Discrete Mathematics

COMS 3203 – Fall 2017

http://www.cs.columbia.edu/~amoretti/3203

Homework # 0

Due Sunday, September 24th

Solve any ten problems for full marks (no extra credit here). You are encouraged to form study groups and discuss with your classmates, come to office hours and post on Piazza but the write up must be your own. Problems denoted (+) are more difficult, and ones designated by (!) may be more interesting or instructive as opposed to emphasizing computation.

Problem 0.1 (Warm Up)

Tell us a little bit about yourself. Why are you majoring in computer science (or something else)? What is the relationship between mathematics and computer science? That is, why bother with a course called "discrete mathematics?" Why the emphasis on proofs rather than learning how to carry out mathematical procedures? What do you hope to get out of the course?

Problem 0.2

Can you prove that for every integer x and for every integer y, if x is odd and y is odd then xy is odd? Translate into symbols using quantifiers.

Problem 1.8

During the morning section of a physics course, two of nine women and two of ten men receive an A. In the evening section, six of the nine women and nine of the fourteen men receive A. Confirm that in each section, a higher proportion of women than of men receive A, but in the combined course, a lower proportion of women than men receive A. Can you give a formal explanation?

Problem 1.21

Let $a, b, c \in \mathbb{R}$ with $a \neq 0$. Find the flaw in the "proof" below that $-b/2a$ is a solution to $ax^2 + bx + c = 0$:

> Let x and y be solutions to the equation. Subtracting $ay^2 + by + c = 0$ from $ax^2 + bx + c = 0$ yields $a(x^2 - y^2) + b(x - y) = 0$, which we rewrite as $a(x + y)(x - y) + b(x - y) = 0$. Hence $a(x + y) + b = 0$, and thus $x + y = -b/a$. Since x and y can be any solutions, we can apply this computation letting y have the same value as x. With $y = x$, we obtain $2x = -b/a$ or $x = -b/2a$.

Problem 1.26

(+) Two post office workers meet along their paths and discuss the following. A: "I know you have three sons. How old are they?" B: "If you take their ages, expressed in years, and multiply those numbers, the result will equal your age." A: "But that's not enough to tell me the answer!" B: "The sum of these three numbers equals the number of windows in that building." A: "Hmm [pause]. But it's still not enough!" B: "My middle son is red-haired." A: "Ah, now it's clear!" How old are the sons? (Hint: The ambiguity at the earlier stages is needed to determine the solution for the full conversation.) (G.P. Klimov).

Problem 1.31

(+) In class we proved the arithmetic-geometric means inequality. Extend this result to the following:

1. Prove that $4xyzw \leq x^4 + y^4 + z^4 + w^4$ for real numbers x, y, z, w. Hint: Use the inequality $2tu \leq t^2 + u^2$ repeatedly.
2. Prove that $3abc \leq a^3 + b^3 + c^3$ for non negative a, b, c. Hint: In the inequality of part (1), set w equal to the cube root of xyz.

Problem 1.36

Let $\mathcal{S} = [3] \times [3]$ (the Cartesian product of $\{1, 2, 3\}$ with itself). Let $\mathcal{T}$ be the set of ordered pairs $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ such that $0 \leq 3x + y - 4 \leq 8$. Prove that $\mathcal{S} \subseteq \mathcal{T}$. Does equality hold?

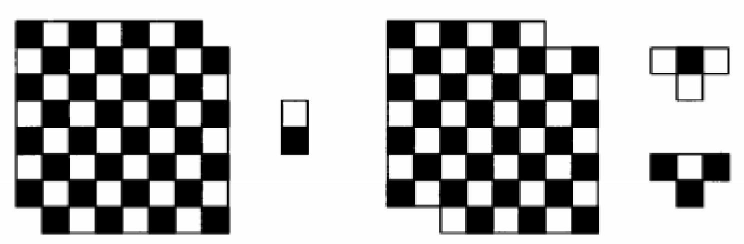
Problem 1.41

Let A, B, C be sets. Explain the relationships below. Use the definitions of set operations and containment along with Venn diagrams to guide the argument.

1. $A \subseteq A \cup B$ and $A \cap B \subseteq A$.
2. $A - B \subseteq A$
3. $A \cap B = B \cap A$, and $A \cup B = B \cup A$.
4. $A \subseteq B$ and $B \subseteq C$ imply $A \subseteq C$.
5. $A \cap (B \cap C) = (A \cap B) \cap C$.
6. $A \cup (B \cup C) = (A \cup B) \cup C$.

Problem 1.55

(+) Let $\mathcal{F}$ be a field consisting of exactly three elements $0, 1, x$. Prove that $x + x = 1$ and that $x \cdot x = 1$. Obtain the addition and multiplication tables for $\mathcal{F}$.



Problem 2.37

Given a real number x, let A be the statement "$\frac{1}{2} < x < \frac{5}{2}$", let B be the statement $x \in \mathbb{Z}$, let C be the statement $x^2 = 1$, and let D be the statement $x = 2$. Which statements below are true for all $x \in \mathbb{R}$?

1. $A \to C$.
2. $B \to C$.
3. $(A \wedge B) \to C$.
4. $(A \wedge B) \to (C \vee D)$.
5. $C \to (A \wedge B)$.
6. $D \to [A \wedge B \wedge (\neg C)]$.
7. $(A \vee C) \to B$.

Problem 2.40

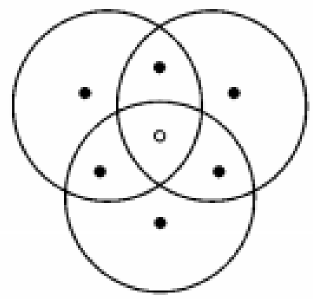
(!) *Checkerboard problems.* (Hint: Use the method of contradiction.)

1. Two opposite corner squares are removed from an eight by eight checkerboard. Prove that the remaining squares cannot be covered exactly by dominoes (rectangles that consist of two adjacent squares).
2. Two squares from each of two opposite corners are deleted shown on the right above. Prove that the remaining squares cannot be covered exactly by copies of the "T-shape" and its rotations.

Problem 2.47

Let $P(x)$ be the assertion "x is odd", and let $Q(x)$ be the assertion "$x^2 - 1$ is divisible by 8". Determine whether the following statements are true:

1. $(\forall x \in \mathbb{Z})[P(x) \to Q(x)]$
2. $(\forall x \in \mathbb{Z})[Q(x) \to P(x)]$



Problem 2.48

Let $P(x)$ be the assertion "x is odd", and let $Q(x)$ be the assertion "x is twice an integer". Determine whether the following statements are true:

1. $(\forall x \in \mathbb{Z})[P(x) \rightarrow Q(x)]$
2. $(\forall x \in \mathbb{Z})\big(P(x)\big) \rightarrow (\forall x \in \mathbb{Z})\big(Q(x)\big)$

2.54

(+) Consider three circles in the plane shown below. Each bounded region contains a token what is white on one side and black on the other. At each step, we can either (a) flip all four tokens inside one circle, or (b) flip the tokens showing white inside one circle to make all four tokens in that circle show black. From the starting configuration with all tokens showing black, can we reach the indicated configuration with all showing black except the token in the central region? (Hint: Consider parity conditions and work backwards from the desired configuration.)