all employers reason perfectly. If n days from now you have concluded that your apprentice is a thief, you will come to the village square at noon that day to denounce your apprentice.” The villagers gather at noon every day thereafter. If in fact $k \geq 1$ of the apprentice are thieves, when will they be denounced, and how do their employers reason? (Hint: Study small values of k, and use induction to prove the pattern for all k).

3.39

(!) Let S_n be the hexagonal arrangement consisting of n rings of dots, as illustrated above for $n \in \{1, 2, 3\}$. Let a_n be the number of dots in S_n. Find formulas for a_n and $\sum_{k=1}^{n} a_k$ (simplifying all sums).

4.12

Determine which of the following statements are true. Give proofs for the true statements and counterexamples for the false statements.

1. Every decreasing function $f : \mathbb{R} \to \mathbb{R}$ is surjective.
2. Every nondecreasing function $f : \mathbb{R} \to \mathbb{R}$ is injective.
3. Every injective function $f : \mathbb{R} \to \mathbb{R}$ is monotone.
4. Every surjective function $f : \mathbb{R} \to \mathbb{R}$ is unbounded.

5. Every unbounded function $f : \mathbb{R} \to \mathbb{R}$ is surjective.

4.17

(+) *The Game of Nim*. A position in Nim consists of some piles of coins. Two players alternate, with each move removing a portion of one pile. The winner is the player who takes the last coin.

Suppose that the starting piles have sizes $n_1, \cdots, n_k$. Prove that Player 2 has a winning strategy if and only if for every j, an even number of $n_1, \cdots, n_k$ have a 1 in position j in their binary representation. For example, when the sizes are $1, 2, 3$, the binary representations are $1, 10, 11$, and the condition holds.

4.24

Let f and g be surjections from $\mathbb{Z}$ to $\mathbb{Z}$, and let $h = fg$ be their product (Definition 1.25). Must h also be surjective? Give a proof or a counterexample.

4.25

Determine which formulas below define surjections from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$.

1. $f(a, b) = a + b$.
2. $f(a, b) = ab$.
3. $f(a, b) = ab(b + 1)/2$.
4. $f(a, b) = (a + 1)b(b + 1)/2$.
5. $f(a, b) = ab(a + b)/2$.

4.32

Let $\mathcal{F}$ be a field. Define f on $\mathcal{F}$ by $f(x) = -x$, and define g on $\mathcal{F} - \{0\}$ by $g(x) = x^{-1}$. Prove that f is a bijection from $\mathcal{F}$ to $\mathcal{F}$ and that g is a bijection from $\mathcal{F} - \{0\}$ to $\mathcal{F} - \{0\}$.

4.47

Prove that the natural numbers, the even natural numbers, and the odd natural numbers form sets of the same cardinality (they are countable).

4.51

(!) Construct an explicit bijection from the open interval $(0, 1)$ to the closed interval $[0, 1]$.